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Three-Pion States in the $K_L \to \mu^+\mu^-$ Puzzle

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Contributions to the absorptive $K_L \to 2\mu$ amplitude coming from intermediate 3π states are estimated on the basis of recent soft-pion results for the process $3\pi \to 2\gamma$. These contributions turn out to be far too small, by 4 orders of magnitude, to resolve the $K_L \to 2\mu$ puzzle.

All theoretical resolutions so far proposed for the $K_L \to 2\mu$ puzzle[1] are forced to call upon cancellation effects which have to be regarded as accidental at the present level of understanding. Apart from this, the various schemes differ widely with respect to introduction of qualitatively new physics.[2] The most conservative approach is one which dismisses the possibility that CP violation or new kinds of particles or interactions play an important role in the puzzle. Instead, the burden is placed on 3π intermediate states, which are supposed to provide terms which largely cancel the contribution from the 2γ state in the unitarity equation for the absorptive $K_L \to 2\mu$ amplitude. The strain on credulity here lies in the magnitude required of the 3π contribution, a magnitude which has to be appreciably larger than first rough estimates would suggest.[3] In the present note we add our contribution to this strain, in the form of an estimate of 3π contributions based on soft-pion considerations.

In order to assess the 3π effects in a framework which ignores CP violation and accepts standard photon-lepton electrodynamics, one requires information on the amplitudes for $3\pi \to 2\gamma$ and $3\pi \to 2\mu$. In our conventional framework, the latter is fully specified if the former is known for virtual as well as real photons. All the remaining ingredients of a unitarity analysis based on 2γ and 3π intermediate states are well enough known: the $2\gamma \to 2\mu$ amplitude from standard electrodynamic theory, the $K_L \to 2\gamma$ and $K_L \to 3\pi$ amplitudes (or rather, their moduli) from experiment. Throughout the unitarity discussion we ignore all other intermediate states. To lowest order in the fine-structure constant the $2\pi\gamma$ and, strictly speaking, also the $3\pi\gamma$ intermediate states ought to be considered. However, the former has been shown to be unimportant,[4] and the latter can reasonably be expected, on phase-space considerations alone, to be even more negligible. We shall have a brief comment on this later on.

At theoretical issue then are the amplitudes for $3\pi^0$, $\pi^+\pi^-\pi^0 \to$ two real or virtual photons. These objects are of course interesting in their own right, even apart from their role in the $K_L \to 2\mu$ puzzle. In particular, the application of soft-pion considerations has been discussed by Aviv, Hari Dass, and Sawyer[5]; and the subject has since been taken up by other authors.[6-10] Interesting issues concerning current algebra, partial conservation of axial-vector current (PCAC), and Ward-identity anomalies arise here. Especially relevant for our present purposes is the idea, proposed by Aviv and Sawyer,[11] that the soft-pion approximation might provide a reasonable basis for estimating contributions from the 3π states in the unitarity analysis of $K_L \to 2\mu$ decay. It must be said at once that, kinematically, the pions in $K_L \to 3\pi$ decay cannot all three be so very soft, unless one regards the K-meson mass to be "small" on a hadronic scale. With appropriate reservations on this score, one may nevertheless hope that the soft-pion methods provide more reliable estimates than can be gained from purely dimensional and phase-space arguments.

The Aviv-Sawyer analysis[11] of $K_L \to 2\mu$ decay was based on the $3\pi \to 2\gamma$ amplitudes of Refs. 5 and 6. We believe that these amplitude results are in error and that the correct soft-pion expressions are as in Ref. 8. We have therefore repeated the analysis. Despite these corrections, we find with Aviv and Sawyer that the 3π states play a negligible role in the absorptive amplitude for $K_L \to 2\mu$ decay. The "naive" unitarity bound, based solely on the 2γ intermediate state, is corrected at most (depending on phases) by a factor of order 10^{-4} in the decay rate. A brief account follows.

The $K_L \to 2\mu$ amplitude has the structure

$$\mathrm{Amp}(K_L \to 2\mu) = g\bar{u}(p)\gamma_5 v(\bar{p}), \tag{1}$$

where p and $\bar{p}$ denote the μ^- and μ^+ momenta. The decay rate is given by

$$\Gamma(K_L \to 2\mu) = \frac{M}{8\pi} v |g|^2 , \qquad (2)$$

where

$$v = (1 - 4m^2/M^2)^{1/2}$$

is the muon velocity, with m the μ mass, and M the K mass. The object is to estimate the absorptive amplitude Img, on the basis of unitarity considerations, in order to set a lower bound for $K_L \to 2\mu$ decay. To get at the unitarity contribution from the intermediate 2γ state we have to consider $K_L \to 2\gamma$ decay, whose amplitude has the structure

$$\mathrm{Amp}(K_L \to 2\gamma) = G\epsilon_{\mu\nu\rho\sigma}\epsilon_\mu^{(1)} k_\nu^{(1)} \epsilon_\rho^{(2)} k_\sigma^{(2)} , \qquad (3)$$

where $k^{(i)}$ and $\epsilon^{(i)}$ are the momentum and polarization vectors of the ith photon. The decay rate is

$$\Gamma(K_L \to 2\gamma) = \frac{M^3}{64\pi} |G|^2 . \qquad (4)$$

The contribution to Img coming from the 2γ state is given by

$$\mathrm{Im} g|_{2\gamma} = \frac{m\alpha}{4v} \ln\left(\frac{1+v}{1-v}\right) \mathrm{Re} G . \qquad (5)$$

Now the modulus $|G|$ is known from empirical information on the $K_L \to 2\gamma$ decay rate. If unitarity contributions coming from 3π states are systematically ignored for both $K_L \to 2\gamma$ and $K_L \to 2\mu$ decay, then Img = Im$g|_{2\gamma}$, Re$G = |G|$, and one finds the "naive" unitarity bound

$$\Gamma(K_L \to 2\mu)/\Gamma(K_L) \gtrsim 6 \times 10^{-9} . \qquad (6)$$

Our task here is to compute the direct 3π contributions to Img, and also their contributions to ImG. For these purposes we require the amplitudes for $3\pi^0$, $\pi^+\pi^-\pi^0 \to$ two real or virtual photons. We adopt, but do not reproduce here, the soft-pion expressions of Ref. 8. These expressions contain three parameters, of which two are well established experimentally: F^π, the constant which describes $\pi^0 \to 2\gamma$ decay; and f, the PCAC constant. The remaining parameter, x, measures the isotensor component of the "σ term" in the current-algebra treatment of π-π scattering. One usually supposes, as we shall do here, that $x=0$. Unless x is unbelievably large, of order 10^3–10^4, this neglect will not qualitatively alter our conclusion that the 3π states do not resolve the $K_L \to 2\mu$ puzzle. Indeed, given the formulas of Ref. 8, and with a little thought about the structure of the unitarity equations and the size of phase space for three pions, one can readily arrive at this qualitative conclusion from rough dimensional arguments. Nevertheless, since we have in fact carried out the numerical work in detail, and because a certain delicacy appears in the details, we shall comment here on a few technical points. For the unitarity calculations we require not only the $3\pi \to 2\gamma$ and $3\pi \to 2\mu$ amplitudes, but also the full complex amplitudes for $K_L \to 3\pi$. The latter are known from experiment only in modulus. However, we can get upper bounds on the 3π contributions by replacing all amplitudes in the unitarity equations with their moduli. It is these upper bounds that we shall report. The computations for ImG are now completely straightforward. For the $3\pi^0$ and $\pi^+\pi^-\pi^0$ contributions we find

$$\begin{aligned} \mathrm{Im} G|_{3\pi^0} &\lesssim 3\times 10^{-5}|G| , \\ \mathrm{Im} G|_{\pi^+\pi^-\pi^0} &\lesssim 2\times 10^{-5}|G| . \end{aligned} \qquad (7)$$

It is evident that the 3π effects here are totally negligible.

Computation of the direct 3π contributions to Img, the absorptive $K_L \to 2\mu$ amplitude, is somewhat less straightforward. The formulas of Ref. 8 are supposed to apply (in the soft-pion limit) for virtual as well as real photons, and they therefore provide a basis for computation of the $3\pi \to 2\mu$ amplitude. On inspection of the formulas for $3\pi \to$ two real or virtual photons one observes two kinds of terms: those which describe emission of a photon by an external pion (bremsstrahlung terms) and those which do not. Correlation of these descriptive expressions with explicit terms in the formulas should be evident and is left to the reader. The $3\pi^0 \to 2\gamma$ amplitude is purely of the nonbremsstrahlung type, whereas the $\pi^+\pi^-\pi^0$ amplitude has both kinds of terms. Computation of the bremsstrahlung-term contributions to $3\pi \to 2\mu$ presents no difficulties, although it is tedious. The calculation here has a structure of the kind associated with a one-loop box diagram and was carried out numerically. For practical purposes we found it convenient to use dispersion-relation methods, taking the invariant squared mass of the 3π system as the dispersion variable. One encounters no anomalous thresholds here, thanks to the masslessness of the physical photons in the intermediate state $3\pi \to 2\gamma \to 2\mu$. For the nonbremsstrahlung terms, the calculation of the $3\pi \to 2\mu$ amplitude has a structure of the kind associated with a one-loop triangle diagram. But here one encounters a logarithmically divergent integral. This comes about because the corresponding amplitudes for $3\pi \to$ two virtual photons do not have any damping as the virtual-photon masses become very large. The soft-pion approximation is unsatisfactory in this regard. However, since the divergence is only logarithmic, we do not think it misleading to employ a cutoff. We again employ dispersion-relation methods. The dispersion integral is logarithmically divergent and we simply cut it off, at an invariant squared mass taken rather arbitrarily to be 1 GeV^2.

Once the $3\pi \rightarrow 2\mu$ amplitudes have been estimated, computation of the 3π contributions to the absorptive $K_L \rightarrow 2\mu$ amplitude is now a simple matter. We present the results in the form of comparison of the 3π and 2γ contributions to Img,

$$\begin{aligned} \mathrm{Im}g\big|_{3\pi^0} &\lesssim 3\times 10^{-5}\,\mathrm{Im}g\big|_{2\gamma}, \\ \mathrm{Im}g\big|_{\pi^+\pi^-\pi^0} &\lesssim 3\times 10^{-5}\,\mathrm{Im}g\big|_{2\gamma}. \end{aligned} \tag{8}$$

In summary, the 3π states, at least when treated in the soft-pion approximation, do nothing to resolve the $K_L \rightarrow 2\mu$ puzzle.[12]

Finally, we comment briefly on the $\pi^+\pi^-\pi^0\gamma$ intermediate state, which is the remaining intermediate state which can contribute at this order in α. Although the decay $K_L \rightarrow \pi^+\pi^-\pi^0\gamma$ has not been observed, to leading order in the photon momentum (the bremsstrahlung approximation) the amplitude for this process is related by gauge invariance to the amplitude for $K_L \rightarrow \pi^+\pi^-\pi^0$. The current-algebra coupling[8] of a photon to $\pi^+\pi^-\pi^0$ can then be used to compute $\pi^+\pi^-\pi^0\gamma \rightarrow \mu^+\mu^-$. An estimate of the relevant integrations indicates a contribution to Img of essentially the same size as that coming from the 3π intermediate state. So the $3\pi\gamma$ contribution is also at least four orders of magnitude too small to resolve the $K_L \rightarrow 2\mu$ puzzle.

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[4]See, under Ref. 2, the papers by Gaillard and by Farrar and Treiman.

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[12]After this work was completed, we received a report on the same subject from M. Pratap, J. Smith, and Z. E. S. Uy [Phys. Rev. D 5, 269 (1972).] Our conclusions are similar.