Lower Bound on λ_+ in K_{l3} Decay

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A lower bound on λ_+ is derived under the assumption that the chiral $SU(2)\times SU(2)$-breaking correction term to the soft-pion theorem in K_{l3} decay is positive. We adopt earlier estimates of phenomenological parameters by Okubo and obtain $\lambda_+ \gtrsim 0.05$.

Exact bounds for K_{l3} decay parameters have been recently derived by Li and Pagels[1] and by Okubo.[2] The general technique of obtaining such bounds has been discussed at some length by Okubo in a later paper.[3] Essentially, the problem boils down to this: Given a real analytic function $F(\xi)$ of a complex variable ξ satisfying the inequality

$$\frac{1}{\pi}\int_{t_0}^{\infty} dt\, \omega(t)\,|F(t)|^2 \leq I\,, \tag{1}$$

where $\omega(t)$ is a given non-negative function of t and $F(t)$ has a branch cut running from t_0 to ∞, one seeks bounds on $F(t)$ and its derivatives for values of t *less than* t_0.

In the problem of K_{l3} decay the attention is focused on the scalar form factor $D(t)\equiv(m_K{}^2-m_\pi{}^2)\times f_+(t)+tf_-(t)$, and the bounds obtained in Refs. 1, 2, and 3 are valid for $t<t_0$:

$$|D(t)|\leq K, \tag{2}$$

$$\left|\frac{4(t_0-t)}{D(t)}D'(t)+(2n-2)-(2n+2)\frac{t_0{}^{1/2}}{t_0{}^{1/2}+(t_0-t)^{1/2}}+\frac{(t_0-t_1)^{1/2}}{(t_0-t_1)^{1/2}+(t_0-t)^{1/2}}\right|\leq\left[\left(\frac{K}{D(t)}\right)^2-1\right]^{1/2}, \tag{3}$$

where

$$K=4(\tfrac{1}{3}\pi\Delta_n)^{1/2}(t_0-t)^{(n-1)/2}\left[1+\left(\frac{t_0}{t_0-t}\right)^{1/2}\right]^{n+1}\left[1+\left(\frac{t_0-t_1}{t_0-t}\right)^{1/2}\right]^{-1/2} \tag{4}$$

and

$$\Delta_n=\int_{t_0}^{\infty}dt\,t^{-n}\rho(t),\quad t_0=(m_K+m_\pi)^2,\quad t_1=(m_K-m_\pi)^2. \tag{5}$$

In (5), $\rho(t)$ is the Källén-Lehmann spectral function for the propagator of the divergence of the strangeness-changing vector current and n is a positive integer. Okubo makes the choice $n=1$. Equation (2) then gives $|f_+(0)|\leq 1.01$ if one estimates Δ_1 from the $(3,3^*)+(3^*,3)$ model of Gell-Mann, Oakes, and Renner[4] and of Glashow and Weinberg.[5] This result is, of course, consistent with the Ademollo-Gatto theorem.[5] Equation (3) is somewhat more interesting and yields the inequality

$$0.12\leq\xi(0)+12.3\lambda_+\leq 0.30, \tag{6}$$

where Okubo[3] has used $f_+(0)=0.85$, which follows from the $(3,3^*)+(3^*,3)$ model if the experimental value $F_K/F_\pi f_+(0)\approx 1.28$ is employed.

Equation (6) restricts the range of permissible values of $\xi(0)$ for a given *input* of λ_+. The experimental situation is still somewhat foggy.[6,7] If one takes $\lambda_+\approx 0.06$, then (6) gives $-0.61\leq\xi\leq-0.43$.

It is natural to raise the question about the possibility of finding a bound on λ_+ which would be useful [in conjunction with (6)] to set bounds on $\xi(0)$ itself. This is the object of the present paper. We show that if the $SU(2)\times SU(2)$-breaking correction term to the Callan-Treiman-Mathur-Okubo-Pandit[8] soft-pion relation

$$f_+(m_K{}^2)+f_-(m_K{}^2)=\frac{F_K}{F_\pi}+O(\epsilon_\pi) \tag{7}$$

is *positive*, then there exists a *lower* bound on λ_+. The sign of the correction term $O(\epsilon_\pi)$ in (7) is not firmly established, but an earlier treatment of the problem by Brink and the author[9] on the soft-pion corrections in K_{l3} decay indicates that the term in question is positive. In any case, we *assume* here that this is the case and proceed to derive the lower bound on λ_+.

In view of Eq. (7) and the assumption $O(\epsilon_\pi)>0$, we may write

$$D(m_K{}^2) \geq m_K{}^2 \frac{F_K}{F_\pi} - m_\pi{}^2 f_+(m_K{}^2). \quad (8)$$

Hence,

$$|D(m_K{}^2)| \geq \left| m_K{}^2 \frac{F_K}{F_\pi} - m_\pi{}^2 f_+(m_K{}^2) \right|$$

$$\geq \left| m_K{}^2 \frac{F_K}{F_\pi} \right| - \left| m_\pi{}^2 f_+(m_K{}^2) \right|. \quad (9)$$

We now appeal to the Li-Pagels-Okubo inequality [Eq. (2)] to write

$$K(m_K{}^2) \geq |D(m_K{}^2)| \geq \left| m_K{}^2 \frac{F_K}{F_\pi} \right| - \left| m_\pi{}^2 f_+(m_K{}^2) \right|, \quad (10)$$

where

$$K(m_K{}^2) = 4[\tfrac{1}{3}\pi\Delta(0)]^{1/2} \left[1 + \left(\frac{(m_K + m_\pi)^2}{m_\pi(2m_K + m_\pi)} \right)^{1/2} \right]^2$$

$$\times \left[1 + \left(\frac{4 m_K}{2m_K + m_\pi} \right)^{1/2} \right]^{-1/2} \quad (11)$$

and

$$[\Delta(0)]^{1/2} \approx 1.01 m_\pi F_\pi. \quad (12)$$

From

$$f_+(t) = f_+(0)\left(1 + \frac{\lambda_+}{m_\pi{}^2} t \right)$$

and Eq. (10), we obtain

$$\left| 1 + \lambda_+ \frac{m_K{}^2}{m_\pi{}^2} \right| \geq |f_+(0)|^{-1} \left(\frac{m_K{}^2}{m_\pi{}^2} \frac{F_K}{F_\pi} - \frac{K(m_K{}^2)}{m_\pi{}^2} \right), \quad (13)$$

which is the desired lower bound, since $\lambda_+ > 0$.

Following Okubo,[3] we set

$$F_K/F_\pi \approx 1.1, \quad f_+(0) = 0.85. \quad (14)$$

Equation (13) then gives $\lambda_+ \gtrsim 0.05$. As emphasized elsewhere by Okubo,[3] the bound is rather sensitive to the values chosen for F_K/F_π and $\Delta(0)$. Okubo has presented a critical discussion of the various methods of determining $\Delta(0)$. We merely note here that a recent communication of Mathur[10] shows that if the chiral SU(3)×SU(3) and scale-invariant limits were to coincide, then the chiral-symmetry breaking by $(3, 3^*) + (3^*, 3)$ terms[11] would provide a consistent and very satisfactory description of the symmetry-breaking mechanism, incorporating the recent results of Cheng and Dashen[12] (on the σ term) and of Ref. 11. In fact, Mathur has shown that the value of $c = \epsilon_8/\epsilon_0$ used in arriving at the estimate of $\Delta(0)$ [Eq. (12)] remains unchanged (see Ref. 3) if scale-invariance and chiral-symmetry limits coincide. Hence, one may venture to take Eq. (12) and the resultant numerical value of the lower bound on λ_+ fairly seriously. If we combine $\lambda_+ \gtrsim 0.05$ with Eq. (6), we are led to $\xi(0) \leq -0.32$. In conclusion, we wish to remark that one could have obtained a more rigorous bound from Eq. (13) by invoking the upper bound on $|f_+(0)|$ following from Eq. (2). It is easy to see, however, that this tends to weaken the lower bound on λ_+. We have, therefore, chosen to stick to Okubo's estimates[3] [Eq. (14)] so as to obtain a more useful bound. However, it should be emphasized that the bound *is* sensitive to the specific assumptions made in the text and hence is somewhat speculative in nature.

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