best tests of T invariance in $\Delta S=1$ decays are to be found in processes which either involve a Σ^0 or Λ in the initial or final state, or which occur between one V multiplet and another.

ACKNOWLEDGMENT

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High-Energy Effects in the N/D Method*

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A discussion of high-energy effects in the N/D method is presented and arguments are given that difficulties of the type described by Coulter can be avoided.

In a recent article,[1] Coulter has given an account of an interesting study of high-energy effects in the N/D method. Starting with a standard resonance form, the generalized potential[2]

$$B(s)=\frac{1}{\pi}\oint_L \frac{\operatorname{Im} A(s')}{s'-s}\,ds'$$

is calculated for $s\geq 4$ directly from the phase shifts. Then $B(s)$ is used in an N/D calculation, and it is found that when the original phase shifts have an asymptotic behavior corresponding to a Castillejo-Dalitz-Dyson (CDD) pole, $\delta\to\pi$ as $s\to\infty$, the method completely fails to reproduce them. But when the asymptotic behavior is changed to a non-CDD form, $\delta\to 0$ as $s\to\infty$, the resonance is reproduced in a satisfactory way. The use of a narrow-width approximation or a Breit-Wigner form in the crossed channel implies the existence of CDD effects, and it is suggested that an attempt to obtain a direct-channel non-CDD resonance from one of these is inconsistent and can be meaningful only if high-energy effects are negligible. Since the amplitude was the same throughout the entire low-energy region in both of Coulter's calculations, it seems that high-energy effects are important, and that the use of this type of resonance in the crossed channel is in serious error.

Here we would like to suggest that the importance of the high-energy form of the direct-channel phase shift does not necessarily imply equal importance to the high-energy form in the crossed channel.[3] The usual input is obtained from the t-channel partial-wave series, which is convergent only for $s>-32$. Since the left-hand cut is given by an integral over t from 4 to $4-s$, the only t values that contribute directly are $4<t<36$. Thus only small changes can result from changing a high-t phase shift from a CDD form to a non-CDD form. Of course, one could use a truncated series for $s<-32$ (with a cutoff to avoid divergences in the integration), so that large t would formally contribute. However, the divergence of the full series makes this procedure highly suspect.

In any realistic calculation, only the part of the left-hand cut $-32<s<0$ should be directly calculated from the crossed-channel partial-wave expansion. One of us (J.D.) has recently discussed[4] a modified form of the Balázs method by which the effects of the remainder of the cut can be estimated. The part of the left-hand cut with $s<-32$ is replaced by poles whose residues are calculated in the usual way by matching the N/D results with a known form in the gap. The method uses closed-form solutions for N and D, retains explicitly the near part of the left-hand cut, and with the matching criteria for the gap is free of arbitrary parameters.[4] The necessary inputs are $\mathrm{Im}\,A(s)$ for $-32<s<0$ and the real function, $A(s)$, for $0<s<4$, so that no separate calculation of $B(s)$ is necessary. In a real calculation, the amplitude used for matching in the gap is again obtained from the t-channel partial-wave expansion (which converges in $0<s<4$ for all $t>4$). $A(s)$ is given by an integral in t from 4 to ∞, but now the integrand is damped by a factor t^{-l-1} which suggests that the low-t region still dominates.

There remains the question of whether the small changes that might occur in the near part of the cut due to a change of high-t forms, or the small changes that might occur in the gap despite the t^{-l-1} damping, might have some significant effect. There is the possibility that CDD and non-CDD direct-channel forms might have similar cuts and/or gaps, so that small changes in these inputs would be crucial. These questions can be studied by a more detailed study of the properties of functions producing phase shifts corresponding to CDD and non-CDD resonances. We can express a unitary partial-wave amplitude in the general form

$$A(s)=[\rho(s)f(s)]^{-1}, \tag{1}$$

where

$$\rho(s)=[(s-4)/s]^{1/2}, \quad f(s)=\cot\delta(s)-i, \tag{2}$$

and $f(s)$ is chosen so as to give the amplitude the correct analytic and threshold properties. A convenient form for $\cot\delta$ which ensures the correct threshold behavior is

$$\cot\delta(s)=\frac{h(s)}{(s-4)^{l+1/2}}, \tag{3}$$

where $h(4)\neq 0$. In addition, we want to require that $A(s)$ be free of CDD poles and also ghost poles on the physical sheet. To obtain conditions for this, we can use a contour around both cuts, closed with a large circle, and calculate

$$I=\frac{1}{2\pi i}\oint ds\,\frac{1}{f(s)}\,\frac{df}{ds} =N_z-N_p, \tag{4}$$

where N_z and N_p are, respectively, the number of zeros and poles of $f(s)$ inside the contour. Taking the behavior of $f(s)$ to be

$$f(s)\underset{s\to\infty}{\sim} s^r, \qquad f(s)\underset{s\to4}{\sim}(s-4)^{-l-1/2}, \tag{5}$$

and taking the contour to these limits, one obtains

$$[\beta(\infty)-\beta(4)]+[\beta(0)-\beta(-\infty)]=\pi(N_z-N_p-l-r-\tfrac{1}{2}), \tag{6}$$

where β is the phase of f above the cut. In terms of the phase of the amplitude itself, $\delta=-\beta$ on top of the cuts, this becomes[5]

$$[\delta(4)-\delta(\infty)]+[\delta(-\infty)-\delta(0)]=\pi(N_z-N_p-l-r-\tfrac{1}{2}). \tag{7}$$

We will restrict our attention here to the case $l=1$ and forms of $f(s)$ having no poles and $r=\frac{1}{2}$, which gives

$$\delta(-\infty)-\delta(0)=\pi(N_z-2)-[\delta(4)-\delta(\infty)]. \tag{8}$$

The form used by Coulter corresponds to the choice

$$h(s)=as(s_R-s). \tag{9}$$

Here $\delta(-\infty)-\delta(0)=0$ and $\delta(4)-\delta(\infty)=-\pi$ which implies $N_z=1$; there is, in fact, one pole of A, a bound state. (Recall that N_z is the number of zeros of f; hence, these represent poles of A.)

By inspecting (8) we see that if we continue to require no ghost poles, fix the number of bound states, and alter the high-energy phase shifts to avoid CDD effects, then we must also alter the left-hand cut such that certain oscillations will occur in both the real and imaginary parts of A to give $\delta(-\infty)-\delta(0)$ the right value. Since this condition involves only the ends of the cut, the required changes can, in principle, occur anywhere; however, in the actual cases we have constructed, the near parts are definitely affected.

For example, a resonant form having no bound states or ghosts, and no CDD effects, is

$$h(s)=a+bs^{1/2}+cs+ds^{3/2}+es^2, \tag{10}$$

with $a=19.875$, $b=75.0875$, $c=15.5375$, $d=0.05875$, $e=0.0125$, where the constants have been chosen to give a resonance at about the ρ mass and also to satisfy (8). In Fig. 1, we compare the imaginary part of the amplitude constructed from this function with that of Coulter's choice, (9), and show that the respective left-hand cuts are indeed quite different. In Coulter's calculation, the transformation from CDD to non-CDD form is carried out numerically so that the analytic form of the altered (non-CDD) amplitude is unknown and the left-hand cuts of his amplitudes cannot be compared. The arguments given here, however, indicate that the cuts differing to the extent of those in Fig. 1 would correspond to very different

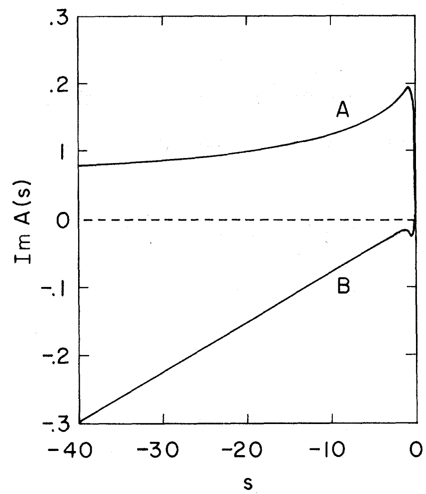

FIG. 1. Left-hand cuts of CDD and non-CDD resonances. *A* is taken from Eq. (9), which is a CDD form, and *B* is from Eq. (10), which is a non-CDD form.

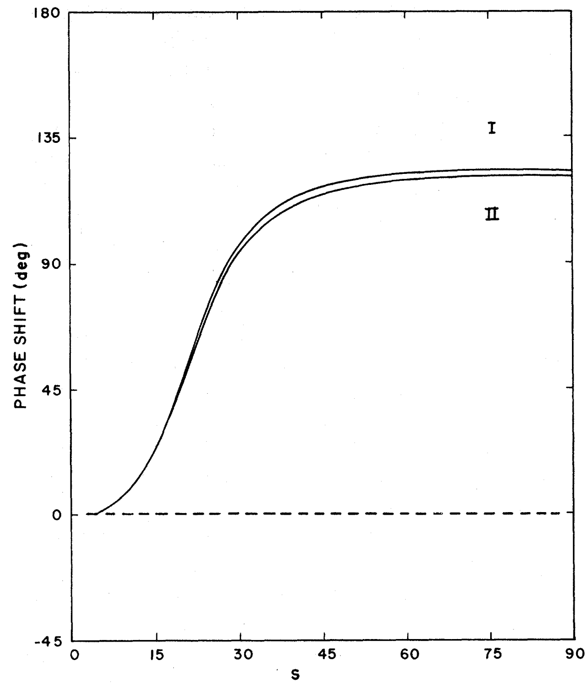

FIG. 2. *N/D* continuation of the amplitude obtained from Eq. (10), a non-CDD form, from its known values in the gap and on the near part of the left-hand cut. I is the known phase shift of the given function, and II is the phase shift from the *N/D* equations.

t-channel processes. Or, stated another way, since the low-energy resonances from crossed-channel partial waves are the dominant feature in this portion of the cut, it is most unlikely that one could get from the CDD form to the non-CDD form by changing nothing but the high-t phase shifts.

We have verified directly that a non-CDD resonance can be accurately reproduced by the present form of the N/D equations. Using (10), one can directly calculate $A(s)$ for $0<s\leq 4$ and $\text{Im}A(s)$ for $-32<s\leq 0$, which are the basic inputs to the calculation.

Although this calculation does not involve the t channel, we are using the same quantities for inputs which would be obtained from the t channel in a problem with crossing symmetry. In this sense, we are simulating a practical calculation to verify that our N/D equations do indeed give an approximate continuation to the region $s>4$. The pole parameters were computed by the usual matching procedure in the gap,[4] and there was no instability with respect to the matching points. The pole positions were varied so as to optimize the agreement in the gap, and excellent fits were thus obtained. The output phase shifts are shown in Fig. 2 (curve II), where they are compared with those obtained directly from the known function (curve I). The agreement is good, and the non-CDD function is well represented by these N/D equations. Our method fails, of course, when a CDD form such as (9) is used, which agrees with the results obtained by Coulter.

We now return to effects in the gap produced by changing the high-energy behavior of the resonances in the t channel. As was noted previously, the high-t contributions are all damped by a factor t^{-l-1}, so that it seems unlikely *a priori* that the output will be significantly altered by such changes. However, there is no need to speculate on this point since a direct calculation is possible. For this purpose, we compare the direct-channel output that is produced by two different t-channel input forms. One of these has CDD (t-channel) phase shifts, and the other does not. However, they both contain the same low-energy resonance, which is taken at the ρ mass. Equation (10) is used for the non-CDD case, while a suitable CDD resonance is given by the unitary p-wave Breit-Wigner form[6]

$$A(t)=\frac{\Gamma\rho^2}{t_R-t-i\Gamma\rho^3}\,,\qquad t_R=28.09,\quad \Gamma^2=250\,, \tag{11}$$

where the parameters[2] have been chosen to produce low-energy agreement between (10) and (11). Thus the t-channel resonance that was actually used is much wider than the ρ.

Both (10) and (11) were used to compute $\text{Im}A(s)$ for $-32<s<0$, and $A(s)$ for $0<s<4$. These were used as inputs to the same kind of N/D calculations as were carried out with the known functions. The two forms were quite similar in the gap, but the real comparison should be made with the di-

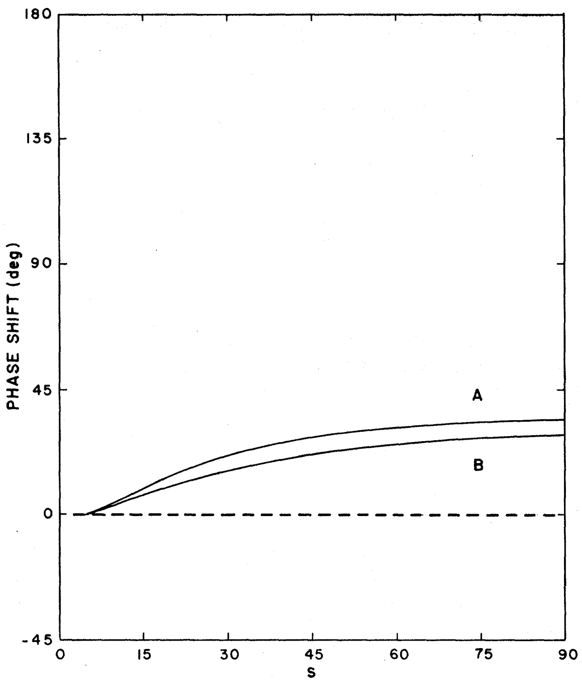

FIG. 3. Direct-channel output comparing two forms of t-channel resonance input. A is obtained from a CDD type of input, and B comes from the non-CDD case.

rect-channel phase shifts, which are given in Fig. 3. Form A is obtained from (11), and B is obtained from (10). Although there is, of course, the cutoff at $s=-32$, this position is dictated by the convergence of the t-channel partial-wave series and is not arbitrary. Nor are there any other arbitrary parameters to be adjusted to bring about agreement in the output; all parameters are adjusted to bring about the match in the gap. That the results, A and B, are quite comparable again suggests that the detailed form used at high t is of no great importance.

The outputs of both calculations are, at face value, disappointing, since neither produces a resonance in the output. However, this is due to the neglect of other partial waves, not any defect in (11) at large t. In fact, we have verified directly[7] that when other partial waves are also kept in the crossed channel, an s-channel $l=1$ resonance appears with the approximate ρ parameters.

To summarize, a given left-hand cut of a partial-wave amplitude should lead clearly to either a CDD or a non-CDD form of the phase shift. We have discussed a form of the N/D equations which uses only the near part of this cut and the gap as inputs, and the main features of these inputs are determined by the low-t resonances in a problem with crossing symmetry. The numerical examples given here suggest that the CDD and non-CDD cuts are indeed different, but that there is very little high-t contribution to the necessary N/D inputs. Thus, while our N/D equations will not reproduce a CDD form when the inputs are taken from the direct channel – an expected result since these equations assume the absence of CDD effects – the use of a crossed-channel CDD form, such as a Breit-Wigner form, has little adverse effect on the direct-channel output.

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[3]Our discussion here is specifically in the context of Coulter's paper where the high-energy behavior is taken from the asymptotic tail of low-energy resonances. Realistically, of course, most processes are presumably dominated by other mechanisms, Regge poles and cuts, or perhaps diffractive effects, which could change the picture considerably. Nevertheless, the question raised by Coulter is a serious one, since if the detailed high-energy form is crucial in one case it may well be in another as well, and no matter what sort of model is used, it is, after all, only going to be an estimate. Our point is that the rather minor change in high-energy behavior of the type discussed by Coulter is in fact not crucial; the importance of more drastic changes is another matter altogether.

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