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Inclusive- and Exclusive-Cross-Section Functionals: Conservation Constraints and Multiplicity Distributions

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A generating functional is constructed that relates inclusive and exclusive multiparticle cross sections. In a special case it reduces to the generating function of the multiplicity distribution. The generating functional is used to derive the complete set of constraints imposed by energy-momentum and charge conservation. The nature of these constraints in the high-energy limit is discussed. We show that, if there is no correlation between a pair of pions of the same charge, then the width of the multiplicity distribution of all charged pions should be about twice that of a Poisson distribution, a result in good agreement with present experiment. We also conjecture that, in the high-energy limit, the leading terms of all the moments of the multiplicity distribution of a particular produced particle become independent of the production mechanism.

I. INTRODUCTION

The natural way to investigate high-energy collision processes is the inclusive way: One first measures total cross sections, then the number of particles of a given type that are produced in various momentum intervals, next the correlations between a pair of particles produced in different momentum intervals, and so forth. In this way one arrives at, in addition to a single-particle number density, a hierarchy of correlation functions that describe multiparticle inclusive cross sections. The inclusive and exclusive multiparticle differential cross sections are easily related to one another by means of a simple generating functional that reduces to the generating function of the multiplicity distribution when its parametric function is replaced by a constant parameter.

The generating functional may be used to derive the entire set of constraints imposed by the additive laws of energy-momentum and charge conservation. These constraints are of a nontrivial nature. For example, an integral of the two-particle correlation function in the central region, the region in rapidity space where both particles are well separated from both the projectile and target, is related to the single-particle number density. Thus, such conservation laws impose constraints even in momentum regions that are far away from kinematical boundaries, and in no region are the correlation functions free of constraint.

The conservation of charge places a nontrivial constraint on the multiplicity distribution of charged particles. The production of K mesons at high energy is quite small, and it can be neglected. Moreover, the correlation between pions and protons should also be negligibly small, since it involves some sort of baryon exchange. Finally, the correlation between pions of like charge may well be small (perhaps a small anticorrelation), since they are not in a resonant state. Under these conditions, charge conservation requires that the width of the multiplicity distribution of all charged pions be about twice that of a Poisson distribution,

$$\langle (n_{\text{ch}} - \langle n_{\text{ch}} \rangle)^2 \rangle \approx 2 \langle n_{\text{ch}} \rangle .$$

This prediction is in good agreement with the Echo Lake experiment[1] that is in the energy region 200–400 GeV.

If the number density scales at high energy, the average multiplicity grows logarithmically with the energy.[2] Recently it has been suggested[3] that the coefficient of the logarithm should depend only on the type of particle produced and should be independent of the specific type of target or projectile. If the multiparticle correlation functions scale at high energy,[4] then the logarithm of the generating function for the entire multiplicity distribution grows logarithmically with the energy.[5] Here we conjecture that all the coefficients of this logarithmic energy growth also depend only on the type of the produced particle and not on the production mechanism. In particular, the logarithmic growth of the width of the multiplicity distribution should depend only on the type of particle produced but not upon the specific nature of the projectile or target.

II. INCLUSIVE AND EXCLUSIVE GENERATING FUNCTIONALS

We shall, for simplicity, consider explicitly the production of identical particles, but we will indicate briefly later the extension of our work to the production of various types of particles. We may write the exclusive differential cross section for the production of n particles in momentum intervals $(d\vec{q}_1)\cdots(d\vec{q}_n)$ as

$$d\sigma_n^{\rm exc}=\prod_{a=1}^{n}\frac{(d\vec{q}_a)}{q_a^0}\,\delta\left(\sum_{b=1}^{n}q_b-P\right)|T_n|^2\,. \qquad (1)$$

Here P^μ is the total four-momentum of the initial state and T_n the transition amplitude for the process including the incident-flux factor. The corresponding exclusive generating functional appears as

$$E[\phi(q)]=\sum_{n=2}^{\infty}\frac{1}{n!}\int\prod_{a=1}^{n}\frac{(d\vec{q}_a)}{q_a^0}\delta\left(\sum_{b=1}^{n}q_b-P\right)\times|T_n|^2\phi(q_1)\cdots\phi(q_n)\,, \qquad (2)$$

since we have

$$d\sigma_n^{\rm exc}=\prod_{a=1}^{n}(d\vec{q}_a)\frac{\delta_3}{\delta\phi(q_a)}E[\phi(q)]\bigg|_{\phi=0}\,. \qquad (3)$$

Note that the total cross section can be written as

$$\sigma_{\rm tot}=E[\phi=1]\,. \qquad (4)$$

In the inclusive process one sums over all final particles that are not explicitly detected. Hence, the inclusive cross section for the detection of n particles accompanied by whatever else might be produced is given by

$$d\sigma_n^{\rm inc}=\prod_{a=1}^{n}\frac{(d\vec{q}_a)}{q_a^0}\sum_{m=1}\frac{1}{m!}\times\int\prod_{b=1}^{m}\frac{(d\vec{q}_b)}{q_b^0}\delta\left(\sum_{c=1}^{n+m}q_c-P\right)|T_{n+m}|^2\,. \qquad (5)$$

We can write this inclusive cross section in terms of the exclusive generating functional, for

$$d\sigma_n^{\rm inc}=\prod_{a=1}^{n}(d\vec{q}_a)\frac{\delta_3}{\delta\phi(q_a)}E[\phi(q)]\bigg|_{\phi=1}\,. \qquad (6)$$

Thus, if we define the inclusive generating function in a conventional manner,

$$d\sigma_n^{\rm inc}=\prod_{a=1}^{n}(d\vec{q}_a)\frac{\delta_3}{\delta\phi(q_a)}I[\phi(q)]\bigg|_{\phi=0}\,, \qquad (7)$$

it is simply related[6] to the exclusive generating functional by a translation of the parametric function $\phi(q)$:

$$I[\phi(q)]=E[\phi(q)+1]\,. \qquad (8)$$

Note that now

$$\sigma_{\rm tot}=I[0]\,. \qquad (9)$$

The extension of this method to the general situation where several different kinds of particles can be produced is straightforward, and we shall not spell this out in detail. It should suffice to remark that in this general case we introduce a separate parametric function for each type of particle. We may do this by using a vector field $\phi_A(q)$, where the suffix A labels the particle type. The exclusive generating functional $E[\phi_A(q)]$ has the general form (2), but with the sum extended to a sum over all particle types with separate denominator factorials for the number of particles of each type and separate products for each component of $\phi_A(q)$. The inclusive generating functional in this general case, $I[\phi_A(q)]$, is again simply related to the exclusive functional:

$$I[\phi_A(q)]=E[\phi_A(q)+1]\,. \qquad (10)$$

For the sake of simplicity, we shall discuss explicitly the case of the production of identical particles for much of our work.

The set of inclusive cross sections defines a set of number-density functions:

$$\frac{d\sigma_1^{\rm inc}}{\sigma_{\rm tot}}=\frac{(d\vec{q})}{q^0}N_1(q)$$

gives the average number of particles produced in the momentum interval $(d\vec{q})$ in a single collision,

$$\frac{d\sigma_2^{\rm inc}}{\sigma_{\rm tot}}=\frac{(d\vec{q}_1)}{q_1^0}\frac{(d\vec{q}_2)}{q_2^0}N_2(q_1,q_2)$$

gives the average number of pairs of particles produced in the momentum interval $(d\vec{q}_1)(d\vec{q}_2)$ in a single collision, and so forth. In terms of these functions, which are completely symmetrical in their momentum variables $q_1, q_2, \ldots,$ we have

$$I[\phi(q)]=\sigma_{\rm tot}\left(1+\sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{a=1}^{n}\frac{(d\vec{q}_a)}{q_a^0}N_n(q_1,\ldots,q_n)\times\phi(q_1)\cdots\phi(q_n)\right)\,. \qquad (11)$$

The number densities, as we have defined them, refer to numbers of particles produced per collision, including elastic collisions. It is sometimes convenient to omit the elastic part of the process and normalize the number densities in terms of numbers of produced particles per inelastic event. This is accomplished by omitting the elastic contribution in the generating functionals, and by replacing the total cross section $\sigma_{\rm tot}$ by the inelastic cross section $\sigma_{\rm incl}$. Such a change of normalization has no effect on the relationships that we shall consider.

For high-energy collisions it is convenient to use a set of correlation functions C_n rather than number-density functions N_n, since particles with widely separated momenta should not be correlated. The number and correlation functions are related by the cluster expansion that is often used in statistical mechanics,

$$N_2(q_1, q_2) = N_1(q_1)N_1(q_2) + C_2(q_1, q_2), \tag{12a}$$

$$\begin{aligned} N_3(q_1, q_2, q_3) &= N_1(q_1)N_1(q_2)N_1(q_3) \\ &+ N_1(q_1)C_2(q_2, q_3) + 2 \text{ perms} \\ &+ C_3(q_1, q_2, q_3), \end{aligned} \tag{12b}$$

$\ldots.$

The complete set of relations is easily expressed in functional language as

$$I[\phi(q)] = \sigma_{\text{tot}} \exp\left(\int \frac{(d\vec{q})}{q^0} N_1(q)\phi(q) + \sum_{n=2}^{\infty} \frac{1}{n!} \int \prod_{a=1}^{n} \frac{(d\vec{q}_a)}{q_a^0} C_n(q_1, \ldots, q_n)\phi(q_1)\cdots\phi(q_n)\right). \tag{13}$$

We should remark that at a finite energy, since a finite number of particles are produced, only a finite number of number-density functions N_n are nonvanishing. However, at a finite energy all the correlation functions are generally nonvanishing, with the higher correlation functions serving only to cancel the lower correlation functions' contribution to the number-density functions that vanish. This is vividly presented in the functional relation $I[\phi=-1]=E[0]=0$. Accordingly, the correlation function C_n is meaningful only if the number of particles correlated, n, is considerably less than the average multiplicity at the energy considered.

When the parametric function $\phi(q)$ is replaced by a constant z, the generating functionals reduce to generating functions for the multiplicity distribution. Indeed, since the probability for producing n particles p_n is given by

$$p_n = \sigma_n^{\text{exc}}/\sigma_{\text{tot}}, \tag{14}$$

we have

$$\begin{aligned} \sum_{n=0}^{\infty} p_n(1+z)^n &= \sigma_{\text{tot}}^{-1} E[z+1] \\ &= \sigma_{\text{tot}}^{-1} I[z]. \end{aligned} \tag{15}$$

In terms of the correlation-function decomposition this appears as

$$\sum_{n=0}^{\infty} p_n(1+z)^n = \exp\left(\langle n\rangle z + \sum_{n=2}^{\infty} \langle C_n\rangle z^n/n!\right), \tag{16}$$

where

$$\langle n\rangle = \int \frac{(d\vec{q})}{q^0} N(q) \tag{17}$$

is the average multiplicity, and

$$\langle C_n\rangle = \int \prod_{a=1}^{n} \frac{(d\vec{q}_a)}{q_a^0} C_n(q_1, \ldots, q_n). \tag{18}$$

If all the $\langle C_n\rangle$ were to vanish, we would obtain the generating function of a Poisson distribution. However, this is not likely to occur even at high energy,[5] particularly in view of the constraints to which we now turn.

III. CONSERVATION CONSTRAINTS

We consider first the constraints imposed by the conservation of energy and momentum in the n-particle exclusive process,

$$\sum_{a=1}^{n} q_a^{\mu} = P^{\mu}. \tag{19}$$

It is convenient to introduce four parameters x^{μ} and write the energy-momentum conservation law in the form

$$\prod_{a=1}^{n} e^{iq_a x} = e^{iPx}, \tag{20}$$

for, on referring to the exclusive generating functional (2), we see that the factors $e^{iq_a x}$ are obtained if we replace $\phi(q_a)$ with $e^{iq_a x}\phi(q_a)$. Hence, the constraints imposed by this conservation law may be expressed as

$$E[e^{iqx}\phi(q)] = e^{iPx}E[\phi(q)]. \tag{21}$$

This structure contains redundant information, since the exponential form (20) is a consequence of the linear law (19). We need, therefore, consider only the linear term in x^{μ}:

$$\int (d\vec{q}\,')q'^{\mu}\phi(\vec{q}\,')\frac{\delta_3}{\delta\phi(q')}E[\phi(q)] = P^{\mu}E[\phi(q)]. \tag{22}$$

Indeed, the iteration of this linear constraint produces the exponential form (21),

$$\begin{aligned} e^{iPx}E[\phi(q)] &= \exp\left(i\int(d\vec{q}\,')q'x\phi(q')\frac{\delta_3}{\delta\phi(q')}\right)E[\phi(q)] \\ &= E\left[\exp\left(i\int(d\vec{q}\,')q'x\phi(q')\frac{\delta_3}{\delta\phi(q')}\right)\phi(q)\right. \\ &\qquad \left.\times\exp\left(-i\int(d\vec{q}\,')q'x\phi(q')\frac{\delta_3}{\delta\phi(q')}\right)\right] \\ &= E[e^{iqx}\phi(q)]. \end{aligned} \tag{23}$$

We may express the constraint in terms of the inclusive functional if we recall that $I[\phi]=E[\phi+1]$,

$$\int(d\vec{q}\,')q'^{\mu}\{\phi(q')+1\}\frac{\delta_3}{\delta\phi(q')}I[\phi(q)] = P^{\mu}I[\phi(q')]. \tag{24}$$

Writing out the expansion (11) in terms of number densities gives a sequence of relations that convey the complete content of energy-momentum conservation:

$$P^\mu=\int \frac{(d\vec{q})}{q^0}\, q^\mu N_1(q), \tag{25a}$$

$$(P^\mu - q^\mu)N_1(q) = \int \frac{(d\vec{q}\,')}{q'^0}\, q'^\mu N_2(q,q'), \tag{25b}$$

$$\ldots .$$

This form of the energy-momentum constraint has a simple physical interpretation. The first condition, Eq. (25a), simply states that the energy and momentum carried off by the produced particles must add up to the initial energy and momentum.[7] The second condition, Eq. (25b), states that the four-momentum of the produced particles that are not detected $(P^\mu - q^\mu)$ times the number density of the detected particle is an integral over the momenta of an undetected particle weighted by the number density for pair production times the four-momentum of the undetected particle. The general form of the constraint,

$$\left(P^\mu - \sum_{a=1}^{n} q_a^\mu\right) N_n(q_1,\ldots,q_n) = \int \frac{(d\vec{q}_{n+1})}{q_{n+1}^0}\, q_{n+1}^\mu N_{n+1}(q_1,\ldots,q_n,q_{n+1}), \tag{26}$$

has a similar interpretation. If we multiply Eq. (25b) by q^ν, integrate over the momentum, and use Eq. (25a), we obtain

$$P^\mu P^\nu = \int \frac{(d\vec{q})}{q^0}\, q^\mu q^\nu N_1(q) + \int \frac{(d\vec{q}_1)}{q_1^0}\frac{(d\vec{q}_2)}{q_2^0}\, q_1^\mu q_2^\nu N_2(q_1,q_2). \tag{27}$$

Energy-momentum constraints of this form have been discussed recently.[8]

The extension of these results to the general case where several types of particles can be produced should be clear. In the general case we label the particle types by 1, 2, ..., and the inclusive generating functional has the structure

$$I[\phi_A(q)] = \sigma_{\text{tot}}\Bigg(1 + \sum_{n_1 n_2 \cdots} \frac{1}{n_1!}\frac{1}{n_2!}\cdots \int \prod_{a=1}^{n_1} \frac{(d\vec{q}_a)}{q_a^0} \prod_{b=1}^{n_2} \frac{(d\vec{p}_b)}{p_b^0}\cdots \times N(q_1,\ldots,q_{n_1},p_1,\ldots,p_{n_2},\ldots)\phi_1(q_1)\cdots\phi_1(q_{n_1})\phi_2(p_1)\cdots\phi_2(p_{n_2})\cdots\Bigg). \tag{28}$$

We can simplify the notation if we introduce a number density $N_{AB}\ldots(q_A, q_B, \ldots)$ which is completely symmetrical under the combined interchange of any two indices together with the two corresponding momenta, with

$$N_{1\cdots 1,2\cdots 2,\ldots}(q_1,\ldots,q_{n_1},p_1,\ldots,p_{n_2},\ldots) = N(q_1,\ldots,q_{n_1},p_1,\ldots,p_{n_2},\ldots). \tag{29}$$

We have, on using the multinomial theorem,

$$I[\phi_A(q)] = \sigma_{\text{tot}}\left(1 + \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{a=1}^{n} \frac{(d\vec{q}_a)}{q_a^0} \sum_{A_1\cdots A_n} N_{A_1\cdots A_n}(q_1,\ldots,q_n)\phi_{A_1}(q_1)\cdots\phi_{A_n}(q_n)\right). \tag{30}$$

The energy-momentum constraints now appear as

$$P^\mu = \sum_A \int \frac{(d\vec{q})}{q^0}\, q^\mu N_A(q), \tag{31}$$

and generally

$$(P^\mu - q_A^\mu - q_B^\mu - \cdots)N_{AB}\ldots(q_A,q_B,\ldots) = \sum_x \int \frac{(d\vec{q}_x)}{q_x^0}\, q_x^\mu N_{AB\cdots x}(q_A,q_B,\ldots q_x). \tag{32}$$

The method that we have used is not restricted to the conservation of momentum, but applies to any additively conserved quantity – for example, electrical charge. We denote the charge of the initial state by e and that of a produced particle of type A by e_A, so that the conservation of charge for some particular exclusive n-particle process reads

$$e_A + e_B + \cdots = e. \tag{33}$$

The derivation is now precisely parallel to that of the momentum constraints, with the result that

$$\sum_{A'} \int (d\vec{q}\,')\, e_{A'}\, \phi_{A'}(q')\frac{\delta_3}{\delta\phi_{A'}(q')}\, E[\phi_A(q)] = eE[\phi_A(q)]. \tag{34}$$

In terms of the inclusive number densities, we have

$$e = \sum_A \int \frac{(d\vec{q})}{q^0}\, e_A N_A(q), \tag{35a}$$

$$(e-e_A)N_A(q)=\sum_B\int\frac{(d\vec{q}\,')}{q'^0}\,e_B N_{AB}(q,q')\,, \qquad (35b)$$

and generally

$$(e-e_A-e_B-\cdots)N_{AB\cdots}(q_A,q_B,\ldots)=\sum_x e_x\int\frac{(d\vec{q}_x)}{q_x^0}\,N_{AB\cdots x}(q_A,q_B,\ldots,q_x)\,. \qquad (36)$$

The physical interpretation of these constraints is entirely similar to that of the energy-momentum constraints.

IV. HIGH-ENERGY COLLISIONS

We have already remarked that at high energy it is convenient to use a set of correlation functions rather than number densities, since particles with widely separated momenta should not be correlated. In particular, if we put

$$N_{AB}(q,q')=N_A(q)N_B(q')+C_{AB}(q,q') \qquad (37)$$

in Eq. (35b) and use Eq. (35a), we obtain

$$-e_A N_A(q)=\sum_B\int\frac{(d\vec{q}\,')}{q'^0}\,C_{AB}(q,q')e_B\,. \qquad (38)$$

We can make use of this constraint if we make several approximations. We may neglect K-meson production at high energy since it is very small, and we should be able to neglect the correlation between pions and protons since this involves some sort of baryon exchange except for small regions in phase space near kinematical boundaries. With these approximations only charged pions contribute to the sum rule (38), and we have

$$-N_{\pi^+}(q)=\int\frac{(d\vec{q}\,')}{q'^0}\,[C_{\pi^+\pi^+}(q,q')-C_{\pi^+\pi^-}(q,q')] \qquad (39a)$$

and

$$+N_{\pi^-}(q)=\int\frac{(d\vec{q}\,')}{q'^0}\,[C_{\pi^-\pi^+}(q,q')-C_{\pi^-\pi^-}(q,q')]\,. \qquad (39b)$$

These constraints can be applied to the multiplicity distribution. It is a straightforward matter to generalize the discussion of the multiplicity distribution given in Sec. II to the case of production of several particle types and establish that

$$\langle n_A\rangle=\int\frac{(d\vec{q})}{q^0}\,N_A(q) \qquad (40)$$

and

$$\langle(n_A-\langle n_A\rangle)(n_B-\langle n_B\rangle)\rangle=\langle n_A\rangle\delta_{AB}+\langle C_{AB}\rangle\,, \qquad (41)$$

where

$$\langle C_{AB}\rangle=\int\frac{(d\vec{q})}{q^0}\frac{(d\vec{q}\,')}{q'^0}\,C_{AB}(q,q')=\langle C_{BA}\rangle\,. \qquad (42)$$

It now follows from Eqs. (39) that

$$\langle n_{\pi^+}\rangle-\langle n_{\pi^-}\rangle=\langle C_{\pi^-\pi^-}\rangle-\langle C_{\pi^+\pi^+}\rangle\,. \qquad (43)$$

Since we are essentially neglecting the production of charged particles other than pions, we must assume that an equal number of positive and negative pions are produced,[9] with the consequence that

$$\langle C_{\pi^-\pi^-}\rangle=\langle C_{\pi^+\pi^+}\rangle\,. \qquad (44)$$

Hence, according to Eq. (41), the widths of the multiplicity distributions of π^- and π^+ must be the same:

$$\langle(n_{\pi^+}-\langle n_{\pi^+}\rangle)^2\rangle=\langle(n_{\pi^-}-\langle n_{\pi^-}\rangle)^2\rangle=\langle n_{\pi^+}\rangle+\langle C_{\pi^+\pi^+}\rangle\,. \qquad (45)$$

Moreover, on using the constraints (39), we can put the width of the charged-pion distribution $n_{\text{ch}\pi}=n_{\pi^+}+n_{\pi^-}$ in the form

$$\langle(n_{\text{ch}\pi}-\langle n_{\text{ch}\pi}\rangle)^2\rangle=2\langle n_{\text{ch}\pi}\rangle+4\langle C_{\pi^+\pi^+}\rangle\,. \qquad (46)$$

It seems reasonable to assume that the correlation between pions of like charge is small[10] (perhaps slightly negative) since they are not in a resonant state. In this case, we find that the width of the multiplicity distribution of pions of the same charge is about that of a Poisson distribution,

$$\langle(n_{\pi^\pm}-\langle n_{\pi^\pm}\rangle)^2\rangle\approx\langle n_{\pi^\pm}\rangle\,, \qquad (47)$$

but the width for all charged pions is about twice that of a Poisson distribution,

$$\langle(n_{\text{ch}\pi}-\langle n_{\text{ch}\pi}\rangle)^2\rangle\approx 2\langle n_{\text{ch}}\rangle\,. \qquad (48)$$

The derivation of these simple results does not require all the elaborate machinery that we have built up. This is exhibited in a simple model. Since we assume that the net charge of the produced pions vanishes, we can describe the distribution by the probability p_m for the production of m pairs of pions with opposite charge. We have

$$\langle n_{\text{ch}\pi}\rangle=\sum_m 2mp_m\,. \qquad (49)$$

Now, according to our hypothesis that the correlation of pions of the same charge vanishes, the average number of pion pairs $n_{\pi^+\pi^+}$ is given by

$$\langle n_{\pi^+\pi^+}\rangle=\langle n_{\pi^+}\rangle^2=\tfrac{1}{4}\langle n_{\text{ch}\pi}\rangle^2\,. \qquad (50)$$

On the other hand, the number of $\pi^+\pi^+$ pairs detected when m neutral $\pi^+\pi^-$ pairs are produced is given by $m(m-1)$, and so

$$\langle n_{\pi^+\pi^+}\rangle = \sum_m m(m-1)p_m$$
$$= \tfrac{1}{4}\langle n_{\mathrm{ch}\pi}{}^2\rangle - \tfrac{1}{2}\langle n_{\mathrm{ch}\pi}\rangle \ . \tag{51}$$

The agreement of these two calculations requires that the width of the charged pion distribution be twice that of a Poisson distribution in accord with Eq. (48). It also requires that the width of the $\pi^+\pi^-$ pair distribution p_m be that of a Poisson distribution.

A Poisson distribution in pairs of charged particles[11] does fit the multiplicity distribution found in the Echo Lake experiment[1] in the energy range 200–400 GeV. However, the detailed shape of the distribution is not measured well and, moreover, it is necessarily difficult to predict theoretically since it entails knowledge about higher-order correlation functions. Hence, the best procedure is to consider a few low-order moments of the distribution, particularly the average number $\langle n\rangle$ and width $\langle (n-\langle n\rangle)^2\rangle$. Unfortunately, the Echo Lake experiment does not distinguish protons from pions and measures the number of all charged particles n_{ch}. We can account for this in a rough way, for, since the proton multiplicity is quite small,[2] $\langle n_p\rangle \simeq 1.5$, we should be able to neglect the width of the proton distribution and any correlation between protons and pions so that

$$\langle (n_{\mathrm{ch}\pi}-\langle n_{\mathrm{ch}\pi}\rangle)^2\rangle \approx \langle (n_{\mathrm{ch}\pi}+n_p-\langle n_{\mathrm{ch}\pi}+n_p\rangle)^2\rangle$$
$$\approx \langle (n_{\mathrm{ch}}-\langle n_{\mathrm{ch}}\rangle)^2\rangle \ . \tag{52}$$

Thus, on replacing $\langle n_{\mathrm{ch}\pi}\rangle$ by $\langle n_{\mathrm{ch}}\rangle - \langle n_p\rangle$, our prediction (48) becomes

$$\langle (n_{\mathrm{ch}}-\langle n_{\mathrm{ch}}\rangle)^2\rangle \approx 2(\langle n_{\mathrm{ch}}\rangle - 1.5) \, . \tag{53}$$

We must note that the particle numbers which we are now considering are numbers per inelastic collision. We have already mentioned in Sec. II that we are free to normalize either to the total number of collisions including elastic scattering or to only inelastic events. Nonetheless, even at 400 GeV, elastic scattering occurs in about 20% of the collisions and appears as a prominent spike in the multiplicity distribution. If we were to include it, we would lose considerable sensitivity to the inelastic distribution, for a major part of the width would then come from the displacement of the average multiplicity from the elastic spike at $n=2$. Moreover, the correlation functions defined by the two normalizations are different since the normalization appears linearly in N_{AB} but quadratically in the $N_A N_B$ contribution. Thus a lack of correlation in terms of an inelastic normalization produces a correlation in terms of a normalization to all events including elastic scattering. We use the inelastic normalization, since we expect the inelastic events to be more nearly uncorrelated, and treat inelastic scattering as a separate process.

We present a comparison of our prediction (53) with the Echo Lake data[1] in the following table:

E_{lab} (GeV)	$\langle n_{\mathrm{ch}}\rangle$	$\langle (n_{\mathrm{ch}}-\langle n_{\mathrm{ch}}\rangle)^2\rangle$	$2(\langle n_{\mathrm{ch}}\rangle - 1.5)$
203	5.9	8.1	8.8
291	6.2	8.9	9.4
424	6.5	10.8	10.0

In view of the many approximations that we have made and of the sensitivity of the width to small error, the excellent agreement of the figures may be a little fortuitous.

We turn now to consider the effects that our constraints have in the high-energy scaling limit. We shall, for the sake of simplicity, return to the case of the production of identical particles. It is convenient to write the center-of-mass longitudinal momentum of a produced particle $q_\parallel$ in terms of its rapidity y,

$$q_\parallel = m_\perp \sinh y \, , \tag{54}$$

where $m_\perp$ is a "transverse mass" dependent upon the momentum perpendicular to the beam $q_\perp$,

$$m_\perp = (q_\perp{}^2 + m^2)^{1/2} \ . \tag{55}$$

At high energy, the rapidity has the kinematical limits

$$-\ln\sqrt{s} \lesssim y \lesssim \ln\sqrt{s} \ , \tag{56}$$

in which $\sqrt{s}$ is the over-all center-of-mass energy. In the central region, where rapidities are finite or separated by a large finite amount from their kinematical limits, the single-particle density and correlation functions are assumed to be independent of the average, over-all rapidity. Thus we have[4]

$$N_1 \to N_1(\vec{q}_\perp) \, , \tag{57a}$$

$$C_2 \to C_2(y_1-y_2, \vec{q}_{1\perp}, \vec{q}_{2\perp}) \, , \tag{57b}$$

and so forth.

The first of the energy-momentum constraints, Eq. (25a), emphasizes the integration regions near the kinematical boundaries and gives no constraint in the central region. The higher constraints are, however, effective in the central region. For example, in terms of the correlation function, Eq. (25b) gives

$$-q_1^\mu N_1(q_1) = \int \frac{(d\vec{q}_2)}{q_2^0} q_2^\mu C_2(y_1-y_2, \vec{q}_{1\perp}, \vec{q}_{2\perp}) \tag{58}$$

or

$$-\begin{Bmatrix} m_\perp(q_1)\cosh y_1 \\ m_\perp(q_1)\sinh y_1 \\ \vec{q}_{1\perp} \end{Bmatrix} N_1(q_1) = \int dy_2 \int d^2q_{2\perp} \begin{Bmatrix} m_\perp(q_{2\perp})\cosh y_1 \\ m_\perp(q_{2\perp})\sinh y_1 \\ \vec{q}_{2\perp} \end{Bmatrix} C_2(y_1 - y_2, \vec{q}_{1\perp}, \vec{q}_{2\perp}). \tag{59}$$

If we change the integration variable to $y = y_1 - y_2$, use the addition formulas for the hyperbolic functions, and note that C_2 is even in y, we get two independent constraints:

$$-m_\perp(q_1)N_1(q_{1\perp}) = \int dy \int d^2q_{2\perp} m_\perp(q_2) \times \cosh y\, C_2(y, \vec{q}_{1\perp}, \vec{q}_{2\perp}) \tag{60a}$$

and

$$-\vec{q}_{1\perp}N_1(q_{1\perp}) = \int dy \int d^2q_{2\perp}\vec{q}_{2\perp} C_2(y, \vec{q}_{1\perp}, \vec{q}_{2\perp}). \tag{60b}$$

We note that the constraint on the transverse momentum (60b) requires that the two transverse momenta be correlated at least to the extent that C_2 must contain the first harmonic in the angle between $\vec{q}_{1\perp}$ and $\vec{q}_{2\perp}$.

The scaling limit gives a simple structure to generating functions of the multiplicity distribution.[5] We recall that this function involves the correlation functions integrated over all momenta, Eqs. (16)–(18). In the scaling limit, the integral over the over-all, average rapidity produces $\ln s$ and constant terms, with the generating function becoming

$$\sum_{n=0}^{\infty} p_n(1+z)^n \to \exp\left(\sum_{n=1}^{\infty} (c_n \ln s + d_n) z^n/n!\right). \tag{61}$$

This gives

$$\langle n\rangle \to c_1 \ln s + d_1, \tag{62a}$$

$$\langle (n - \langle n\rangle)^2\rangle \to (c_1 + c_2)\ln s + (d_1 + d_2), \tag{62b}$$

and so forth. Recently it has been suggested[3] that the coefficient c_1 depends only on the type of particle produced and not upon the production mechanism. Such a behavior follows from the multiperipheral model where the central region is uncorrelated with the target and projectile particles. It also follows from a more formal Regge-pole analysis in which the inclusive process is viewed[12] as the discontinuity of a multiparticle scattering process. We note here that the same arguments apply to all of the coefficients c_n: At a sufficiently high energy we expect any c_n to depend only upon the particular particle produced and to be independent of the specific type of target or projectile. However, it is clear that the higher correlation functions C_n will scale only if the number of particles being correlated, n, is considerably less than the average multiplicity $\langle n\rangle$. Since the latter increases only logarithmically with energy, in practice only a few correlation functions and corresponding moments of low order should become independent of the production mechanism.

Note added in proof. Recently, Le Bellac [M. Le Bellac, Phys. Letters 37B, 413 (1971)] has observed that the high-energy limit given by Eq. (61) is in conflict with ordinary Pomeranchukon exchange where the Pomeranchukon is a simple pole with intercept unity. This conflict is avoided if the Pomeranchukon has an intercept less than unity, or if it is associated with cuts in the angular momentum plane. It is also avoided if, as advocated by Wilson [K. Wilson (unpublished report)], the scattering process consists of two distinct parts, a diffractive part and a "multiperipheral" part, with the high-energy limit (61) applying only to the latter. Le Bellac's argument is easily presented within the context of our work.

We have

$$\langle (n - \langle n\rangle)^k\rangle = \left(z\frac{d}{dz} - \langle n\rangle\right)^k I(z-1)\Big|_{z=1},$$

and thus a generating function

$$M(\lambda) = \sum_k \frac{1}{k!}\lambda^k \langle (n-\langle n\rangle)^k\rangle = \exp\left[\lambda\left(z\frac{d}{dz} - \langle n\rangle\right)\right] I(z-1)\Big|_{z=1} = e^{-\lambda\langle n\rangle} I(e^\lambda - 1).$$

In the high-energy limit (61) we obtain

$$M(\lambda) \sim \exp[(a_2\lambda^2 + a_3\lambda^3 + \cdots)\ln s]$$

and

$$\langle (n - \langle n\rangle)^{2N}\rangle = \left(\frac{d}{d\lambda}\right)^{2N} M(\lambda)\Big|_{\lambda=0} \sim (\ln s)^N.$$

On the other hand, since

$$\langle n\rangle \sim c_1 \ln s,$$

we also have

$$\langle (n - \langle n\rangle)^{2N}\rangle \sim \sum_n p_n (n - c_1 \ln s)^{2N},$$

which is a sum of positive terms. Hence, on comparing the two versions of the 2Nth moment, we conclude that, with n fixed, the asymptotic bound

$$s \to \infty:\ p_n < (\text{const})(\ln s)^{-N}$$

obtains for arbitrary N. In particular, since N is arbitrary, this violates the ordinary high-energy Pomeranchukon limit

$$p_2 = \sigma_{el}/\sigma_{tot} \sim (\ln s)^{-1}.$$

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[1]L. W. Jones *et al.*, Phys. Rev. Letters 25, 1679 (1970).

[2]Some experimental evidence for this is discussed by N. F. Bali, L. S. Brown, R. D. Peccei, and A. Pignotti, Phys. Rev. Letters 25, 557 (1970), and by N. F. Bali, L. S. Brown, and R. D. Peccei, Phys. Rev. D 4, 2760 (1971).

[3]See, for example, the second paper in Ref. 2.

[4]H. D. I. Abarbanel, Phys. Rev. D 3, 2227 (1971).

[5]A. H. Mueller, Phys. Rev. D 4, 150 (1971).

[6]After the completion of this work, a report (unpublished) by K. J. Biebl and J. Wolf appeared in which this relationship is discussed.

[7]T. T. Chou and C. N. Yang, Phys. Rev. Letters 25, 1072 (1970).

[8]C. E. DeTar, D. Z. Freedman, and G. Veneziano, Phys. Rev. D 4, 906 (1971); E. Predazzi and G. Veneziano, Lett. Nuovo Cimento 2, 749 (1971).

[9]An indication of the accuracy of this assumption can be obtained from the multiplicity of protons produced in proton-proton collisions at high energy. It is about 1.5 (see Ref. 2). Hence, neglecting strange-particle production, charge conservation requires that $\langle n_{\pi^+}\rangle - \langle n_{\pi^-}\rangle \approx 0.5$.

[10]We need assume, of course, that only the average correlation $\langle C_{\pi^+\pi^+}\rangle$ is small. There may well be some momentum-dependent correlation between pions of like charge so as to ensure the satisfaction of the energy-momentum constraints.

[11]C. P. Wang, Phys. Rev. 180, 1463 (1969).

[12]A. H. Mueller, Phys. Rev. D 2, 2963 (1970).