where ξ takes typical values $0<\xi<1$. Two useful relations used in evaluation of f_1 and f_2 at $k^2=m_K^2$ are

$$m_a{}^2 f_a{}^2 \Gamma^{abc}_{\beta\gamma}(k,p) = -iC^a_A k^\alpha \Gamma^{abc}_{\alpha\beta\gamma}(k,p) + i[\Delta^{(A)}_{\beta\gamma}(p)]^{-1} - i[\Delta^{(V)}_{\beta\gamma}(k+p)]^{-1} \tag{B8}$$

and

$$m_a{}^2 f_a{}^2 \Pi^{abcd}_{\nu\lambda\tau}(k,p,q) \cong iC^a_A k^\mu \Pi^{abcd}_{\mu\nu\lambda\tau}(k,p,q)\,, \tag{B9}$$

which relate the primitives to the proper vertex functions containing one divergence. In Eq. (B8) the superscripts V and A in the propagators refer to the vector and axial-vector propagators for the indices b and c.

*Research supported in part by the National Science Foundation.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 3 1 FEBRUARY 1972

How Accurate Are Quark-Model Double-Scattering Relations?*

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The quark-model multiple-scattering relations have been rederived using the formalism of Goldberger and Watson instead of the Glauber formalism used by all authors previously. It is shown that deep binding of the quarks need not invalidate the use of the impulse approximation and that definite limits may be placed on the angular range of validity of the formulas. For an incident momentum of 5 GeV/c in the center-of-momentum frame, the conventional multiple-scattering formulas are estimated to be reasonably accurate for a momentum transfer squared less than 3 (GeV/c)^2.

I. INTRODUCTION

Since Anisovitch[1] and Lipkin and Scheck[2] first used additivity in the quark amplitudes to obtain relations among the total cross sections for hadron-hadron scattering, many remarkable results have been obtained, agreeing with experiment typically within about 10% to 20%. The author[3] and Franco[4] have, under slightly differing assumptions, inserted corrections to additivity due to multiple scattering of the quarks, with some improvement in the fit to experimental values of the total cross sections. These two papers and all subsequent work have used the Glauber[5] multiple-scattering formalism familiar because of its application to scattering from deuteron targets. Several authors[6] have applied the multiple-scattering idea to the differential cross sections, with good results. The most remarkable success of this procedure is the concise explanation of the origin of the "breaks"[7] in the proton-proton differential scattering cross sections.

However, there are some difficulties with the Glauber formalism, and with additivity in general. The most frequent explanation for the experimental elusiveness of the quarks has been that they have masses of at least 10 GeV. If this explanation is accepted, it is very difficult to understand physically why additivity should work; we are certainly very far from the situation of very small binding energy compared to kinetic energy of the projectile, which is the situation usually used to justify the impulse approximation. Then there is the additional problem of the use of the Glauber approximation at relatively large angles. To what values of q^2 can we use the usual multiple-scattering formulas in view of the fact that the Glauber approximation is known[5,8] to break down away from the forward direction? It is not, however, necessarily true that there must be a breakdown in the multiple-scattering formulas at larger angles, since the usual Glauber representation of the quark-quark amplitudes will break down, but the formulation might nevertheless handle the purely multiple-scattering aspects accurately.

There are two possible ways of discussing the situation. We might investigate the multiple-scattering situation in an improved eikonal model, or we might use the multiple-scattering formalism of Goldberger and Watson.[9] The first approach has been taken, for nonrelativistic projectiles in nuclear physics, by Hüfner,[10] using the improved eikonal expressions of Sugar and Blankenbecler.[11]

However, both questions are also readily answered in the multiple-scattering formalism of Goldberger and Watson.[9] In this approach it is possible to estimate corrections to the neglect of the binding in the quark-quark amplitudes, and we will find that this neglect may well be justified if the potential is sufficiently flat, regardless of its depth. Moreover, since this formalism handles the individual amplitudes exactly, we will be able to isolate the approximations involved in the multiple-scattering formulas and to estimate the range of angles over which the formulas are reasonably accurate.

The Goldberger-Watson formalism gives a useful alternate means of discussing the accuracy of the multiple-scattering formulas as normally used in quark-model discussions. It has the advantage of displaying the use of two-body amplitudes explicitly, and of postponing approximations until the last possible moment; it may not, however, give as easily multiple-scattering formulas for intermediate angles. In principle the multiple-scattering descriptions given in the Goldberger-Watson and Glauber formalisms are not term by term the same, but as used in quark-model work so far, they are parallel. We have chosen to work in the Goldberger-Watson formalism.

We will consistently make several assumptions throughout the discussion. First, we will work with spinless quarks. The discussion is much simpler without spin, without affecting in a significant manner any of the questions we are concerned with. The generalization to quarks with spin can be made when it is desired to use spin-dependent amplitudes in comparison with experiment. Second, we will, for definiteness, always work specifically with meson-baryon scattering. The generalization to baryon-baryon scattering is immediate. Third, we assume that the range of the quark-quark scattering interaction is small compared to the radius of the quark distribution in hadrons, as is necessary in order to believe in the convergence of the multiple-scattering series. Fourth, we assume that a Hamiltonian for the scattering system exists and that there are no essential difficulties in defining an asymptotic Hamiltonian when the hadrons are far apart. Finally, we neglect (quark-) pair production at intermediate stages of the scattering process. Since we are considering elastic scattering only, such processes amount to long-range interactions between quarks, which we are already neglecting. We also suppress any variables indicating explicitly the operator, nonlocal character of interactions and amplitudes, but no manipulations depend on this simplification so that no restrictions on the validity of our results are introduced.

In Sec. II, we will discuss the neglect of the binding potential in the quark-quark amplitudes, finding that such neglect is marginally justified, at least for small q^2. Our general effort to avoid details of the binding will be least successful in this section. In Sec. III, we repeat the Goldberger-Watson derivation of the multiple-scattering series specifically for quarks, in order to isolate the necessary approximations. Then in Sec. IV we show that the corrections to the usual multiple-scattering formulas may be estimated, and we obtain angular limits for the accuracy of the formulas. Finally, in Sec. V we summarize and discuss our results.

II. THE NEGLECT OF THE QUARK BINDING

In this section and Sec. III, we will be following the general development of the scattering theory given in Goldberger and Watson[9] rather closely, but adapting it for the possibly large binding energy of the quarks. We will examine first the scattering of one particular pair of quarks and will assume that the effect of the binding can be adequately represented by one-body operators in any given coordinate system. Then we can start from the definitions of the "bound" T matrix $\mathcal{T}_b$ and a "free" T matrix $\mathcal{T}$ which does not involve the binding, given in Eqs. (11–13) and (11–18) of Goldberger and Watson:

$$\mathcal{T}_b = V + V(d - V)^{-1}V , \tag{1a}$$

$$\mathcal{T} = V + V(d_0 - V)^{-1}V , \tag{1b}$$

where

$$d = E_a - K_i - K_t - U + i\epsilon , \tag{2a}$$

$$d_0 = E_a^0 - K_i - K_t + i\epsilon , \tag{2b}$$

and where V is the scattering interaction (i.e., the difference between the full Hamiltonian and the asymptotic Hamiltonian), U is the binding interaction, K_i and K_t are the Hamiltonians of the incident and target particles, respectively, with both U and V turned off, and where E_a and E_a^0 are defined by $(K_i + K_t + U)\Psi = E_a\Psi$ and $(K_i + K_t)\phi = E_a^0\phi$. It is also convenient to have a quantity R_a defined by $R_a = E_a - E_a^0$. We will have no need, however, to refer explicitly to Ψ and ϕ.

The relationship between $\mathcal{T}_b$ and $\mathcal{T}$ may be found from Eqs. (1) and (2). If we subtract Eq. (2b) from Eq. (2a) we get $d_0 - d = U - R_a$. Then by subtracting Eq. (1b) from Eq. (1a) and using the identity $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$, we get

$$\mathcal{T}_b = \mathcal{T} + \mathcal{T} d_0^{-1}(U - R_a)d^{-1}\mathcal{T}_b . \tag{3}$$

Equation (3) is really an integral equation for $\mathcal{T}_b$ in terms of $\mathcal{T}$; the first two terms of an iterative solution of this equation are

$$\mathcal{T}_b = \mathcal{T} + \mathcal{T} d_0^{-1}(U - R_a)d_0^{-1}\mathcal{T} + \cdots . \tag{4}$$

Now what we would like to do is to neglect the second and all subsequent terms in Eq. (4), so that $\mathcal{T}_b \approx \mathcal{T}$. The scattering will then not be directly affected by the binding. We will now show, however, that it is not sufficient simply to assume that U is small enough to be ignored altogether, even for small-mass quarks. We will be interested in the matrix elements of $\mathcal{T}$ between plane-wave quark states, and we may use translational invariance to rewrite the matrix elements of $\mathcal{T}$ between such states as

$$\langle \vec{k}\vec{Q} | \mathcal{T} | \vec{p}\vec{P} \rangle = \delta(\vec{k} + \vec{Q} - \vec{p} - \vec{P}) \langle \vec{k}\vec{Q} | T | \vec{p}\vec{P} \rangle . \tag{5}$$

In the states $|\vec{p}\vec{P}\rangle$, the first momentum belongs to the incident quark, the second to the target. Now we assume that U and R_a are comparable in size, so that we may estimate $U - R_a$ by $2R_a$, and we also assume that we may neglect intermediate states having more than two quarks as contributing at worst an amount of the same order as the two-quark intermediate states. Then from Eq. (4) we get

$$\langle \vec{k}\vec{Q} | \mathcal{T}_b - \mathcal{T} | \vec{p}\vec{P} \rangle \simeq 2R_a \delta(\vec{k} + \vec{Q} - \vec{p} - \vec{P}) \int d^3p' d^3P' \delta(\vec{P}' - \vec{p} - \vec{P} + \vec{p}') \langle \vec{k}\vec{Q} | T | \vec{p}'\vec{P}' \rangle \times (\epsilon_{i,p} + \epsilon_{t,P} - \epsilon_{i,p'} - \epsilon_{t,P'} + i\epsilon)^{-2} \langle \vec{p}'\vec{P}' | T | \vec{p}\vec{P} \rangle . \tag{6}$$

The $\epsilon_{i,p}$ and $\epsilon_{t,P}$ are given by $K_i|\vec{p}\rangle = \epsilon_{i,p}|\vec{p}\rangle$ and $K_t|\vec{P}\rangle = \epsilon_{t,P}|\vec{P}\rangle$; we assume that

$$\epsilon_{i,p} = (p^2 + m_i^2)^{1/2} , \quad \epsilon_{t,P} = (P^2 + m_t^2)^{1/2} , \tag{7}$$

but we will avoid commitment as to the size of the quark masses or their relationship to the "free" quark mass. We may now estimate the integral in Eq. (6) by substituting for T an average or a maximum value, ignoring the angular dependence of T. This procedure will work simply if the denominator in Eq. (6) is not angle-dependent. For general m, the denominator will be independent of angles only in the center-of-mass system; we will therefore work in that system, with $\vec{p} = -\vec{P}$. With these approximations and conditions, the angular integration just yields 4π, and we may convert the integration over dp' to an integration over $d\epsilon_{i,p'}$, extend the integration to $\epsilon_{p'} = -\infty$, and perform the integral by contours to get

$$|\langle \vec{k}\vec{Q} | \mathcal{T}_b - \mathcal{T} | \vec{p}\vec{P} \rangle| \lesssim 16\pi^3 R_a \delta(\vec{k} + \vec{Q} - \vec{p} - \vec{P}) \times \left(1 + \frac{d\epsilon_{t,P}}{d\epsilon_{i,p}}\right)^{-2} \frac{d}{d\epsilon_{i,p}} \left(p^2 T^2 \frac{dp}{d\epsilon_{i,p}}\right) . \tag{8}$$

Now if we neglect the energy dependence of T (which is reasonable at high energies), identify $T\delta(\vec{k} + \vec{Q} - \vec{p} - \vec{P})$ with a "typical" value of $\mathcal{T}$, and utilize the optical theorem $\sigma_{\text{tot}} = -2(2\pi)^3 v^{-1} T$ with T taken to be primarily imaginary, we get

$$|(\mathcal{T}_b - \mathcal{T})/\mathcal{T}| \lesssim R_a(\epsilon_{i,p}^2 + P^2)\sigma_{\text{tot}}/(\pi\epsilon_{i,p}) . \tag{9}$$

If, then, we substitute the maximum possible value of $2\epsilon_{i,p}$ for $(\epsilon_{i,p}^2 + P^2)/\epsilon_{i,p}$ and use a quark-quark total cross section of 4 mb at about 20 GeV,[3] we must have $R_a \ll 5$ MeV in order to have $|(\mathcal{T}_b - \mathcal{T})/\mathcal{T}| \ll 1$. This requirement is of course ridiculous, and we conclude that it is not possible to ignore U in attempting to justify the impulse approximation in the quark model. In fact, with a requirement on R_a that is this severe, no amount of patching [for instance, the quark amplitudes are presumably forward-peaked[6] so that only a small angular region contributes in Eq. (6)] will be likely to rescue the argument. Hence, in accord with intuition it is not reasonable to attempt to justify additivity in the quark model on the basis of neglecting the binding interaction altogether.

However, it is not necessary to completely neglect the binding in order to have the "bound" T matrix be essentially the free one. If the binding interaction acts like a potential which is relatively flat, the bound particle will be in a state which most of the time is quite similar to that of a free particle. Under these circumstances, it is possible to use the "free" T matrix even for a deeply bound quark. The essential physics of this approximation is that an over-all constant in the potential energy can have no physical significance, so that the particle appears to have small potential energy. Formally, an estimate of the error in using $\mathcal{T}$ for $\mathcal{T}_b$ may be obtained by using $\langle U-R_a\rangle$ instead of $2R_a$ as an estimate of the size of U in Eq. (6). Then the condition for obtaining $|(\mathcal{T}_b-\mathcal{T})/\mathcal{T}|\ll 1$ becomes $|\langle U-R_a\rangle_{\text{g.s.}}|\ll 10$ MeV, where the expectation value is to be taken in the (moving) ground state of the hadron. Now recalling that $(K+U)\Psi_{\text{g.s.}}=E_a\Psi_{\text{g.s.}}$ and $E_a^0=(p^2+m^2)^{1/2}$, and writing $K=[(P^{\text{op}})^2+m^2]^{1/2}$, we get a condition

$$|(p^2+m^2)^{1/2}-\langle[(P^{\text{op}})^2+m^2]^{1/2}\rangle_{\text{g.s.}}|\ll 10\,\text{MeV}. \tag{10}$$

The quantity p is of course a parameter. In practice a Fourier analysis of the ground state is performed, and the condition in Eq. (10) must be valid for all significant Fourier components. Typically, components involving $P_{\text{c.m.}}-\rho<p<P_{\text{c.m.}}+\rho$, where ρ is some relative momentum, will be important. Now the condition in Eq. (10) may be made a little less restrictive if it is assumed that the quark amplitude can be neglected outside of an angular region of about 20°; then the right-hand side of Eq. (10) becomes about 100 MeV, and the resulting condition on ρ is $\rho\ll 100$ MeV. While this condition on ρ is still rather restrictive, it is at least not immediately impossible. We must also remember that it is not really necessary for the success of the quark-model relationships to achieve $\mathcal{T}_b\simeq\mathcal{T}$; we must only have $\mathcal{T}_b$ roughly the same in all hadrons. Thus we believe that the impulse approximation in the quark model is not unreasonable.

III. THE MULTIPLE-SCATTERING FORMULAS

We assume that it is indeed possible to neglect the effect of the binding interaction, so that we may start the discussion of the multiple-scattering expansion from Eq. (1b). If we write the scattering interaction V as a sum of pair interactions V_{ij}, we have

$$\mathcal{T}=\sum_{ij}V_{ij}+\sum_{iji'j'}V_{ij}\Big(d_0-\sum_{kl}V_{kl}\Big)^{-1}V_{i'j'}\,. \tag{11}$$

We would like to write $\mathcal{T}$ as a function of the individual t matrices t_{ij} defined by

$$t_{ij}=V_{ij}+V_{ij}(d_0-V_{ij})^{-1}V_{ij}\,. \tag{12}$$

First we define

$$\mathcal{T}^{(ab)}=\sum_{\substack{ij\\(ij)\neq(ab)}}\Big[V_{ij}+\sum_{kl}V_{ij}\Big(d_0-\sum_{mn}V_{mn}\Big)^{-1}V_{kl}\Big], \tag{13}$$

where $(ij)\neq(ab)$ means that either $i\neq a$ or $j\neq b$. Then by subtracting Eq. (12) from Eq. (13) and using the identity

$$\Big(d_0-\sum_{mn}V_{mn}\Big)^{-1}=(d_0-V_{ij})^{-1}+(d_0-V_{ij})^{-1}\sum_{\substack{pq\\(pq)\neq(ij)}}V_{pq}\Big(d_0-\sum_{mn}V_{mn}\Big)^{-1}, \tag{14}$$

it is easy to show that

$$\mathcal{T}^{(ab)}-\sum_{\substack{ij\\(ij)\neq(ab)}}t_{ij}=\sum_{\substack{ij\\(ij)\neq(ab)}}(d_0-V_{ij})^{-1}\sum_{\substack{kl\\(kl)\neq(ij)}}\Big[V_{kl}+V_{kl}\Big(d_0-\sum_{mn}V_{mn}\Big)^{-1}\sum_{pq}V_{pq}\Big]. \tag{15}$$

But we also have the identity

$$(d_0-V_{ij})^{-1}=d_0^{-1}+(d_0-V_{ij})^{-1}V_{ij}d_0^{-1}, \tag{16}$$

which allows us to convert the first factor in Eq. (15) into $t_{ij}d_0^{-1}$; the last factor is just $\mathcal{T}^{(ij)}$, so

$$\mathcal{T}^{(ab)}=\sum_{\substack{ij\\(ij)\neq(ab)}}(t_{ij}+t_{ij}d_0^{-1}\mathcal{T}^{(ij)}). \tag{17}$$

Now if we let the superscript (00) represent the absence of any restriction on the sum in Eq. (13), so that $\mathcal{T}^{(00)}=\mathcal{T}$, iteration of Eq. (17) gives the desired expression of $\mathcal{T}$ in terms of t_{ij}

$$\mathcal{T}=\sum_{ij}t_{ij}+\sum_{\substack{ij\\(ij)\neq(kl)}}\sum_{kl}t_{ij}d_0^{-1}t_{kl}+\sum_{\substack{ij\\(ij)\neq(kl),\,(kl)\neq(mn)}}\sum_{kl}\sum_{mn}t_{ij}d_0^{-1}t_{kl}d_0^{-1}t_{mn}+\cdots\,. \tag{18}$$

Note that in the last term quoted in this equation, it is possible to have $(ij)=(mn)$. Equation (18) represents the scattering of a pair, followed by the propagation of the system, followed by the scattering of a third pair (which may be the same as the first pair, but *not* the same as the second), and so forth.

Equation (18), which is actually familiar from nuclear physics, has the advantage of involving not the Glauber approximation to the two-body scattering operators, but the exact operators themselves. We will be forced to make small-angle approximations to reduce the equation to a usable form, but these approximations will be confined to the denominators d_0, and the accuracy of the approximation may be estimated, so that we will be able to make some statement about the range of validity of our results; this information is hard to obtain in the Glauber formalism.

To see how the argument goes, we will derive the usual multiple-scattering formulas for meson-baryon scattering in the quark model ignoring spin. The generalization to spin-dependent amplitudes is in principle simple, and, as long as we work only through double scattering, the generalization to baryon-baryon scattering is immediate, so in practice we are not seriously limiting the derivation. We also assume that only the quark variables need be considered in calculating the scattering, and that intermediate states involving more or fewer quarks than the initial state may be ignored. In that case we have five spatial and five momentum variables: two each $(\vec{x}_1, \vec{x}_2, \vec{p}_1, \vec{p}_2)$ for the quarks in the meson and three each $(\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{q}_1, \vec{q}_2, \vec{q}_3)$ for the quarks in the baryon. A more convenient set of variables are the following:

$$\begin{aligned}
\vec{Z}_1 &= \vec{x}_1 - \vec{x}_2,\\
\vec{Z}_2 &= \vec{y}_1 - \vec{y}_3,\\
\vec{Z}_3 &= \vec{y}_2 - \vec{y}_3,\\
\vec{Z}_4 &= \vec{x}_1 - \vec{y}_1,\\
\vec{Z}_5 &= \tfrac{1}{5}(\vec{x}_1 + \vec{x}_2 + \vec{y}_1 + \vec{y}_2 + \vec{y}_3),\\
\vec{Q}_1 &= \tfrac{1}{5}\vec{p}_1 - \tfrac{4}{5}\vec{p}_2 + \tfrac{1}{5}\vec{q}_1 + \tfrac{1}{5}\vec{q}_2 + \tfrac{1}{5}\vec{q}_3,\\
\vec{Q}_2 &= \tfrac{2}{5}\vec{p}_1 + \tfrac{2}{5}\vec{p}_2 + \tfrac{3}{5}\vec{q}_1 - \tfrac{3}{5}\vec{q}_2 - \tfrac{3}{5}\vec{q}_3,\\
\vec{Q}_3 &= -\tfrac{1}{5}\vec{p}_1 - \tfrac{1}{5}\vec{p}_2 - \tfrac{1}{5}\vec{q}_1 + \tfrac{4}{5}\vec{q}_2 - \tfrac{1}{5}\vec{q}_3,\\
\vec{Q}_4 &= \tfrac{3}{5}\vec{p}_1 + \tfrac{3}{5}\vec{p}_2 - \tfrac{2}{5}\vec{q}_1 - \tfrac{2}{5}\vec{q}_2 - \tfrac{3}{5}\vec{q}_3,\\
\vec{Q}_5 &= \vec{p}_1 + \vec{p}_2 + \vec{q}_1 + \vec{q}_2 + \vec{q}_3,
\end{aligned} \tag{19}$$

with an inverse transformation

$$\begin{aligned}
\vec{x}_1 &= \tfrac{1}{5}\vec{Z}_1 + \tfrac{2}{5}\vec{Z}_2 - \tfrac{1}{5}\vec{Z}_3 + \tfrac{3}{5}\vec{Z}_4 + \vec{Z}_5,\\
\vec{x}_2 &= -\tfrac{4}{5}\vec{Z}_1 + \tfrac{2}{5}\vec{Z}_2 - \tfrac{1}{5}\vec{Z}_3 + \tfrac{3}{5}\vec{Z}_4 + \vec{Z}_5,\\
\vec{x}_3 &= \tfrac{1}{5}\vec{Z}_1 + \tfrac{2}{5}\vec{Z}_2 - \tfrac{1}{5}\vec{Z}_3 - \tfrac{2}{5}\vec{Z}_4 + \vec{Z}_5,\\
\vec{x}_4 &= \tfrac{1}{5}\vec{Z}_1 - \tfrac{3}{5}\vec{Z}_2 + \tfrac{4}{5}\vec{Z}_3 + \tfrac{2}{5}\vec{Z}_4 + \vec{Z}_5,\\
\vec{x}_5 &= \tfrac{1}{5}\vec{Z}_1 - \tfrac{3}{5}\vec{Z}_2 - \tfrac{1}{5}\vec{Z}_3 - \tfrac{2}{5}\vec{Z}_4 + \vec{Z}_5,\\
\vec{p}_1 &= \vec{Q}_1 + \vec{Q}_4 + \tfrac{1}{5}\vec{Q}_5,\\
\vec{p}_2 &= -\vec{Q}_1 + \tfrac{1}{5}\vec{Q}_5,\\
\vec{q}_1 &= \vec{Q}_2 - \vec{Q}_4 + \tfrac{1}{5}\vec{Q}_5,\\
\vec{q}_2 &= \vec{Q}_3 + \tfrac{1}{5}\vec{Q}_5,\\
\vec{q}_3 &= -\vec{Q}_2 - \vec{Q}_3 + \tfrac{1}{5}\vec{Q}_5.
\end{aligned} \tag{20}$$

The momenta have been chosen so that

$$\sum_i \vec{x}_i \cdot \vec{p}_i + \sum_i \vec{y}_i \cdot \vec{q}_i = \sum \vec{Z}_i \cdot \vec{Q}_i \tag{21}$$

and the Jacobian of the transformation is one. In this new set of variables our equations will be longer than they would have been if written in terms of x, y, p, and q, but it will be much clearer how to perform most of the integrations.

From translational invariance it is clear that

$$\Psi_M(\vec{x}_1, \vec{x}_2) = \exp[\tfrac{1}{2} i \vec{P}_M \cdot (\vec{x} + \vec{x}_2)]\, \psi_M(\vec{Z}_1), \tag{22}$$

$$\Psi_B(\vec{y}_1, \vec{y}_2, \vec{y}_3) = \exp[\tfrac{1}{3} i \vec{P}_B \cdot (\vec{y}_1 + \vec{y}_2 + \vec{y}_3)]\, \psi_B(\vec{Z}_2, \vec{Z}_3),$$

and we will of course work in the center-of-mass system, where $\vec{P}_M + \vec{P}_B = 0$. The eigenstates of $K_i + K_t$ in Eq. (2b) will be taken to be

$$\phi \equiv |a\rangle = (2\pi)^{-15/2} \exp[i(\vec{p}_1 \cdot \vec{x}_1 + \vec{p}_2 \cdot \vec{x}_2 + \vec{q}_1 \cdot \vec{y}_1 + \vec{q}_2 \cdot \vec{y}_2 + \vec{q}_3 \cdot \vec{y}_3)], \tag{23}$$

and we will assume that the energies associated with the momenta $\vec{p}_i$ and $\vec{q}_i$ are

$$E_i = (p_i^2 + m_M^2)^{1/2}, \quad E_\alpha = (q_\alpha^2 + m_B^2)^{1/2}. \tag{24}$$

A typical single-scattering contribution to $\mathcal{T}$, $T_{11}^{(1)}$, is now very simply obtained. (The subscripts on T_{11} indicate that the first baryon quark and the first meson quark are involved in the scattering.) If the initial momenta are $\vec{P}_M$ and $\vec{P}_B$ and the final momenta are $\vec{P}'_M$ and $\vec{P}'_B$, and if the initial and final states are denoted $|i\rangle$ and $|f\rangle$, respectively, we have

$$T_{11}^{(1)} = \langle f|a'\rangle \langle a'|t_{11}|a\rangle \langle a|i\rangle = (2\pi)^{-30} \int d^3Z_1'' \cdots d^3Z_5\, d^3Q_1' \cdots d^3Q_5$$

$$\times\Big[\exp\{-i[(\vec{P}_M'+\vec{P}_B')\cdot\vec{Z}_5''+\vec{\tau}'\cdot(-3\vec{Z}_1''+4\vec{Z}_2''-2\vec{Z}_3''+6\vec{Z}_4'')]\}\psi_M^*(\vec{Z}_1'')\psi_B^*(\vec{Z}_2'',\vec{Z}_3'')$$
$$\times\exp\Big(i\sum_{i=1}^{5}\vec{Q}_i'\cdot(\vec{Z}_i''-\vec{Z}_i')\Big)t_{11}(\vec{Z}_4')\exp\Big(i\sum_{i=1}^{5}\vec{Q}_i\cdot(\vec{Z}_i'-\vec{Z}_i)\Big)$$
$$\times\exp\{i[(\vec{P}_M+\vec{P}_B)\cdot\vec{Z}_5+\vec{\tau}\cdot(-3\vec{Z}_1+4\vec{Z}_2-2\vec{Z}_3+6\vec{Z}_4)]\}\psi_M(\vec{Z}_1)\psi_B(\vec{Z}_2,\vec{Z}_3)\Big],\tag{25}$$

where $\vec{\tau}=\frac{1}{10}\vec{P}_M-\frac{1}{15}\vec{P}_B$. A large list of integrals can now be done, either yielding δ functions or using the δ functions of previous integrations. If we do these integrals, and if we define $\vec{q}=\vec{P}_B-\vec{P}_B'=\vec{P}_M'-\vec{P}_M$ and observe that $\vec{\tau}'-\vec{\tau}=\frac{1}{6}\vec{q}$, we get

$$T_{11}^{(1)}=(2\pi)^3\delta(\vec{P}_M+\vec{P}_B-\vec{P}_M'-\vec{P}_B')\Big(\int d^3Z_4'\,e^{-i\vec{q}\cdot\vec{Z}_4}t_{11}(\vec{Z}_4')\Big)\Big(\int e^{i\vec{q}\cdot\vec{Z}_1/2}\psi_M^*(\vec{Z}_1)\psi_M(\vec{Z}_1)d^3Z_1\Big)$$
$$\times\Big(\int d^3Z_2d^3Z_3\,e^{-i\vec{q}\cdot(2\vec{Z}_2/3-\vec{Z}_3/3)}\psi_B^*(\vec{Z}_2,\vec{Z}_3)\psi_B(\vec{Z}_2,\vec{Z}_3)\Big)\ .\tag{26}$$

Each of the factors in large parentheses is the appropriate form factor for the t matrix or the wave function, so Eq. (26) is the standard form for single scattering. It is worth emphasizing that Eq. (26) is valid at all angles, so long as the approximation of neglecting the effects of the binding does not break down.

The double-scattering amplitudes are more difficult. There are three different kinds of terms: (1) A quark in the meson scatters successively from two different quarks in the baryon; (2) a quark in the baryon scatters successively from two different quarks in the meson; (3) the two scatterings involve different quarks in both the meson and the baryon. Moreover, when the intermediate states are inserted in the double-scattering contributions $T^{(2)}$, giving [since $|a\rangle$ is an eigenstate of $(E-K_i-K_t+i\epsilon)^{-1}$]

$$T^{(2)}=\langle f|a''\rangle\langle a''|t_{i\alpha}|a'\rangle(E_a^0-E_{a'}^0+i\epsilon)^{-1}\langle a'|t_{j\beta}|a\rangle\langle a|i\rangle,\tag{27}$$

it makes a difference to the final form of the expressions which amplitude appears to the right and which to the left. If we represent $T^{(2)}$ above by the schematic notation $T^{(2)}=t_{i\alpha}\otimes t_{i\beta}$, we have six typical double-scattering contributions (the others are obtained by permuting indices):

$$\begin{aligned}
&T_A^{(2)}=t_{12}(\vec{x}_1'-\vec{y}_2')\otimes t_{11}(\vec{x}_1-\vec{y}_1),\quad T_B^{(2)}=t_{11}\otimes t_{12},\\
&T_C^{(2)}=t_{21}(\vec{x}_2'-\vec{y}_1')\otimes t_{11}(\vec{x}_1-\vec{y}_1),\quad T_D^{(2)}=t_{11}\otimes t_{21},\\
&T_E^{(2)}=t_{22}(\vec{x}_2'-\vec{y}_2')\otimes t_{11}(\vec{x}_1-\vec{y}_1),\quad T_F^{(2)}=t_{11}\otimes t_{22}.
\end{aligned}\tag{28}$$

The physical amplitudes are of course the sum of the amplitudes differing only in the ordering of the scattering, since it is not possible to tell by any measurement which scattering occurred "first."

It is possible to handle all six of these amplitudes together, rather than separately. To do this, we choose

$$\vec{v}_a=\sum_{i=1}^{4}a_i\vec{Z}_i,\qquad \vec{v}_b=\sum_{i=1}^{4}b_i\vec{Z}_i,\tag{29}$$

and pick the constants a_i and b_i so that $\vec{v}_a$ is the argument of the first t and $\vec{v}_b$ the argument of the second. (The arguments of all the t_{ij} have been chosen such that a_4 and b_4 are always unity.) The argument of a typical exponential in Eq. (27) may now be written

$$\sum_{i=1}^{4}\vec{Q}_i\cdot\vec{Z}_i=\sum_{i=1}^{3}(\vec{Q}_i-a_i\vec{Q}_4)\cdot\vec{Z}_i+\vec{Q}_4\cdot\vec{v}_a.\tag{30}$$

If we call the typical double-scattering contribution

$$T_{ab}^{(2)}=t_a(\vec{v}_a)\otimes t_b(\vec{v}_b),\tag{31}$$

and substitute properly in Eq. (27), we get an expression analogous to Eq. (25), only more complicated. In addition to the analogs of the factors in Eq. (25), there is an energy denominator F_{ab} given by

$$F_{ab}^{-1}(\vec{p}_i,\vec{q}_\alpha)=\sum_{i=1}^{2}[(p_i'^2+m_M^2)^{1/2}-(p_i^2+m_M^2)^{1/2}]+\sum_{\alpha=1}^{3}[(q_\alpha'^2+m_B^2)^{1/2}-(q_\alpha^2+m_B^2)^{1/2}]+i\epsilon.\tag{32}$$

Again a large number of integrals will be immediate, either yielding δ functions or using δ functions obtained in earlier integrations. When these integrals are all evaluated, we have the result

$$T_{ab}^{(2)}=(2\pi)^{-9}\delta(\vec{P}_M+\vec{P}_B-\vec{P}_M'-\vec{P}_B')\int d^3Z_1'\cdots d^3Z_3d^3v_ad^3v_bd^3Q_1'\cdots d^3Q_4'$$

$$\times \exp\left[i\left(\sum_{i=1}^{3}(\vec{Q}_i' - a_i\vec{Q}_4' + 6a_i\vec{\tau}')\cdot\vec{Z}_i' + (3\vec{Z}_1' - 4\vec{Z}_2' + 2\vec{Z}_3')\cdot\vec{\tau}'\right)\right]\psi_M^*(\vec{Z}_1')\psi_B^*(\vec{Z}_2', \vec{Z}_3')$$
$$\times \exp[i(\vec{Q}_4' - 6\vec{\tau}')\cdot\vec{v}_a]\, t_a(\vec{v}_a)F_{ab}\exp[-i(\vec{Q}_4' - 6\vec{\tau})\cdot\vec{v}_b]t_b(\vec{v}_b)\psi_M(\vec{Z}_1)\psi_B(\vec{Z}_2, \vec{Z}_3)$$
$$\times \exp\left[-i\left(\sum_{i=1}^{3}(\vec{Q}_i' - b_i\vec{Q}_4' + 6b_i\vec{\tau})\cdot\vec{Z}_i + (3\vec{Z}_1' - 4\vec{Z}_2' + 2\vec{Z}_3')\cdot\vec{\tau}\right)\right], \quad (33a)$$

where

$$F_{ab}(P_M, P_B, \vec{Q}_1', \vec{Q}_2', \vec{Q}_3', \vec{Q}_4') = (\{[\tfrac{1}{5}(\vec{P}_M + \vec{P}_B) + \vec{Q}_1' + 6\vec{\tau} + b_1(6\vec{\tau} - \vec{Q}_4')]^2 + m_M{}^2\}^{1/2} + \{[\tfrac{1}{5}(\vec{P}_M + \vec{P}_B) - \vec{Q}_1' - b_1(6\vec{\tau} - \vec{Q}_4')]^2 + m_M{}^2\}^{1/2}$$
$$+ \{[\tfrac{1}{5}(\vec{P}_M + \vec{P}_B) - 6\vec{\tau} - \vec{Q}_2' - b_2(\vec{Q}_4' - 6\vec{\tau})]^2 + m_B{}^2\}^{1/2} + \{[\tfrac{1}{5}(\vec{P}_M + \vec{P}_B) + \vec{Q}_3' + b_3(6\vec{\tau} - \vec{Q}_4')]^2 + m_B{}^2\}^{1/2}$$
$$+ \{[\tfrac{1}{5}(\vec{P}_M + \vec{P}_B) - \vec{Q}_2' + b_2(\vec{Q}_4' - 6\vec{\tau}) - \vec{Q}_3' + b_3(\vec{Q}_4' - 6\vec{\tau})]^2 + m_B{}^2\}^{1/2})$$
$$- (\text{the same with } \vec{Q}_4' \text{ substituted everywhere for } 6\vec{\tau}) + i\epsilon. \quad (33b)$$

Notice that integrations over d^3v_a and d^3v_b have replaced the integrations over d^3Z_4 and d^3Z_4'.

Now the states ψ_M and ψ_B are nonrelativistic in their respective rest frames and therefore have internal momenta that are small compared with $\vec{P}_M$ and $\vec{P}_B$. However, the exponentials in Eq. (33a) involve $\vec{\tau}$ and $\vec{\tau}'$, which are large momenta; thus the momentum variables in the integration are large. We would like to have the integrations be over momenta that are restricted to small values by the constraints imposed by ψ_M and ψ_B; thus we make the substitutions

$$\vec{Q}_1' = \vec{Q}_1 - 3\vec{\tau} = \vec{Q}_1 - 3\vec{\tau}' + \tfrac{1}{2}\vec{q}, \quad \vec{Q}_3' = \vec{Q}_3 - 2\vec{\tau} = \vec{Q}_3 - 2\vec{\tau}' + \tfrac{1}{3}\vec{q},$$
$$\vec{Q}_2' = \vec{Q}_2 + 4\vec{\tau} = \vec{Q}_2 + 4\vec{\tau}' - \tfrac{2}{3}\vec{q}, \quad \vec{Q}_4' = \vec{Q}_4 + 6\vec{\tau} = \vec{Q}_4 + 6\vec{\tau}' - \vec{q}. \quad (34)$$

The resulting expression for the amplitude is

$$T_{ab}^{(2)} = (2\pi)^{-9}\delta(\vec{P}_M + \vec{P}_B - \vec{P}_M' - \vec{P}_B')\int d^3Z_1' \cdots d^3Z_3 d^3v_a d^3v_b d^3Q_1 \cdots d^3Q_4$$
$$\times \exp\left[i\sum_{i=1}^{3}(\vec{Q}_i - a_i\vec{Q}_4 + a_i\vec{q})\cdot\vec{Z}_i' + (\tfrac{1}{2}\vec{Z}_1' - \tfrac{2}{3}\vec{Z}_2' + \tfrac{1}{3}\vec{Z}_3')\cdot\vec{q}\right]\psi_M^*(\vec{Z}_1')\psi_B^*(\vec{Z}_2', \vec{Z}_3')$$
$$\times \exp[i(\vec{Q}_4 - \vec{q})\cdot\vec{v}_a]t_a(\vec{v}_a)F_{ab}(\vec{P}_M, \vec{P}_B, \vec{Q}_i)\exp(-i\vec{Q}_4\cdot\vec{v}_b)t_b(\vec{v}_b)$$
$$\times \exp\left[-i\left(\sum_{i=1}^{3}(\vec{Q}_i - b_i\vec{Q}_4)\cdot\vec{Z}_i\right)\right]\psi_M(\vec{Z}_1)\psi_B(\vec{Z}_2, \vec{Z}_3), \quad (35)$$

where F is to be obtained by substituting from Eq. (34) in (33b).

Before we obtain an explicit form for F and discuss the approximations necessary to bring Eq. (35) into manageable form, we need to discuss the amplitude $T_{ba}^{(2)}$. As discussed earlier, $T_{ab}^{(2)}$ and $T_{ba}^{(2)}$, since they differ only by the order of the scattering processes, are not distinguishable from each other physically. The relevant quantity is therefore their sum, which we will denote by $T_{ab+ba}^{(2)}$. The amplitude $T_{ba}^{(2)}$ may be obtained from Eq. (35) by simply interchanging a_i with the corresponding b_i. It is convenient to confine the resulting changes to the factor F. This may be done by changing the dummy momentum variables in Eq. (35) according to

$$\vec{\bar{Q}}_1 = \vec{\bar{Q}}_i - a_i\vec{Q}_4 + a_i\vec{q}, \quad i = 1, 2, 3$$
$$\vec{\bar{Q}}_4 = -\vec{\bar{Q}}_4 + \vec{q}, \quad (36)$$

after the interchange of a_i and b_i. There are no b_i in Eq. (36) because a comparison of Eqs. (29) and (28) reveals that for the relevant amplitudes $T_A^{(2)}$, $T_C^{(2)}$, and $T_E^{(2)}$, all the b_i are zero.

A little algebra now shows that the exponentials for $T_{ba}^{(2)}$ are the same functions of $\vec{\bar{Q}}_1$, $\vec{\bar{Q}}_2$, $\vec{\bar{Q}}_3$, and $\vec{\bar{Q}}_4$ as those for $T_{ab}^{(2)}$ are of $\vec{Q}_1$, $\vec{Q}_2$, $\vec{Q}_3$, and $\vec{Q}_4$. Hence the changes in Eq. (35) when $T_{ba}^{(2)}$ is calculated instead of $T_{ab}^{(2)}$ are confined to F, and we may obtain $T_{ba}^{(2)}$ directly from Eq. (35) by the substitution

$$F_{ba}(\vec{P}_M, \vec{P}_B, \vec{Q}_1, \ldots, \vec{Q}_4) = F_{ab}(\vec{P}_M, \vec{P}_B, \vec{Q}_1 - a_1\vec{Q}_4 + a_1\vec{q}, \ldots, -\vec{Q}_4 + \vec{q}). \quad (37)$$

It is of course most convenient to work with $T_{ab+ba}^{(2)}$, which may be done simply by taking $F_{ab+ba} = F_{ab} + F_{ba}$.

Now F is defined originally in Eq. (33b) and must be subjected to the changes in variables in Eqs. (34) and (36). Once this is done, and all the (now irrelevant) b_i have been set equal to zero, we have

$$F_{ab}^{-1} = [(\tfrac{1}{2}\vec{P}_M + \vec{Q}_1)^2 + m_M^2]^{1/2} + [(\tfrac{1}{3}\vec{P}_B + \vec{Q}_2)^2 + m_B^2]^{1/2} - [(\tfrac{1}{2}\vec{P}_M + \vec{Q}_1 + \vec{Q}_4)^2 + m_M^2]^{1/2} - [(\tfrac{1}{3}\vec{P}_B + \vec{Q}_2 - \vec{Q}_4)^2 + m_B^2]^{1/2} + i\epsilon . \quad (38)$$

At this point we may simplify our expressions considerably if we use our expectation that the meson and baryon states involve internal momenta that are small compared with the total momenta $\vec{P}_M$ and $\vec{P}_B$. Specifically, if $\vec{q}$ is small compared with $\vec{P}_M$ and $\vec{P}_B$, the wave functions in Eq. (35), involving small relative momenta, ensure that $\vec{Q}_1$, $\vec{Q}_2$, $\vec{Q}_3$, and $\vec{Q}_4$ are all small compared with $\vec{P}_M$ and $\vec{P}_B$. It is therefore permissible to expand F in terms of powers of internal momenta divided by total momentum. For the moment we will keep only the lowest term (corrections will be discussed in the next section), so that we approximate

$$F_{ab} \approx (\vec{Q}_4 \cdot \vec{c} + i\epsilon)^{-1} , \quad (39)$$

where we define

$$\vec{c} = \tfrac{1}{3}\vec{P}_B(\tfrac{1}{9}P_B^2 + m_B^2)^{-1/2} - \tfrac{1}{2}\vec{P}_M(\tfrac{1}{4}P_M^2 + m_M^2)^{-1/2} . \quad (40)$$

The quantity $\vec{c}$ is the relative velocity of the quarks, ignoring internal motion. Using Eq. (37) we see that

$$F_{ab+ba} \approx \frac{1}{\vec{Q}_4 \cdot \vec{c} + i\epsilon} + \frac{1}{(-\vec{Q}_4 + \vec{q}) \cdot \vec{c} + i\epsilon} . \quad (41)$$

Since the lowest-order term in F_{ab+ba} is independent of $\vec{Q}_1$, $\vec{Q}_2$, and $\vec{Q}_3$, there are now three more integrations possible in Eq. (35) after the insertion of the approximation for F. In addition, for small q, $\vec{q}$ is nearly perpendicular to the incident direction, so that $\vec{q} \cdot \vec{c} \approx 0$, and we may write

$$F_{ab+ba} \approx (\vec{Q}_4 \cdot \vec{c} + i\epsilon)^{-1} - (\vec{Q}_4 \cdot \vec{c} - i\epsilon)^{-1} = \frac{-2\pi i}{c}\delta\left(\frac{\vec{Q}_4 \cdot \vec{c}}{c}\right), \quad (42)$$

allowing one further integration to be done in Eq. (35). After all these integrations are performed, we obtain a form-factor version of the double-scattering amplitude,

$$T^{(2)}_{ab+ba} = \frac{-2\pi i}{c}\delta(\vec{P}_M + \vec{P}_B - \vec{P}'_M - \vec{P}'_B)\int d^3v_a d^3v_b d^3Z_1 d^3Z_2 d^3Z_3 d^2Q_\perp \exp\{i[-a_1\vec{Q}_\perp + (a_1 + \tfrac{1}{2})\vec{q}] \cdot \vec{Z}_1\}\psi_M^*(\vec{Z}_1)\psi_M(\vec{Z}_1)$$
$$\times \exp\{i[-a_2\vec{Q}_\perp + (a_2 - \tfrac{2}{3})\vec{q}] \cdot \vec{Z}_2 + i[-a_3\vec{Q}_\perp + (a_3 + \tfrac{1}{3})\vec{q}] \cdot \vec{Z}_3\}\psi_B^*(\vec{Z}_2, \vec{Z}_3)\psi_B(\vec{Z}_2, \vec{Z}_3)$$
$$\times \exp[i(\vec{Q}_\perp - \vec{q}) \cdot \vec{v}_a]t_a(\vec{v}_a)\exp(-i\vec{Q}_\perp \cdot \vec{v}_b)t_b(\vec{v}_b) , \quad (43)$$

where $Q_\parallel = \vec{Q} \cdot \vec{c}/c$ and $\vec{Q}_\perp = \vec{Q} - Q_\parallel \vec{c}/c$. The subscript on the remaining $\vec{Q}$ ($\vec{Q}_4$) has been dropped.

We have not yet made any statement about the relative size of the total momentum and the quark mass m_M or m_B. However, in comparison of the double-scattering formulation with experiment[3] it has been found that the coefficient of the term quadratic in the amplitudes is not dependent on the energy; thus in practice we must have c independent of energy, which will be true only if $p_M \gg m_M$ and $p_B \gg m_B$. Since c looks like the relative velocity of the quarks, which we would expect to be the same as the relative velocity of the hadrons (neglecting internal momenta), the value of $c = 2$, which is obtained by assuming $p_M \gg m_M$, $p_B \gg m_B$, is very reasonable, if not compelling. We therefore conclude that the binding of the quarks is probably such that the effective mass in the bound state is small.

The specific form of each of the three types of double scattering can be found by taking $c \approx 2$ and substituting the appropriate values for the a_i. These forms all agree with the forms conventionally used[6-8] in quark-model analyses, provided $q \approx 0$.

IV. THE ACCURACY OF THE MULTIPLE-SCATTERING FORMULAS

In order to discuss the accuracy of the amplitudes given in Eq. (43), we must retain higher-order terms in the expansion on F_{ab}^{-1} in Eq. (38). In order to avoid immediate assumptions about the sizes of P_M and P_B relative to m_M and m_B we take the large quantities in the square roots to be $E_M^2 = \tfrac{1}{4}P_M^2 + m_M^2$ and $E_B^2 = \tfrac{1}{9}P_B^2 + m_B^2$ and assume that all relative momenta are small compared with either of these. Then making use of the parallel and perpendicular component notation given below Eq. (43) we get

$$F_{ab}^{-1} = c\vec{Q}_\parallel + i\epsilon + \frac{1}{2E_M}[\vec{Q}_{1\perp}^2 - (\vec{Q}_1 + \vec{Q}_4)_\perp^2] + \frac{1}{2E_B}[(\vec{Q}_2 + \vec{Q}_3)_\perp^2 - 2\vec{Q}_{2\perp} \cdot \vec{Q}_{3\perp} - (\vec{Q}_2 - \vec{Q}_4)_\perp^2 - \vec{Q}_{3\perp}^2]$$
$$+ \frac{\delta_M^2}{2E_M}[\vec{Q}_{1\parallel}^2 - (\vec{Q}_1 + \vec{Q}_4)_\parallel^2] + \frac{\delta_B^2}{2E_B}[(\vec{Q}_2 + \vec{Q}_3)_\parallel^2 - 2\vec{Q}_{2\parallel}\vec{Q}_{3\parallel} - (\vec{Q}_2 - \vec{Q}_4)_\parallel^2 - \vec{Q}_{3\parallel}^2], \quad (44)$$

where

$$\delta_M^2 = 1 - P_M^2/4E_M^2 \text{ and } \delta_B^2 = 1 - P_B^2/9E_B^2 .$$

The quantity F_{ba} may be readily obtained from Eqs. (36), (37), and (44); it is roughly the same form as F_{ab} but depends on $\vec{q}$ as well as on $\vec{Q}_1$, $\vec{Q}_2$, $\vec{Q}_3$, and $\vec{Q}_4$; the result is

$$\begin{aligned} F_{ba}^{-1} = {} & c(-\vec{Q}_4+\vec{q})_\parallel + i\epsilon + \frac{1}{2E_m}\{[\vec{Q}_1 - a_1(\vec{Q}_4-\vec{q})]_\perp{}^2 - [\vec{Q}_1+(a_1+1)(\vec{q}-\vec{Q}_4)]_\perp{}^2\} \\ & + \frac{1}{2E_B}\{[\vec{Q}_2+\vec{Q}_3+(a_2+a_3)(\vec{q}-\vec{Q}_4)]_\perp{}^2 - 2[\vec{Q}_2+a_2(\vec{q}-\vec{Q}_4)]_\perp\cdot[\vec{Q}_3+a_3(\vec{q}-\vec{Q}_4)]_\perp \\ & \qquad - [\vec{Q}_2+(a_2+1)(\vec{q}-\vec{Q}_4)]_\perp{}^2 - [\vec{Q}_3+a_3(\vec{q}-\vec{Q}_4)]_\perp{}^2\} \\ & + \frac{\delta_M{}^2}{2E_M}\{[\vec{Q}_1 - a_1(\vec{Q}_4-\vec{q})]_\parallel{}^2 - [\vec{Q}_1+(a_1+1)(\vec{q}-\vec{Q}_4)]_\parallel{}^2\} \\ & + \frac{\delta_B{}^2}{2E_B}\{[\vec{Q}_2+\vec{Q}_3+(a_2+a_3)(\vec{q}-\vec{Q}_4)]_\parallel{}^2 - 2[\vec{Q}_2+a_2(\vec{q}-\vec{Q}_4)]_\parallel[\vec{Q}_3-a_3(\vec{q}-\vec{Q}_4)]_\parallel \\ & \qquad - [\vec{Q}_2+(a_2+1)(\vec{q}-\vec{Q}_4)]_\parallel{}^2 - [\vec{Q}_3+a_3(\vec{q}-\vec{Q}_4)]_\parallel{}^2\}\,. \end{aligned} \tag{45}$$

Because we now have dependence on all internal momenta, we are unable to perform integrations over $\vec{Q}_1$, $\vec{Q}_2$, and $\vec{Q}_3$ after substitution of Eqs. (44) and (45) into the expression for $T^{(2)}$, Eq. (35). However, the integration over $dQ_{4\parallel}$ can be done by contours, and the effect of the higher-order terms then becomes more obvious. In particular, we will be able to see in a clearer fashion the dependence of the corrections on angle (through their dependence on $\vec{q}$) and obtain an estimate for the largest angle for which Eq. (43) is reasonably accurate.

To avoid serious awkwardness in notation in the remainder of this section we denote $Q_{4\parallel}$ by the letter x. Then to perform the integration over dx, we observe that we may write

$$\begin{aligned} F_{ab}{}^{-1} &= Ax^2+Bx+D+i\epsilon\,, \\ F_{ba}{}^{-1} &= A'x^2+B'x+D'+i\epsilon\,, \end{aligned} \tag{46}$$

where D and D' are of second order in relative momentum divided by total momentum and where B and B' are $\pm c$ plus terms of first order. If we substitute the expressions for F_{ab} and F_{ba} from Eq. (46) into Eq. (35) and extract all factors that depend on x, we get an integral

$$\begin{aligned} I = \int dx \exp\{iu_\parallel x\,[(Ax^2+Bx+D+i\epsilon)^{-1} \\ + (A'x^2+B'x+D'+i\epsilon)^{-1}]\}\,, \end{aligned} \tag{47}$$

where $\vec{u} = \sum_{i=1}^{3} a_i \vec{Z}_i + \vec{v}_a - \vec{v}_b$. The integral I may be performed by contours if the denominators are factored:

$$\begin{aligned} & Ax^2+Bx+D = A(x-x_1)(x-x_2), \\ & A'x^2+B'x+D' = A'(x-x_1')(x-x_2'), \\ & x_1 = -\frac{D}{B} - \frac{AD^2}{B^3} + \cdots, \qquad x_2 = -\frac{B}{A} + \frac{D}{B} + \frac{AD^2}{B^3} + \cdots, \\ & x_1' = -\frac{D'}{B'} - \frac{A'D'^2}{B'^3} + \cdots, \quad x_2' = -\frac{B'}{A'} + \frac{D'}{B'} + \frac{A'D'^2}{B'^3} + \cdots. \end{aligned} \tag{48}$$

Each denominator of course has two poles; however, $x_2 >> x_1$ and $x_2' >> x_1'$ so that the residues of the poles at x_2 and x_2' are exponentials which vary rapidly as $u_\parallel$ changes compared with the variation of the corresponding exponentials obtained from the residues at x_1 and x_1'. Hence when we eventually integrate over $u_\parallel$, we expect the residues of the poles at x_2 and x_2' to contribute only a small amount to the final result; we therefore ignore those poles in the evaluation of I. Then the contributing poles are, from the first denominator, $x = x_1 - i\epsilon$, and from the second $x = x_1' + i\epsilon$. For $u_\parallel > 0$ we must complete the contour upward and only the second denominator contributes, while for $u_\parallel < 0$ we must complete the contour downward and only the first denominator contributes; when all the algebra is done and terms are kept consistently to second order, we get

$$\begin{aligned} I = -\frac{2\pi i}{c}\Big[\theta(u_\parallel)\Big(1+\frac{1}{c}\tilde{B}'+\frac{1}{c^2}\tilde{B}'^2-\frac{2}{c^2}A'D'\Big)e^{iD'u_\parallel/c} \\ + \theta(-u_\parallel)\Big(1-\frac{1}{c}\tilde{B}+\frac{1}{c^2}\tilde{B}^2-\frac{2}{c^2}AD\Big)e^{-iDu_\parallel/c}\Big]; \end{aligned} \tag{49}$$

where $\tilde{B} = B - c$ and $\tilde{B}' = B' + c$ and where $\theta(y) = 1$ for $y > 0$ and vanishes otherwise. Clearly the first approximation to I is to set the two exponentials equal to unity and to ignore all but the unity terms in the prefactors; we then recover our earlier lowest-order expression, Eq. (43).

The higher-order terms arising from the exponentials and from the prefactors in Eq. (49) should be considered separately. Let us look first at the prefactors and specifically at $\tilde{B}$. By comparing Eqs. (44) and (46) we see that a typical term in $\tilde{B}$ is $\delta_M Q_{1\parallel}/E_M$ and that all other terms in $\tilde{B}$ are similar in form. The presence of a factor $Q_{1\parallel}$ prevents any immediate integration over $dQ_{1\parallel}$ so that we cannot get a form-factor description of the amplitude; $\tilde{B}$ in effect introduces a nonlocality into the form factors of the scattering. However, we are assuming that only small internal momenta are involved in the wave functions ψ_M and ψ_B; thus for all values of the integration for which ψ is nonnegligible, $Q_{1\parallel}/E_M \ll 1$. This means that the nonlocality in the form factors is small and may be neglected, independent of the angle of scattering. A similar argument holds for all the terms in $\tilde{B}$ and AD and for most of the terms in $\tilde{B}'$ and $A'D'$.

However, $\tilde{B}'$ and $A'D'$ also involve $\vec{q}$, whose size is set by the angle of scattering rather than being limited by the size of the wave functions. These terms grow larger as the scattering angle increases and so in principle limit the angular range in which the lowest-order expression for $T^{(2)}$ is reliable. The effect could be discussed in a manner parallel to the discussion of the similar terms from the exponentials to be given shortly. However, both $\tilde{B}'$ and $A'D'$ contain as factors in every term either $\delta_M{}^2$ or $\delta_B{}^2$. We have already argued that fits to experiment indicate that we should take $P \gg m$. But for large P, both $\delta_M{}^2$ and $\delta_B{}^2$ are near zero; thus we do not expect in practice that these terms in the prefactors will affect $T^{(2)}$ significantly.

Next, we must discuss the effect of the exponentials in Eq. (49). A typical factor obtained from the exponential would be, among those not dependent on $\vec{q}$, $\exp(iQ_{4\perp}{}^2u_\parallel/E_M)$. Since $\vec{Q}_{4\perp}$ is an internal momentum, we expect that it will be important only for values less than or of the order of the inverse of the range of the wave function. But $u_\parallel$ is also important only for values of the order of the range of the wave function or less, so that $Q_{4\perp}u_\parallel$ is of order unity in the regions of the integration that contribute significantly. Then a factor $Q_{4\perp}/E_M \ll 1$ is left in the argument of the exponential, so that the exponential is near unity. The exponential may then be expanded, and by repeating the argument in the preceding paragraph we see that these typical $\vec{q}$-independent terms cause nonlocalities in the form factors but are small at all angles.

Finally, to discuss the effect of the $\vec{q}$-dependent parts of the exponentials in I, we must return to Eq. (35) for the full amplitude where there is additional exponential $\vec{q}$ dependence. If we substitute Eq. (49) for I into Eq. (35) for $T^{(2)}$, keeping only the $\vec{q}$-dependent terms in the exponentials (and only unity for the prefactors), we can perform the integrations over $\vec{Q}_1$, $\vec{Q}_2$, and $\vec{Q}_3$ to get

$$T^{(2)}_{ab+ba} \approx -\frac{2\pi i}{c}\delta(\vec{P}_M+\vec{P}_B-\vec{P}_M{}'-\vec{P}_B{}')\int d^3v_a d^3v_b d^3Z_1 d^3Z_2 d^3Z_3 d^2Q_\perp [e^{i(\vec{Q}_\perp-\vec{q})\cdot\vec{v}_a}t_a(\vec{v}_a)][e^{-i\vec{Q}_\perp\cdot\vec{v}_b}t_b(\vec{v}_b)]$$
$$\times\exp\{i[(a_1+\tfrac{1}{2})\vec{Z}_1+(a_2-\tfrac{2}{3})\vec{Z}_2+(a_3+\tfrac{1}{3})\vec{Z}_3]\cdot\vec{q}\}$$
$$\times\psi^*_M(\vec{Z}_1)\psi^*_B(\vec{Z}_2,\vec{Z}_3)[\theta(u_\parallel)e^{iq_\parallel u_\parallel}+\theta(-u_\parallel)]\psi_M(\vec{Z}_1)\psi_B(\vec{Z}_2,\vec{Z}_3). \quad (50)$$

In deriving Eq. (50), terms involving $q_\perp{}^2$ have been dropped. These terms can easily be shown to always be less than half the size of the terms we have kept. We have also dropped the subscript from $\vec{Q}_4$.

In order to get an estimate of the size of the corrections due to nonzero $q_\parallel$, let us assume that all amplitudes and wave functions are symmetric under reflection through a plane perpendicular to the incident direction in the center-of-mass system, that is

$$\begin{aligned} &t(-v_\parallel)=t(v_\parallel),\\ &\psi^*_M(-x_\parallel)\psi_M(-x_\parallel)=\psi^*_M(x_\parallel)\psi_M(x_\parallel),\\ &\psi^*_B(-y_\parallel,\,-w_\parallel)\psi_B(-y_\parallel,\,-w_\parallel)=\psi^*_B(y_\parallel,w_\parallel)\psi_B(y_\parallel,w_\parallel).\end{aligned} \quad (51)$$

Then we may change the sign of all spatial integration variables in the $\theta(-u_\parallel)$ term in Eq. (50) and obtain a form with only one θ function (for simplicity we also assume short-range scattering interactions so that we may neglect $\vec{v}_a$ and $\vec{v}_b$),

$$T^{(2)}_{ab+ba} \approx \frac{2\pi i}{c}\delta(\vec{P}_M+\vec{P}_B-\vec{P}_M'-\vec{P}_B')\int d^3v_a d^3v_b d^3Z_1 d^3Z_2 d^3Z_3 d^2Q_\perp [e^{i(\vec{Q}_\perp-\vec{q})\cdot\vec{v}_a}t_a(\vec{v}_a)][e^{-i\vec{Q}_\perp\cdot\vec{v}_b}t_b(\vec{v}_b)]$$
$$\times\exp\{i[(a_1+\tfrac{1}{2})\vec{Z}_1+(a_2-\tfrac{2}{3})\vec{Z}_2+(a_3+\tfrac{1}{3})\vec{Z}_3]\cdot\vec{q}_\perp\}\psi^*_M(\vec{Z}_1)\psi^*_B(\vec{Z}_2,\vec{Z}_3)$$
$$\times\theta(u_\parallel)J(q_\parallel)\psi_M(\vec{Z}_1)\psi_B(\vec{Z}_2,\vec{Z}_3), \quad (52)$$

where we define

$$J(q_\parallel)=\exp[iq_\parallel(\tfrac{1}{2}\vec{Z}_1-\tfrac{2}{3}\vec{Z}_2+\tfrac{1}{3}\vec{Z}_3)_\parallel]+\exp\{-iq_\parallel[(a_1+\tfrac{1}{2})\vec{Z}_{1\parallel}+(a_2-\tfrac{2}{3})\vec{Z}_{2\parallel}+(a_3+\tfrac{1}{3})\vec{Z}_{3\parallel}]\}. \quad (53)$$

For q sufficiently small, the exponentials are near 1 everywhere that the wave functions ψ are reasonably large, so that we may expand the exponentials around 1 to get

$$J(q_\parallel)\approx 1+iq_\parallel(-a_1\vec{Z}_{1\parallel}-a_2\vec{Z}_{2\parallel}-a_3\vec{Z}_{3\parallel})\,. \tag{54}$$

For all the amplitudes, each a_i is ± 1, so if we estimate each of the $\vec{Z}_{i\parallel}$ by R, the radius of the quark distribution in a hadron, we get

$$J(q_\parallel)\approx 1+3iRq_\parallel\,. \tag{55}$$

Now if we call E the maximum fractional error permitted in the amplitude, we clearly have

$$\left|\left(\frac{\delta T}{T}\right)_{\max}\right| = E = 3Rq_\parallel\,, \tag{56}$$

and since $\theta\simeq|\vec{q}|/P_B$ and $q_\parallel\approx q^2/(2P_B)=\frac{1}{2}P_B\theta^2$, we get a maximum allowable angle θ_m and momentum transfer q_m given by $\theta_m\approx[2E/(3P_BR)]^{1/2}$ and $q_m{}^2=2EP_B/(3R)$. Now since we are discussing the addition of a real part to a predominantly imaginary amplitude, we can tolerate $E=0.5$; if we also take $P_B=5\ \text{GeV}/c$ and, ignoring Lorentz contraction of the internal state of the hadrons, $R\approx\frac{1}{2}\text{fm}=1/(0.4\ \text{GeV})$ we get $\theta_m\approx 9^\circ$ and $q_m{}^2\approx 0.67\ (\text{GeV}/c)^2$. For comparison, with $P_B=5\ \text{GeV}/c$ the first break in the proton-proton scattering data[7] is at about $q^2=1\ (\text{GeV}/c)^2$. However, the composite particle is really Lorentz-contracted in the direction parallel to $\vec{c}$. At 5 GeV/c we have $\gamma\approx 5$ so that we should really use $R\approx\frac{1}{10}$ fm. Thus more realistic angular limits are $\theta_m\approx 20^\circ$ and $q_m{}^2\approx 3\ (\text{GeV}/c)^2$ in the center-of-momentum frame.

V. CONCLUSIONS AND DISCUSSION

We have seen that it is possible, using the Goldberger-Watson scattering theory, to justify the quark-model multiple-scattering formulas despite the possible deep binding of the quarks. We find also that it is possible to put fairly well-defined limits on the angular validity of the formulas, which cannot be done using the Glauber formalism. The angular limits come from the propagation of the system between scatterings, rather than from the representation of the individual scattering amplitudes, again in contrast to the Glauber results. It is also possible in principle to evaluate corrections, although a rather detailed knowledge of the quark wave functions would be necessary.

Clearly there is no theoretical barrier to the use of quark-model scattering relations at relatively large angles, since the Goldberger-Watson formalism provides a flexible language for discussing the scattering at angles where the simplest analysis is not clearly valid.

ACKNOWLEDGMENTS

The author would like to thank Professor Arthur Halprin and Professor Robert N. Hill for useful conversations during the course of this work.

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*Research supported in part by the University of Delaware Research Foundation.

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