We only sketch the proof for the left half-space, which proceeds in a very similar fashion. Now all the integrals over $\vec{K}$ should be transformed into the integrals over $\vec{k}$ and the commutator function D_n in the homogeneous dielectric medium,

$$D_n(\vec{r}, t) = \frac{1}{(2\pi)^3}\int d^3k \frac{\sin Kt}{K} e^{i\vec{k}\cdot\vec{r}}, \tag{12}$$

replaces the vacuum function D.

The integration contour runs above the real axis, thus avoiding the branch points of the function

$$K_3 = [k_3^2/n^2 - (k_1^2 + k_2^2)(1 - 1/n^2)]^{1/2}.$$

The integral along the large semicircle (Fig. 2) vanishes in the limit. Since there are no singularities inside the contour, the integral vanishes. The result is similar to Eq. (7) but has an additional factor of n^2 on the left-hand side.

Now let us integrate the left-hand side of Eq. (7) over $\vec{r}\,'$ with a square-integrable divergenceless function $\vec{f}$ having support in the right half-space. When $z > 0$ we get $\vec{f}(\vec{r})$ – this has already been proved. But the norm of $\vec{f}$ is certainly not less than the norm of its orthogonal expansion (Bessel inequality). Thus for $z < 0$ we get zero. This proves Eq. (7) in the case when $\vec{r}$ and $\vec{r}\,'$ are on the opposite sides of the plane $z = 0$.

Another expression of the completeness is the validity of the canonical commutation relations between the field operators,

$$[\hat{D}_i(\vec{r}, t), \hat{B}_j(\vec{r}\,', t)] = i\hbar\epsilon_{ikj}\partial_k\delta^3(\vec{r} - \vec{r}\,'), \tag{13}$$

which follows directly from our calculations.

It is also possible to prove the completeness by deriving the Plancherel formulas,

$$\int d^3x\,|\vec{E}(\vec{r})|^2 n^2(\vec{r}) = \frac{2}{(2\pi)^3}\int_{k_3>0} d^3k \sum_s |u_1(\vec{k}, s)|^2 + \frac{2}{(2\pi)^3}\int_{K_3<0} d^3K \sum_s |v_1(\vec{K}, s)|^2 \tag{14}$$

or

$$\int d^3x\,|\vec{B}(\vec{r})|^2 = \frac{2}{(2\pi)^3}\int_{k_3>0} d^3k \sum_s |u_2(\vec{k}, s)|^2 + \frac{2}{(2\pi)^3}\int_{K_3<0} d^3K \sum_s |v_2(\vec{K}, s)|^2, \tag{15}$$

where

$$\begin{aligned} u_1(\vec{k}, s) &\equiv \int d^3x\, n^2(\vec{r})\vec{\mathfrak{E}}_L^*(\vec{k}, s, \vec{r})\cdot\vec{E}(\vec{r}),\\ v_1(\vec{K}, s) &\equiv \int d^3x\, n^2(\vec{r})\vec{\mathfrak{E}}_R^*(\vec{K}, s, \vec{r})\cdot\vec{E}(\vec{r}),\\ u_2(\vec{k}, s) &\equiv \int d^3x\, \vec{\mathfrak{B}}_L^*(\vec{k}, s, \vec{r})\cdot\vec{B}(\vec{r}),\\ v_2(\vec{K}, s) &\equiv \int d^3x\, \vec{\mathfrak{B}}_R^*(\vec{K}, s, \vec{r})\cdot\vec{B}(\vec{r}). \end{aligned} \tag{16}$$

In the calculations it is convenient to divide the field into two fields having supports in the two half-spaces and to separate the transverse electric and transverse magnetic parts. We can use again the contour-integral methods and then reduce the equation to the Plancherel formula for the ordinary Fourier transform.

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Pulsar Data and the Dispersion Relation for Light*

John M. Rawls
Yale University, New Haven, Connecticut 06520
(Received 15 September 1971)

Recent observations of pulsed γ radiation from NP0532 provide a very accurate check on the dispersion relation obeyed by light. The data are sufficient to rule out the mechanism proposed by Pavlopoulos for breaking Lorentz invariance.

Because pulsars[1] emit broad-bandwidth signals that arrive in narrow pulses, they provide an excellent test of the dispersion relation for light. The best known vehicle for such a test is the Crab pulsar NP0532, which has a small (less than 3 msec) pulsewidth and whose pulsed spectrum covers at least thirteen decades, from radio waves[2] to γ rays.[3] The low-frequency emission of NP0532 has previously been used to obtain a limit on the photon mass.[4] The resulting limit, $m_\gamma < 10^{-44}$ g, is inferior to others now available, e.g., that based upon considerations of the galactic magnetic field,[5] the principal difficulty being that the intervening plasma acts like an effective-mass term in the dispersion relation, thus masking the m_γ contribution. Some improvement over the above estimate will come with greater knowledge of the interstellar electron density; one can also attempt to systematically isolate the electronic contribution by looking at different angles to the galactic plane, but it seems unlikely that such a procedure will give a limit comparable to that of Ref. 5, i.e., $m_\gamma < 10^{-56}$ g.

However, the data in the region from optical to γ ray do provide the most powerful test to date of the dispersion relation for light at high frequencies. Assuming that there are no canceling effects, the spread due to dispersion cannot be larger than the observed width of the pulse; hence the difference of the velocities of light at frequencies ω_1 and ω_2 is bounded by

$$\frac{v(\omega_1) - v(\omega_2)}{c} \leqslant \frac{c\Delta t}{D},$$

where Δt is the pulsewidth in seconds, and D is the distance to the source. In the case of NP0532, $\Delta t \approx 2.6$ msec and $D \approx 2$ kiloparsecs, yielding

$$\frac{\Delta v}{c} \leqslant 1.3 \times 10^{-14} \tag{1}$$

for any two frequencies found in the same pulse. But broad-bandwidth experiments have nearly overlapped the region from optical to low-energy γ ray so one can conclude that the velocity of light varies by at most a few parts in 10^{14} over the entire range. The great stability of the period of pulsars actually enables one to make even more accurate statements by use of a synchronized apparatus. Warner and Nather[6] claim to have shown that $\Delta v/c < 5 \times 10^{-18}$ in the wavelength range 0.35–0.55 μ. These results are to be compared with direct measurements of the velocity of light, which are accurate to about a part in 10^6 in the radio-optical range and to a part in 10^3 for γ radiation.[7]

To obtain the corresponding limit on a dispersion relation, suppose that the propagation of light is governed by a noncovariant but rotationally invariant wave equation of the form

$$\left(-l_0^2 \vec{\nabla}^4 + \vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0, \tag{2}$$

where l_0 is a "fundamental" length.[8] Feinberg[4] has also studied the limits imposed by pulsar data on noncovariant terms in the wave equation but, at that date, the most energetic known pulsed spectra were soft x rays and the arguments given here are only made compelling by the higher-energy data. The dispersion relation appropriate to (2) has the form

$$v = \frac{d\omega}{dk} = (1 + 2l_0^2k^2)(1 + l_0^2k^2)^{-1/2}$$
$$\approx c(1 + \tfrac{3}{2}l_0^2k^2), \tag{3}$$

where it has been assumed that $l_0^2k^2 \ll 1$. Then

$$l_0^2 = \frac{2}{3}\frac{v(\omega_1) - v(\omega_2)}{c}(k_1^2 - k_2^2)^{-1}.$$

We apply this formula to the data of Ref. 3, in which pulsed information from NP0532 was observed in the energy range 0.1 to 1 MeV; use of the bound (1) yields

$$l_0 < 1.9 \times 10^{-18} \text{ cm}. \tag{4}$$

Such a fundamental length is about 5 orders of magnitude smaller than that needed in a number of attempts to manufacture cutoffs in relativistic theories. Such attempts have usually involved either a parameter of order of magnitude of the classical electron radius[9] or the pion Compton wavelength.[10] Equation (4) does not of course rule out the possibility that other, perhaps larger, lengths may enter the theory in other ways.[11] Claims of observing higher-energy γ rays from NP0532 have been made[12] but they are somewhat controversial[3,13]; use of those results at face value improves the bound on l_0 by about a factor of 50.

For comparison, we mention the limits on the nonlocality of lepton-photon interactions as obtained from fitting the predictions of quantum electrodynamics to experiment. A recent measurement of the anomalous magnetic moment of the muon[14] provides the bound $\Lambda > 7$ GeV/c on the cutoff defined through the usual modification of the muon propagator,

$$\frac{1}{q^2 - m^2} \to \frac{1}{q^2 - m^2} - \frac{1}{q^2 - m^2 - \Lambda^2}.$$

This is equivalent to a limit on the fundamental length: $l \equiv \hbar/\Lambda c < 3 \times 10^{-15}$ cm. Lee and Wick[15] have proposed that the photon propagator has a negative-metric contribution of the above type. With the largely arbitrary but popular choice of 40 GeV/c^2 for the heavy photon mass, one finds $l \approx 5 \times 10^{-16}$ cm.

The limit we have obtained for l_0 does make quantitative statements about at least one physical theory. If space-time is not described merely by a set of coordinate assignments and is in fact an oriented medium, where vectors are attached to each point, one is dealing with a Finsler space,[16] and the corresponding geometry is more complicated than Riemannian geometry. Mindlin and Tiersten[17] have found that wave propagation in simple oriented media is governed by Eq. (2). Pavlopoulos[18] has proposed that this equation does govern the propagation of light, and further proposes that the length l_0 is universal in character, in the hope of solving the well-known divergence problems of quantum field theory. One must then choose $l_0 \approx 10^{-13}$ cm, which is in contradiction with our limit, Eq. (4).

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