Completeness of Evanescent Waves

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Contour-integration methods are applied to prove the completeness of the solutions of Maxwell equations, when half of the space is filled with a dielectric medium. Our proof fills a gap in a recent article by Carniglia and Mandel on the quantization of evanescent waves. A simple derivation of the commutation relations between the creation and annihilation operators is also given.

In this note we add two comments on a recent paper by Carniglia and Mandel[1] (hereafter referred to as CM) on the quantization of the electromagnetic field in a space which is filled with a homogeneous dielectric of refractive index n to the left of the plane $z=0$, and is empty to the right of this plane.

First, we observe that for every linear conservative system the Heisenberg equations of motion

$$i\hbar \frac{\partial}{\partial t}\hat{O}(t)=[\hat{O}(t),\hat{H}] \tag{1}$$

impose rather stringent conditions on the commutation relations.

In particular, when applied to the case described in CM, these equations enable us to determine immediately the unknown functions $f(\vec{k},s)$ and $g(\vec{K},s)$ which appear in the commutation relations [CM equations (65)–(67)] for the creation and annihilation operators of elementary modes $\hat{u}^\dagger$, $\hat{v}^\dagger$ and $\hat{u}$, $\hat{v}$,

$$\begin{aligned}&[\hat{u}(\vec{k},s),\hat{u}^\dagger(\vec{k}',s')]=f(\vec{k},s)\delta_{ss'}\delta^3(\vec{k}-\vec{k}'),\\&[\hat{v}(\vec{K},s),\hat{v}^\dagger(\vec{K}',s')]=g(\vec{K},s)\delta_{ss'}\delta^3(\vec{K}-\vec{K}').\end{aligned} \tag{2}$$

All remaining commutators are assumed to vanish.

The modes created by $\hat{u}^\dagger$ and $\hat{v}^\dagger$ form triplets of incident, reflected, and transmitted waves and are labeled by the incident wave vector $\vec{k}$ or $\vec{K}$ (depending on whether the wave is incident from the left or from the right) and the polarization index s. The electric field in the modes is denoted by $\vec{\mathfrak{E}}_L(\vec{k},s,\vec{r})$ or $\vec{\mathfrak{E}}_R(\vec{K},s,\vec{r})$ and the magnetic field by $\vec{\mathfrak{B}}_L(\vec{k},s,\vec{r})$ or $\vec{\mathfrak{B}}_R(\vec{K},s,\vec{r})$. If the wave is incident from the left its transmitted part may be evanescent – that is, exponentially decreasing – in the right half-space.

The Hamiltonian and electric field operator expressed in terms of creation and annihilation operators have the form

$$\hat{H}=\sum_{s=1}^{2}\frac{1}{(2\pi)^3}\left(\int_{k_3>0}d^3k\,K\hat{u}^\dagger(\vec{k},s)\hat{u}(\vec{k},s)+\int_{K_3<0}d^3K\,K\hat{v}^\dagger(\vec{K},s)\hat{v}(\vec{K},s)\right), \tag{3}$$

$$\begin{aligned}\hat{\vec{E}}(\vec{r},t)=&\frac{1}{(2\pi)^3}\int_{k_3>0}d^3k\sum_{s=1}^{2}\left(\frac{K}{\epsilon_0}\right)^{1/2}[\hat{u}(\vec{k},s)\vec{\mathfrak{E}}_L(\vec{k},s,\vec{r})e^{-iKt}+\text{H.c.}]\\&+\frac{1}{(2\pi)^3}\int_{K_3<0}d^3K\sum_{s=1}^{2}\left(\frac{K}{\epsilon_0}\right)^{1/2}[\hat{v}(\vec{K},s)\vec{\mathfrak{E}}_R(\vec{K},s,\vec{r})e^{-iKt}+\text{H.c.}].\end{aligned} \tag{4}$$

Inserting both these expressions into Eq. (1) and making use of the linear independence of the functions $\vec{\mathfrak{E}}_L$ and $\vec{\mathfrak{E}}_R$, we obtain the final result

$$f(\vec{k},s)=(2\pi)^3\hbar=g(\vec{K},s). \tag{5}$$

Next we would like to supply the missing proof of the completeness of the modes. Carniglia and Mandel write on this subject: "We shall not here enter into the question of completeness of the set of modes with respect to solutions of the Helmholtz equation, which appears to be a difficult problem." In fact, their article contains a great part of the calculations needed to prove the completeness.

In the space of square-integrable vector functions with vanishing divergence (in the distribution sense) the unit kernel has the form

$$(\delta_{ij}-\Delta^{-1}\partial_i\partial_j)\delta^3(\vec{r}-\vec{r}\,'). \tag{6}$$

Let us consider the right half-space first. In this case, for the family of electric fields $\vec{\mathfrak{E}}_L$ and $\vec{\mathfrak{E}}_R$, the completeness condition reads

$$\frac{2}{(2\pi)^3}\int_{k_3>0} d^3k \sum_s \mathfrak{E}_{Li}(\vec{k},s,\vec{r})\mathfrak{E}^*_{Lj}(\vec{k},s,\vec{r}\,') + \frac{2}{(2\pi)^3}\int_{K_3<0} d^3K \sum_s \mathfrak{E}_{Ri}(\vec{K},s,\vec{r})\mathfrak{E}^*_{Rj}(\vec{K},s,\vec{r}\,') = (\delta_{ij}-\Delta^{-1}\partial_i\partial_j)\delta^3(\vec{r}-\vec{r}\,'). \tag{7}$$

An analogous formula holds for a complete family of magnetic fields. We have chosen the same normalization convention as CM.

We operate on both sides of Eq. (7) with the Laplacian (we also use the Helmholtz equation $[\Delta+K^2n^2(\vec{r})]\vec{\mathfrak{E}}(\vec{r})=0$),

$$\frac{2}{(2\pi)^3}\int_{k_3>0} d^3k \sum_s K^2\mathfrak{E}_{Li}(\vec{k},s,\vec{r})\mathfrak{E}^*_{Lj}(\vec{k},s,\vec{r}\,') + \frac{2}{(2\pi)^3}\int_{K_3<0} d^3K \sum_s K^2\mathfrak{E}_{Ri}(\vec{K},s,\vec{r})\mathfrak{E}^*_{Rj}(\vec{K},s,\vec{r}\,') = (\partial_i\partial_j-\delta_{ij}\Delta)\delta^3(\vec{r}-\vec{r}\,'). \tag{8}$$

In order to evaluate the left-hand side (LHS) of this equation, we follow with appropriate modifications the procedure used in CM to find the commutator $[\hat{E}_i(\vec{r},t),\hat{E}_j(\vec{r}\,',t')]$ [CM equations (76)–(84)]. The result is

$$\begin{aligned}\mathrm{LHS} = {} & \left(\delta_{ij}\frac{\partial^2}{\partial t\partial t'}-\frac{\partial^2}{\partial x_i\partial x_j'}\right)\frac{\partial}{\partial t}D(\vec{r}-\vec{r}\,',t-t')\bigg|_{t=t'} + \frac{2}{(2\pi)^3}\int d^3K\,\frac{K^2}{2}\left[\left(\frac{K_3-k_3}{K_3+k_3}\right)\epsilon_i\epsilon_j+\left(\frac{n^2K_3-k_3}{n^2K_3+k_3}\right)(\vec{c}\times\vec{\epsilon})_i(\vec{c}^{\,(R)}\times\vec{\epsilon})_j\right] \\ & \times\exp\{i[K_1(x-x')+K_2(y-y')+K_3(z+z')]\} \\ & + \frac{2}{(2\pi)^3}\int d^2K\int_0^{[(1-1/n^2)(K_1^2+K_2^2)]^{1/2}} d|K_3|\,K^2\left(\frac{2k_3|K_3|}{k_3^2+|K_3|^2}\epsilon_i\epsilon_j+\frac{2k_3n^2|K_3|}{k_3^2+n^4|K_3|^2}(\vec{c}\times\vec{\epsilon})_i(\vec{c}^{\,*}\times\vec{\epsilon})_j\right)\exp[-|K_3|(z+z')] \\ & \times\exp\{i[K_1(x-x')+K_2(y-y')]\}.\end{aligned} \tag{9}$$

[There are two misprints in CM equation (84). The first and the third terms have wrong signs.]

The two integrals in Eq. (9) can be combined to form a contour integral in the complex K_3 plane,

$$\frac{1}{(2\pi)^3}\int d^2K\int_C dK_3\,K^2\left[\left(\frac{K_3-k_3}{K_3+k_3}\right)\epsilon_i\epsilon_j+\left(\frac{n^2K_3-k_3}{n^2K_3+k_3}\right)(\vec{c}\times\vec{\epsilon})_i\,(\vec{c}^{\,(R)}\times\vec{\epsilon})_j\right]\exp\{i[K_1(x-x')+K_2(y-y')+K_3(z+z')]\}. \tag{10}$$

The contour C (Fig. 1) consists of two halves of the real axis, part of the imaginary axis up to the branch point of the function $k_3=[K_3^2n^2+(K_1^2+K_2^2)\times(n^2-1)]^{1/2}$, and a large semicircle. The two contributions from the integral along the imaginary axis have opposite signs of k_3, in agreement with the convention that on the real axis $\mathrm{sgn}(k_3)=\mathrm{sgn}(K_3)$. Since $z+z'>0$, the integral along the large semicircle vanishes in the limit. There are no singularities inside the contour C and thus the integral along C is zero. Using known properties of the Jordan-Pauli commutator function $D(\vec{r},t)$, we obtain finally

$$\mathrm{LHS}=(\partial_i\partial_j-\delta_{ij}\Delta)\delta^3(\vec{r}-\vec{r}\,'). \quad \text{Q.E.D.} \tag{11}$$

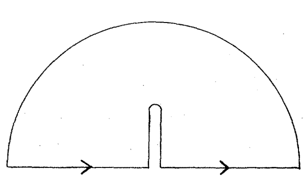

FIG. 1. Integration contour in the K_3 plane.

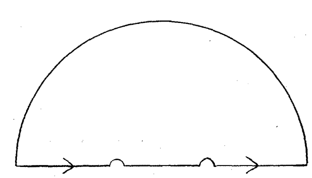

FIG. 2. Integration contour in the k_3 plane.

We only sketch the proof for the left half-space, which proceeds in a very similar fashion. Now all the integrals over $\vec{K}$ should be transformed into the integrals over $\vec{k}$ and the commutator function D_n in the homogeneous dielectric medium,

$$D_n(\vec{r}, t) = \frac{1}{(2\pi)^3}\int d^3k \frac{\sin Kt}{K} e^{i\vec{k}\cdot\vec{r}}, \qquad (12)$$

replaces the vacuum function D.

The integration contour runs above the real axis, thus avoiding the branch points of the function

$$K_3 = [k_3{}^2/n^2 - (k_1{}^2 + k_2{}^2)(1 - 1/n^2)]^{1/2}.$$

The integral along the large semicircle (Fig. 2) vanishes in the limit. Since there are no singularities inside the contour, the integral vanishes. The result is similar to Eq. (7) but has an additional factor of n^2 on the left-hand side.

Now let us integrate the left-hand side of Eq. (7) over $\vec{r}'$ with a square-integrable divergenceless function $\vec{f}$ having support in the right half-space. When $z > 0$ we get $\vec{f}(\vec{r})$ – this has already been proved. But the norm of $\vec{f}$ is certainly not less than the norm of its orthogonal expansion (Bessel inequality). Thus for $z < 0$ we get zero. This proves Eq. (7) in the case when $\vec{r}$ and $\vec{r}'$ are on the opposite sides of the plane $z = 0$.

Another expression of the completeness is the validity of the canonical commutation relations between the field operators,

$$[\hat{D}_i(\vec{r}, t), \hat{B}_j(\vec{r}', t)] = i\hbar\epsilon_{ikj}\partial_k\delta^3(\vec{r} - \vec{r}'), \qquad (13)$$

which follows directly from our calculations.

It is also possible to prove the completeness by deriving the Plancherel formulas,

$$\int d^3x\, |\vec{E}(\vec{r})|^2 n^2(\vec{r}) = \frac{2}{(2\pi)^3}\int_{k_3>0} d^3k \sum_s |u_1(\vec{k}, s)|^2 + \frac{2}{(2\pi)^3}\int_{K_3<0} d^3K \sum_s |v_1(\vec{K}, s)|^2 \qquad (14)$$

or

$$\int d^3x\, |\vec{B}(\vec{r})|^2 = \frac{2}{(2\pi)^3}\int_{k_3>0} d^3k \sum_s |u_2(\vec{k}, s)|^2 + \frac{2}{(2\pi)^3}\int_{K_3<0} d^3K \sum_s |v_2(\vec{K}, s)|^2, \qquad (15)$$

where

$$\begin{aligned} u_1(\vec{k}, s) &\equiv \int d^3x\, n^2(\vec{r}) \vec{\mathfrak{E}}_L^*(\vec{k}, s, \vec{r}) \cdot \vec{E}(\vec{r}), \\ v_1(\vec{K}, s) &\equiv \int d^3x\, n^2(\vec{r}) \vec{\mathfrak{E}}_R^*(\vec{K}, s, \vec{r}) \cdot \vec{E}(\vec{r}), \\ u_2(\vec{k}, s) &\equiv \int d^3x\, \vec{\mathfrak{B}}_L^*(\vec{k}, s, \vec{r}) \cdot \vec{B}(\vec{r}), \\ v_2(\vec{K}, s) &\equiv \int d^3x\, \vec{\mathfrak{B}}_R^*(\vec{K}, s, \vec{r}) \cdot \vec{B}(\vec{r}). \end{aligned} \qquad (16)$$

In the calculations it is convenient to divide the field into two fields having supports in the two half-spaces and to separate the transverse electric and transverse magnetic parts. We can use again the contour-integral methods and then reduce the equation to the Plancherel formula for the ordinary Fourier transform.

[1] C. K. Carniglia and L. Mandel, Phys. Rev. D 3, 280 (1971).

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Pulsar Data and the Dispersion Relation for Light*

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Recent observations of pulsed γ radiation from NP0532 provide a very accurate check on the dispersion relation obeyed by light. The data are sufficient to rule out the mechanism proposed by Pavlopoulos for breaking Lorentz invariance.

Because pulsars[1] emit broad-bandwidth signals that arrive in narrow pulses, they provide an excellent test of the dispersion relation for light. The best known vehicle for such a test is the Crab pulsar NP0532, which has a small (less than 3 msec) pulsewidth and whose pulsed spectrum covers at least thirteen decades, from radio waves[2] to γ rays.[3] The low-frequency emission of NP0532