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Analyticity in the Forward Tube, Complex Scaling, and New Sum Rules for Virtual Compton Amplitudes*

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The forward virtual Compton amplitude is studied as an analytic function of two complex variables. A method is developed for deriving sum rules that follow from analyticity in two variables where the domain of analyticity is the forward tube. Using this method, new sum rules are derived. These sum rules follow from analyticity in the tube and "complex" scaling. They relate a double integral over the inelastic form factor $W_2(\nu, q^2)$ to an integral over the scaling function $F_2(\omega)$. The sum rules provide a tool for directly testing whether scaling occurs when $|q^2| \to \infty$ in complex directions inside the tube domain, and whether this scaling starts at relatively low values of $|q^2|$ as in the real case. The possibility of finding more restrictive sum rules by enlarging the domain of analyticity is also briefly explored and a mathematical but unphysical example is given.

I. INTRODUCTION

In this paper we shall show how one can derive sum rules for the virtual Compton amplitude that follow from analyticity in *two* complex variables. Our main result is a new sum rule that relates a double integral of the inelastic structure function $W_2(\nu, q^2)$ to an integral over the scaling function $F_2(\omega)$. The only input used to obtain this sum rule is analyticity in the forward tube, and complex scaling.

For photons with four-momentum q we use as our two variables $q_0 \equiv \zeta$ and $|\vec{q}| \equiv \eta$. The forward virtual Compton amplitude is then analytic in ζ and η regular in the forward tube domain given by $\text{Im}\,\zeta > |\text{Im}\eta|$. One important property of the tube is that it contains points at which both ζ and η are complex but such that $q^2 \equiv \zeta^2 - \eta^2$ is real and spacelike. It also of course contains closed contours along which q^2 is real and negative but ζ and η are complex. For any given fixed spacelike value of q^2 at least one of the two Compton amplitudes, T_2, satisfies an unsubtracted dispersion relation in ν at fixed q^2; $\nu = q \cdot p$. Thus for any complex q_0, and spacelike q^2, $T_2(\nu, q^2)$ is known from the experimental measurement of $W_2(\nu', q^2)$, and the elastic form factors, and the use of the dispersion relation.

The main question now is: Can we find a contour in the tube along which q^2 is spacelike, so that integrating $T_2(\zeta, \eta)$ along the contour we get a useful sum rule by substituting for T_2 at each point along the contour the value obtained from the dispersion relation? We stress here the fact that given an analytic function of two variables, $f(z_1, z_2)$, regular in some domain, the integral of f over some closed curve C lying completely inside the region of analyticity, $\int_C f(z_1(\tau), z_2(\tau))d\tau$, is *not* necessarily zero. It is of course true that if we fix z_2 (or z_1) then over C' on which z_2 is fixed $\int_{C'} f(z_1, z_2)dz_1 \equiv 0$, if C' is inside the analyticity domain. It is, however, still possible in most cases to find "special" contours such that $\int_C f(z_1(\tau), z_2(\tau))d\tau = 0$.

In Sec. II, we first show how these "special" contours can be found. Then we consider the following problem: Find a "special" closed contour in the four-dimensional tube domain such that along this contour q^2 is spacelike and preferrably variable. For such a contour $\int_C T_2(\zeta(\tau), \eta(\tau))d\tau = 0$, with τ being a parameter that defines the contour. For each value of τ along our integration path, $q^2 = \zeta^2(\tau) - \eta^2(\tau)$ is negative and therefore the value of T_2 at that point is determined by W_2 through the dispersion relation. If one can find such a special contour the above integral will give a powerful sum rule by substituting for $T_2(\zeta(\tau), \eta(\tau))$ the dispersion representation for it.

The first result of Sec. II is to show that, with our analyticity domain limited to the forward tube, the only "special" contours we can find for which q^2 is spacelike along the contour have the feature that $q^2(\tau)$ is also constant along C. These contours lead to trivial identities and no sum rules since by interchanging integrations over τ and ν' one shows they hold trivially for any W_2.

Next we try to construct "special" contours along which $q^2(\tau)$ is restricted to be real and can be both spacelike and timelike. In this case we find many solutions. Each gives us a sum rule for $T_2(\nu, q^2)$. However, these sum rules involve both timelike and spacelike q^2. They are not very useful since $T_2(\nu, q^2)$ for timelike q^2 is not accessible to reasonable experiments.

Up to this point our only input is analyticity in the forward tube which follows directly from the fact that our amplitudes are Fourier transforms of a retarded function. Very little physics has been included and it is not surprising that the results are just methodological.

In Sec. III, we include more physical input and get some useful sum rules. The new physical input we assume is the concept of complex scaling. Namely, we assume that $\nu T_2(\nu, q^2)$ also scales when $|\nu| \to \infty$, $|q^2| \to \infty$ but ν/q^2 remains fixed. We stress that we only make such an assumption for paths that lie completely within the domain of analyticity, the forward tube. With this assumption, which is valid in many models that scale, $\nu T_2(\nu, q^2)$ is also known for some large values of complex ν and complex q^2, and as we shall show, given by the same function of $\omega = 2\nu/(-q^2)$ as one obtains when taking the usual real Bjorken limit[1] $-q^2 \to \infty$, ω fixed, for νT_2.

The problem now is to choose a "special" contour which is made of two parts. Along the first part q^2 is spacelike and variable and goes through the interesting resonance region for some values of the parameter τ. Along the second part of the contour both ν and q^2 are in the complex-scaling region. We find such a special contour and it gives us the sum rule given in Eq. (3.13). Since the analyticity input we use requires no testing at present energies, the main value of the sum rule (3.13) is to test not only complex scaling, which is probably correct if real scaling is true, but also to answer the important physical question of whether complex scaling occurs at relatively low values of $|q^2|$ as it does in the real case. If it does not, then the precocious nature of real Bjorken scaling could well be due to an accidential cancellation of some phases and not more fundamental reasons.

In Sec. III we also write another sum rule which in addition has a region of the contour along which T_2 is in the Regge region, ν large and q^2 small but complex. We hope that this sum rule, if complex scaling holds, will also differentiate between the different Regge fits of the data for νW_2.

Finally, in Sec. IV we go back to the mathematical question posed in Sec. II and ask if it is possible to find "special" contours along which q^2 is spacelike and variable if the domain of analyticity of $T_2(\zeta, \eta)$ is larger than just the tube. We know that the actual domain is larger but we do not (at least this author does not) as yet know what is the maximal domain. But we show that at least from the mathematical side the question is open. Namely, we give an example of a domain D' larger than the tube, for which one can find "special" contours with q^2 spacelike along the contour. However, unfortunately, D' has some very unphysical features. It looks quite doubtful that, at least without a complete study of the analyticity domain of the full four-point function, one could make much progress towards a completely spacelike sum rule. Nevertheless, enlarging the tube domain might be quite helpful in improving the sum rules of Sec. III or deriving similar physical sum rules.

A detailed numerical evaluation of the sum rules (3.13) and (3.15) is being carried out and will be published separately.

II. ANALYTICITY IN TWO VARIABLES AND SUM RULES FOR COMPTON AMPLITUDES

In this section we shall study the analyticity of the forward amplitude for the scattering of virtual photons on protons as a function of two complex variables. We show how in principle this analyticity could lead to potentially useful sum rules.

We write the spin-averaged amplitude for virtual Compton scattering as

$$T_{\mu\nu}(q, p) = i\int d^4x\, e^{iq\cdot x}\theta(x_0)\langle p|[J_\mu(x), J_\nu(0)]|p\rangle . \tag{2.1}$$

Here we take only the connected part and the absence of operator Schwinger terms is assumed. Actually all one needs to assume is that the extra terms in (2.1) due to operator Schwinger terms are polynomials in q_0 and q^2 and automatically have the tube analyticity to be discussed below. We use the retarded product to define $T_{\mu\nu}$ since it is the one that gives an analytic function for $\mathrm{Im}q_0 > 0$. The time-ordered and retarded functions are of course identical for $q_0 > 0$, q_0 real.

Following the usual notation we define the two invariant amplitudes T_1 and T_2 as

$$T_{\mu\nu}(q, p) = (q_\mu q_\nu - g_{\mu\nu}q^2)\frac{T_1}{q^2} + \frac{1}{m^2}\left(p_\mu - q_\mu \frac{q\cdot p}{q^2}\right)\left(p_\nu - q_\nu \frac{q\cdot p}{q^2}\right) T_2 , \tag{2.2}$$

where the well-known inelastic structure functions are given by

$$W_1(\nu, q^2) = \frac{\mathrm{Im}T_1}{2\pi} , \qquad W_2(\nu, q^2) = \frac{\mathrm{Im}T_2}{2\pi} . \tag{2.3}$$

The variable ν is defined by $\nu = q\cdot p$.

There are two properties of T_2 that we would like to recall here. First, we note the fact that as $q^2 \to 0$,

$$W_2(\nu, q^2) \cong O(q^2). \tag{2.4}$$

Second, we assume that for fixed spacelike q^2, $T_2(\nu, q^2)$ satisfies an unsubtracted dispersion relation in ν of the form

$$T_2(\nu, q^2) = \frac{f(q^2)}{\frac{1}{4}q^4 - \nu^2} + 4\int_{\nu_t}^{\infty} d\nu' \, \frac{\nu' W_2(\nu', q^2)}{\nu'^2 - \nu^2}, \qquad (2.5)$$

with

$$\nu_t = \tfrac{1}{2}\mu^2 + \mu m - \tfrac{1}{2}q^2, \quad q^2 < 0. \qquad (2.6)$$

The residue of the Born term, $f(q^2)$, is a known linear combination of $(G_E^p)^2$ and $(G_M^p)^2$.

From (2.5) we see that given W_2 for physical ν' and spacelike q^2 and the elastic form factors, then $T_2(\nu, q^2)$ is known for any complex ν and spacelike q^2. *Thus given any fixed value of* $q^2 < 0$ *we can by measuring* W_2 *for* $\nu > \nu_t(q^2)$ *effectively determine through (2.5) the value of* $T_2(\nu, q^2)$ *for that value of* q^2 *and any complex value of* ν.

Finally, we note that as in (2.4) as $q^2 \to 0$, we have

$$T_2(\nu, q^2) = O(q^2). \qquad (2.7)$$

For the discussion below it turns out to be simpler to use two new variables ζ and η defined by

$$\zeta = q_0, \qquad \vec{q} = \eta\vec{e}, \quad \vec{e}^{\,2} = 1. \qquad (2.8)$$

We choose our frame such that $\vec{e}\cdot\vec{p} = 0$. Actually for most of this paper it suffices to use the laboratory frame, $\vec{p} = 0$. Our old variables are given by

$$\nu = \zeta p_0, \qquad q^2 = \zeta^2 - \eta^2. \qquad (2.9)$$

We consider $T_{\mu\nu}(\zeta, \eta)$ as a function of the two complex variables ζ and η, given by

$$T_{\mu\nu}(\zeta, \eta) = i\int d^4x \, e^{i(\zeta x_0 - \eta\vec{e}\cdot\vec{x})} \theta(x_0) \langle p | [J_\mu(x), J_\nu(0)] | p\rangle. \qquad (2.10)$$

Using the locality of the currents, it follows immediately that every component of $T_{\mu\nu}$ is an analytic function in the two variables ζ and η regular in the domain D,[2]

$$D: \ \mathrm{Im}\zeta > |\mathrm{Im}\eta|. \qquad (2.11)$$

This domain D is just the forward tube when one translates it back to four-vector form. For if we write $q = q_r + iq_i$, then (2.1) defines a function analytic in the forward tube; $q_i^2 > 0$, $q_i^0 > 0$. This is identical with (2.11) since $q_i^2 = (\mathrm{Im}\zeta)^2 - (\mathrm{Im}\eta)^2$.

The continuation of $T_{\mu\nu}(\zeta, \eta)$ to complex values of ζ and η clearly leads to the analytic function defined by the physical amplitude since by definition it agrees with the physical amplitude on the two-dimensional surface, $\mathrm{Im}\zeta = 0$ and $\mathrm{Im}\eta = 0$. This two-dimensional surface is a nonanalytic surface.

It is easy to show that the analyticity of all components of $T_{\mu\nu}$ in ζ and η implies that both $T_1(\zeta, \eta)$ and $T_2(\zeta, \eta)$ are also analytic in the domain D.

For reasons that will soon be evident below, we limit our discussion to the invariant amplitude $T_2(\zeta, \eta)$.

It is trivial to show that there exist points inside the four-dimensional domain D at which both ζ and η are complex, but such that $q^2 = \zeta^2 - \eta^2$ is real and negative (spacelike). At such points we have

$$\zeta_r \zeta_i = \eta_r \eta_i. \qquad (2.12)$$

However, since we are inside D, $\zeta_i > |\eta_i|$, and therefore, $|\eta_r| > |\zeta_r|$. It then follows that $q^2 = (\zeta_r^2 - \eta_r^2) - (\zeta_i^2 - \eta_i^2)$ is negative definite at such points.

At these points q^2 is real and spacelike and ν is complex, and therefore $T_2(\zeta_1, \eta_1)$ is completely determined at each of these points by a measurement of $W_2(\nu', q^2 = \zeta_1^2 - \eta_1^2)$ for all $\nu' > \nu_t$ and the use of (2.5). Thus the complex spacelike points, though they lie inside the forward tube, are still accessible to experiment because of the unsubtracted nature of (2.5).

Suppose now we choose a closed contour inside D defined by some parameter τ,

$$\zeta = \zeta(\tau), \quad \eta = \eta(\tau), \qquad (2.13)$$

such that along this contour $\zeta^2 - \eta^2$ is real and

$$\zeta^2(\tau) - \eta^2(\tau) < 0. \qquad (2.14)$$

If we can choose the contour such that

$$\int_C T_2(\zeta(\tau), \eta(\tau)) d\tau = 0, \qquad (2.15)$$

then we would get an interesting sum rule from (2.15) by substituting (2.5) into (2.15). The sum rule will provide a restriction on W_2 if along the contour $\zeta^2(\tau) - \eta^2(\tau)$ is not only negative but also not constant.

The immediate problem one faces, however, is that given an analytic function of two complex variables, $f(z_1, z_2)$, regular in a four-dimensional domain D, it is *not* true that for any closed contour chosen to lie inside D, $\int_C f(z_1(\tau), z_2(\tau)) d\tau = 0$. However, it is true obviously that if we take a contour along which z_2 is fixed, then $\int_C f(z_1, z_2) dz_1 = 0$ and similarly if we take a contour along which z_1 is fixed and integrate over z_2.

Given a contour C lying completely in D, one way to determine whether (2.15) is true is to find a pair of transformations $\omega_1 = \omega_1(\zeta, \eta)$ and $\omega_2 = \omega_2(\zeta, \eta)$ such that $\omega_2(\zeta, \eta)$ remains constant as ζ and η vary along C, and such that the inverse transformations $\zeta = \zeta(\omega_1, \omega_2)$ and $\eta = \eta(\omega_1, \omega_2)$ are analytic in the domain D' which is the image of D under the mapping $(\zeta, \eta) \to (\omega_1, \omega_2)$.

A simpler way to proceed is as follows. Suppose we can construct functions $\zeta(\tau)$ and $\eta(\tau)$ such that:

(a) $\zeta(\tau)$ and $\eta(\tau)$ are analytic in τ in some domain G bounded by a simple contour C;

(b) $\zeta^2(\tau)-\eta^2(\tau)<0$ for τ on the boundary C;

(c) for all $\tau\in G$,

$$\text{Im}\zeta(\tau)>|\text{Im}\eta(\tau)|\,.$$

We now consider the function $\phi(\tau)$ defined by

$$\phi(\tau)\equiv T_2(\zeta(\tau),\eta(\tau))\,. \tag{2.16}$$

By construction $\phi(\tau)$ is analytic in τ regular for all $\tau\in G$. Hence we can write

$$\oint_C \phi(\tau)d\tau\equiv\oint_C d\tau\, T_2(\zeta(\tau),\eta(\tau))=0\,. \tag{2.17}$$

Along C, $\zeta^2(\tau)-\eta^2(\tau)$ is negative and real by construction and hence we can substitute (2.5) into (2.17) and get

$$\oint_C d\tau\frac{f(\zeta^2(\tau)-\eta^2(\tau))}{\frac{1}{4}[\zeta^2(\tau)-\eta^2(\tau)]-\zeta^2(\tau)m^2} =-4\oint_C d\tau\int_{\nu_t(\tau)}^{\infty}\frac{d\nu'\nu' W_2(\nu',q^2=\zeta^2(\tau)-\eta^2(\tau))}{\nu'^2-\zeta^2(\tau)m^2}\,. \tag{2.18}$$

This sum rule is a trivial identity when $\zeta^2(\tau)-\eta^2(\tau)$ is constant along C. For then both sides of (2.18) would be zero by interchanging integrations. It would be a very powerful sum rule if $\zeta^2(\tau)-\eta^2(\tau)$ varies significantly along C. However, unfortunately, as we shall see below, without enlarging the region of analyticity D given in (2.11), it is impossible to obtain (2.18) except for the trivial case of constant q^2.

We note here that condition (b) on ζ and η is quite restrictive. If the domain G is finite then since $\zeta^2(x)-\eta^2(x)$ is real on the simple closed curve C bounding G, we have $\text{Im}(\zeta^2-\eta^2)=0$ for almost all $\tau\in C$. This a harmonic function in G and hence $\text{Im}(\zeta^2-\eta^2)=0$ for all $\tau\in G$ unless $\zeta^2-\eta^2$ becomes unbounded on some isolated points on the boundary. If it does not then our only solution has the form $\zeta^2(\tau)=\eta^2(\tau)\pm(\text{real const})$. Thus in the interesting cases either $\zeta(\tau)$ or $\eta(\tau)$ must become unbounded on at least one point on the boundary. This might require us to introduce convergence factors into (2.17) and (2.18) as we shall see below.

The problem can be simplified substantially by transforming the hypothetical region G into the upper half-plane. Given any bounded G and simple closed curve C one can always find a conformal transformation,

$$\tau=\tau(z), \tag{2.19}$$

which maps G onto the upper half z plane and C onto the real axis. This way we can concentrate on functions $\zeta(z)$ and $\eta(z)$ which are analytic in the upper half-plane.

We can here state our two main problems:

Problem A. Find two functions $\zeta(z)$ and $\eta(z)$ such that

(i) $\zeta(z)$ and $\eta(z)$ are analytic for $\text{Im}z>0$;

(ii) $\zeta^2(x)-\eta^2(x)\leq 0$, i.e., real and nonpositive on real axis;

(iii) $\text{Im}\zeta(z)>|\text{Im}\eta(z)|$, for all $\text{Im}z>0$.

Conditions (i) and (iii) imply that $\zeta(z)$ is a Herglotz function and hence bounded by $C|z|$ as $|z|\to\infty$ along complex directions. It also follows from (iii) and (i) that $\eta(z)$ is bounded by const $\times|z|$ for large $|z|$ although $\eta(z)$ does not necessarily have to be a Herglotz function.

Problem B. Find two functions $\zeta(z)$ and $\eta(z)$ such that

(i) $\zeta(z)$ and $\eta(z)$ are analytic for $\text{Im}z>0$;

(ii) $\zeta^2(x)-\eta^2(x)$ is real on the real axis;

(iii) $\text{Im}\zeta(z)>|\text{Im}\eta(z)|$, for all $\text{Im}z>0$.

In problem B we allow q^2 on the boundary to be both spacelike and timelike. Although this makes part of our sum rule not measurable, we shall see later that by including some physics, in addition to pure analyticity, we can get some interesting and useful sum rules from case B.

Problem A has an immediate, but for our purposes useless, solution. One takes

$$\zeta(z)=[\eta^2(z)-m_0^{\,2}]^{1/2},\quad m_0 \text{ real.} \tag{2.20}$$

Setting $\eta(z)$ to be a Herglotz function guarantees the absence of branch points in $\zeta(z)$ for $\text{Im}z>0$. It is trivial to show that

$$\text{Im}[\eta^2(z)-m_0^{\,2}]^{1/2}>|\text{Im}\eta(z)|,\quad \text{Im}z>0. \tag{2.21}$$

Note that for any complex number E such that $\text{Im}E>0$, we have $\text{Im}(E^2-m_0^{\,2})^{1/2}>\text{Im}E$, where m_0 is real. For $\text{Im}E=0$, $\text{Im}(E^2-m_0^{\,2})^{1/2}\geq\text{Im}E$. This solution has the feature that q^2 is a constant independent of z, $q^2=\zeta^2-\eta^2=-m_0^{\,2}$.

The inequality (2.21) is satisfied with some room to spare. One would thus think that there must be other solutions of problem A with at least a slowly varying $q^2(z)$. It is remarkable that one can show that the only solutions of problem A are of the form (2.20). In Appendix A we give a proof of this fact which is due to Dyson.[3]

Unfortunately, solutions of the type given by (2.20) with constant spacelike q^2 are useless for our purposes. For example one can choose $\eta(z)$ such that $\eta(\infty)$ is a constant. Then $T_2(\zeta(z),\eta(z))$ will tend to a constant as $|z|\to\infty$. The value of that constant will be $T_2(\zeta(\infty),\eta(\infty))$. One can write

$$\int_{-\infty}^{+\infty}\frac{dx}{(x+i)^2}\,T_2(\zeta(x),\eta(x))=0. \tag{2.22}$$

The factor $(z+i)^2$ is introduced to make the contribution from the large semicircle in the upper half z plane vanish. In the integration above, $\zeta^2(x)-\eta^2(x)=-m_0^2$. If we substitute (2.5) into (2.22), we get an identity for any W_2, G_E, and G_M since we integrate over fixed q^2. All one has to do is interchange the order of integration over x with that over ν'.

Problem B, on the other hand, has an infinite number of solutions. We recall that the difference between the two cases is just that in case B we allow $q^2(x)$ to be positive and negative on the real axis, while in case A $q^2(x)<0$ on the real axis.

We write down a class of solutions of problem B as follows:

$$\zeta(z)=m_0\left(h(z)-\frac{a}{4(z+b)h(z)}\right),\qquad \eta(z)=m_0\left(h(z)+\frac{a}{4(z+b)h(z)}\right), \tag{2.23}$$

where a and b are real *positive* constants, and m_0 is a mass that sets the scale, all other quantities being dimensionless. The function $h(z)$ is given by

$$h(z)=c+\int_0^\infty \frac{\rho(x)}{x-z}\,dx,\quad \rho(x)\geq 0,\quad c>0. \tag{2.24}$$

The function $h(z)$ is thus chosen to be analytic and of Herglotz type. Furthermore, we set $\rho(x)>0$ except at $x=0$ and $x=\infty$. This way from (2.24) it is clear that $h(z)$ has no zeros not only for $\mathrm{Im}z>0$ but on the real axis as well. As $|z|\to\infty$, we have $h(z)\to c$.

Let us now check that (2.23) satisfy the conditions of problem B. The analyticity for $\mathrm{Im}z>0$ is evident. Both $\zeta(z)$ and $\eta(z)$ have a pole at $z=-b$ on the real axis. As for condition (ii), we have

$$q^2(x)=\zeta^2(x)-\eta^2(x)=-m_0^2\left(\frac{a}{x+b}\right). \tag{2.25}$$

Thus on the boundary q^2 is spacelike for all $x>-b$ and timelike for $x<-b$, and q^2 has a pole at $x=-b$. Finally, we check that for all $\mathrm{Im}z>0$,

$$\mathrm{Im}\zeta(z)>|\mathrm{Im}\eta(z)|. \tag{2.26}$$

From (2.23) we write

$$\zeta(z)=m_0[h(z)+G(z)],\qquad \eta(z)=m_0[h(z)-G(z)], \tag{2.27}$$

with

$$G(z)=-\frac{a}{4(z+b)h(z)}. \tag{2.28}$$

We claim that G is also a Herglotz function. We have

$$\mathrm{Im}G(z)=\frac{ab\,\mathrm{Im}[h(z)]+a\,\mathrm{Im}[zh(z)]}{4|z+b|^2|h(z)|^2}. \tag{2.29}$$

From the representation (2.24) of $h(z)$ we see that

$$\mathrm{Im}[zh(z)]=c\,\mathrm{Im}z+\mathrm{Im}z\int_0^\infty\frac{x\rho(x)dx}{|x-z|^2}, \tag{2.30}$$

which is positive for $\mathrm{Im}z>0$. From (2.30) we see that the lower limit of the integral in (2.24) should be zero or positive to ensure positivity in (2.30). Now, $\mathrm{Im}\zeta(z)=m_0[\mathrm{Im}h(z)+\mathrm{Im}G(z)]$ and $\mathrm{Im}\eta(z)=m_0[\mathrm{Im}h(z)-\mathrm{Im}G(z)]$ where both $\mathrm{Im}h$ and $\mathrm{Im}G$ are positive. Hence, inequality (2.26) is satisfied.

Before we can use the functions in (2.23) to obtain a sum rule we have to discuss the behavior of $T_2(\zeta(z),\eta(z))$ as $|z|\to\infty$ and at the singular point $z=-b$. First, we take the $|z|\to\infty$ limit. In this case it follows from (2.23) and (2.24) that

$$\zeta(z)\to cm_0,\qquad \eta(z)\to cm_0,\qquad q^2(z)\to 0. \tag{2.31}$$

In addition we see that for large $|z|$, $q^2(z)=O(1/z)$. As $z\to -b$, we have

$$\zeta(z)\to\infty,\qquad \eta(z)\to\infty,\qquad q^2(z)=\frac{-m_0^2a}{z+b}\to\infty. \tag{2.32}$$

However, the Bjorken variable $\omega=-2\nu/q^2$ tends to a finite limit,

$$\lim_{z\to -b}\omega(z)=\lim_{z\to -b}\frac{-2\nu(z)}{q^2(z)}=\frac{-m}{2m_0}\frac{1}{h(-b)}, \tag{2.33}$$

where by definition $h(-b)$ is a real positive number. Taking the limit $z\to -b$ corresponds to taking the Bjorken scaling limit. Hence if the scaling assumption is correct,

$$\lim_{z\to -b}[\nu(z)T_2(\zeta(z),\eta(z))]=\text{const}, \tag{2.34}$$

where $\zeta(z)$ and $\eta(z)$ are given by (2.23). But since $\nu(z)\to\infty$ as $z\to -b$, we have

$$\lim_{z\to -b}T_2(\zeta(z),\eta(z))=0. \tag{2.35}$$

We stress here that the scaling assumption (2.34) is not necessary to get a sum rule. We shall see below that as long as $T_2(\zeta(z),\eta(z))$ does not blow up worse than a polynomial as $z\to -b$, we can still get a modified sum rule.

The function $T_2(\zeta(z),\eta(z))$ with ζ and η given by (2.23) is now an analytic function of z for all $\mathrm{Im}z>0$; it vanishes as $O(1/z)$ as $|z|\to\infty$, and even though ζ and η are singular at $z=-b$, $T_2(\zeta(z),\eta(z))$ vanishes at that point. By taking a contour just above the real z axis we get the sum rule

$$\int_{-\infty}^{+\infty} \frac{dz}{z+i} T_2(\xi(z), \eta(z)) = 0. \qquad (2.36)$$

The factor $(z+i)^{-1}$ ensures the fact that the contribution from the large semicircle vanishes. If (2.35) is not correct, i.e., the scaling limit does not hold, and T_2 is divergent as $z \to -b$, then we can modify (2.34) and write

$$\int_{-\infty}^{+\infty} \left(\frac{z+b}{z+ib}\right)^N \frac{dz}{z+i} T_2(\xi(z), \eta(z)) = 0, \qquad (2.37)$$

where N is determined by the degree of divergence of $T_2(\xi(z), \eta(z))$ as $z \to -b$. Again we assume the absence of an essential singularity at $z = -b$.

In order to get a sum rule in which only $\mathrm{Im}T_2(\xi(x), \eta(x))$ appears we have to choose a different convergence factor than in (2.34). We note that $T_2(\xi(z), \eta(z))/\sqrt{z}$ is also analytic in the upper half z plane, and as $|z| \to \infty$ it vanishes as $O(1/|z|^{3/2})$. Hence we can write

$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{x}} T_2(\xi(x), \eta(x)) = 0. \qquad (2.36')$$

The Born term satisfies this sum rule trivially and can be taken out. We define T_2' as

$$T_2'(\xi(z), \eta(z)) = T_2(\xi(z), \eta(z)) - \frac{f(-m_0^2 a/(z+b))}{[m_0^4 a^2/4(z+b)^2] - \xi^2(z) m^2}. \qquad (2.38)$$

The denominator of the Born term certainly does not vanish inside the forward tube, $q \cdot p \neq \frac{1}{2}q^2$ if $q = q_r + iq_i$ and $q_i^2 > 0$, $q_i^0 > 0$. The function $f(q^2)$ is analytic in the cut q^2 plane with a timelike cut only.

The sum rule (2.36′) will involve timelike q^2 for $x < -b$ and spacelike q^2 for $x > b$. Writing (2.36′) for $T_2'(\xi(x), \eta(x))$, using (2.5) for the region $x > -b$, and taking the real part of both sides, we get

$$\int_{-\infty}^{-b} \frac{dx}{\sqrt{-x}} \mathrm{Im}[T_2'(\xi(x+i0), \eta(x+i0))]$$
$$= -2\pi \int_{-b}^{0} \frac{dx}{\sqrt{-x}} W_2(\nu = \xi(x)m, q^2 = -m_0^2 a/(x-b)) - \int_0^\infty \frac{dx}{\sqrt{x}} \mathrm{Re}\left(\int_{\nu_t(x)}^\infty 4\nu' d\nu' \frac{W_2(\nu', q^2 = -m_0^2 a/(x+b))}{\nu'^2 - \xi^2(x) m^2}\right), \qquad (2.39)$$

where

$$\nu_t(x) = \tfrac{1}{2}\mu^2 + \mu m + \frac{m_0^2 a}{x+b}. \qquad (2.40)$$

Note that for $x > 0$, $\xi(x)$ almost everywhere has an imaginary part so the real part of the last integral above is *not* just a principal value. In (2.39), $\xi(x)$ and $\eta(x)$ are given by (2.23).

The right-hand side of (2.39) is directly calculable from the data on W_2 for spacelike q^2. The left-hand side depends on the imaginary part of T_2' for timelike values of q^2, a quantity which is not accessible to experiment. The imaginary part on the left includes the contributions of the class II intermediate states. Thus the sum rule (2.39) is practically useless for any direct experimental checks.

The question of course arises of using (2.39) as it stands to check certain model calculations that make predictions about the timelike region. This could be useful but we must add a word of warning. Very little information has gone into (2.39), just analyticity in the tube. *For example any model calculation of $T_2(\nu, q^2)$ that satisfies the Deser-Gilbert-Sudarshan (DGS) representation*[4] *will automatically satisfy (2.39) since it will certainly have the tube analyticity.* However, in some model calculations arbitrary momentum cutoffs are introduced and it is not beyond the ingenuity of theorists to violate two-variable analyticity. In summary, while we do not rule out the possibility that (2.39) might be useful, it certainly is a very weak restriction.

From this point on there are two ways to proceed further. One is to add some more physics to the analyticity information and see if we can get more useful sum rules. This we shall do in Sec. III. The new physics we add is complex scaling inside the tube and Regge behavior.

The second way to proceed further is much more ambitious. One can try to enlarge the region D to find the maximal region of analyticity in ξ and η, and then see if with this new maximal domain it is possible to solve problem A with condition (iii) replaced by $(\xi(z), \eta(z)) \in D_{\max}$ for all $\mathrm{Im}z > 0$. This, if possible, will lead to a powerful sum rule for $W_2(\nu, q^2)$, restricted to spacelike values of q^2, which can be directly compared with the data. In Sec. IV we make a very limited start to explore this direction. We first show that at least mathematically it is possible to define an analyticity domain, $D' \supset D$, such that for D', problem A has nontrivial solutions which we explicitly construct. However, the example of D' which we give has some rather unphysical features.

III. COMPLEX SCALING, ANALYTICITY, AND FINITE-q^2 SUM RULES

We have seen in Sec. II that starting with just analyticity in the tube it was not possible to find solutions to problem A with $\zeta^2(x)-\eta^2(x)$ spacelike on the real axis. The reason we were seeking such functions was because $T_2(\nu, q^2)$ is experimentally accessible to us only for spacelike q^2. Clearly, what we need is additional physical input that determines $T_2(\nu, q^2)$ for some other values of q^2. Then we get around the difficulty by choosing contours in the z plane such that on one part of the contour q^2 is spacelike and on the rest of the contour T_2 is determined by the additional physical input. This additional input we shall, in this section, take to be what we call "complex scaling," a concept which we will define and clarify below.

In Sec. II we found many solutions to problem B where $\zeta^2(x)-\eta^2(x)$ is spacelike on some segments on the real axis and timelike on the rest of the real axis. We shall start with such a solution for which $\zeta^2(x)-\eta^2(x)<0$ in some interval $x_1<x<x_2$. Then for any curve C joining x_1 and x_2 through the upper half z plane, we get

$$\int_{x_1}^{x_2} dx\, \nu(x) T_2(\zeta(x), \eta(x)) = -\int_C dz\, \nu(z) T_2(\zeta(z), \eta(z)). \tag{3.1}$$

The left-hand side of the sum rule is given directly by the data on W_2. To determine the right-hand side we choose $\zeta(z)$, $\eta(z)$, and C such that along C, $|\nu(z)|$ is large, $|q^2|$ is large, $q^2(z)$ complex, but their ratio is finite. Note that we are not proposing to generalize scaling to all complex directions, but only to directions which correspond to q_i remaining inside the forward tube. We certainly avoid the regions with singularities.

To define complex scaling let us recall the definition of the usual Bjorken limit.[1] We shall need to assume that not only νW_2 but also νT_2 scales. We use the variables $\omega = 2\nu/-q^2$, and q^2, and write

$$\mathcal{T}_2(\omega, q^2) \equiv \frac{\nu}{m^2} T_2(\nu, q^2). \tag{3.2}$$

Bjorken scaling states that for any fixed ω, we have

$$\lim_{-q^2\to\infty;\ \omega\,\text{fixed}} \mathcal{T}_2(\omega, q^2) = \mathfrak{F}_2(\omega). \tag{3.3}$$

The limit function $\mathfrak{F}_2$ can be calculated from the scaling form factor $F_2(\omega)$ by *assuming* that one can take the limit inside the integral sign in (2.5). One gets

$$\mathfrak{F}_2(\omega) = 4\omega \int_1^\infty d\omega' \frac{F_2(\omega')}{\omega'^2-\omega^2}, \tag{3.4}$$

where

$$\lim_{-q^2\to\infty} (\nu'/m^2) W_2(\nu', q^2) = F_2(\omega'), \quad \omega' \text{ fixed.}$$

The assumption of the interchangeability of limits is not necessary in models where scaling is the result of the dominance of behavior on the light cone.[5] There, (3.3) and (3.4) are features of the model.

In (3.3) we can certainly take ω to be complex, keeping q^2 real and spacelike, and the limit will still hold and is given by (3.4) with complex ω. This is a trivial extension of scaling.

What is not so trivial is taking the limit for fixed complex ω along complex directions in q^2. Namely, one considers $\lim_{|q^2|\to\infty} \mathcal{T}_2(\omega, q^2)$ with ω fixed and $\arg(q^2)\neq\pi$, where the limit is taken for values of ω and q^2 that correspond to ζ and η remaining inside the region of analyticity D, the forward tube.

On the basis of looking at several models which scale, we shall make the assumption that complex q^2 scaling limits exist and write

$$\lim_{|q^2|\to\infty;\ \omega\,\text{fixed}} \mathcal{T}_2(\omega, q^2) \equiv \tilde{\mathfrak{F}}_2(\omega). \tag{3.5}$$

Again we stress that in (3.5) we choose paths that remain inside the region of analyticity. The above limit can easily be disastrous in the q^2 timelike direction.

The second assumption about complex scaling we make is that it is path-independent at least for paths in the tube, and hence we identify the two limits by writing

$$\tilde{\mathfrak{F}}_2(\omega) \equiv \mathfrak{F}_2(\omega). \tag{3.6}$$

This feature of complex scaling is also true in several models, as will be shown at the end of this section. However, one can do more. In Appendix B we show that if $\mathcal{T}_2(\omega, q^2)$ is polynomially bounded for large $|q^2|$ and fixed ω, then for values of ω lying in the strip $-1<\text{Re}\,\omega<+1$ the limits (3.3) and (3.5) lead to identical results as long as one takes the limit in (3.5) along lines that lie in the $\text{Re}(q^2)<0$ half-plane. In the application we consider in this section, $\text{Re}(q^2)$ does in fact remain negative.

We shall defer the discussion of models which exhibit complex scaling to the end of this section, and proceed now to derive sum rules which will test whether this assumption is correct or not.

An interesting and still puzzling feature of the usual real scaling is its "precocious" nature. Namely, the onset of scaling begins at low values of q^2 of the order of a few BeV^2. An immediate question to ask is, given complex scaling, is its limit reached quickly for not too large values of $|q^2|$ as in the real case? Or is the "precocious" nature of real scaling more of an accident due to some cancellations that occur only for real spacelike q^2? In the sum rule we derive below, we shall assume that the limits (3.3) and (3.5) are reached at roughly the same rate and use the sum rule to test whether this is in agreement with the data.

We now derive a finite-q^2 sum rule which depends on analyticity in the tube and complex scaling. Following the discussion of Sec. II we choose $\zeta(z)$ and $\eta(z)$ as follows:

$$\begin{aligned}\zeta(z)&=m_0\left(h(z)-\frac{(z-1)}{4(z-b)h(z)}\right),\\ \eta(z)&=m_0\left(h(z)+\frac{(z-1)}{4(z-b)h(z)}\right).\end{aligned}\qquad(3.7)$$

Here we take

$$b>2\qquad(3.8)$$

and

$$h(z)=c+\int_b^\infty \frac{\rho(x)}{x-z}\,dx,\quad \rho\geq 0.\qquad(3.9)$$

The lower limit in (2.9) has to be greater than b. With this choice of $\zeta(z)$ and $\eta(z)$, we have

$$q^2(z)=-m_0^{\,2}\left(\frac{z-1}{z-b}\right).\qquad(3.10)$$

Both $\zeta(z)$ and $\eta(z)$ are analytic for $\mathrm{Im}z>0$. We have to also prove that for ζ and η given by (3.7) we have $\mathrm{Im}\zeta(z)>|\mathrm{Im}\eta(z)|$ for all $\mathrm{Im}z>0$. This can easily be done by repeating the same steps followed in Eqs. (2.27) – (2.30) in Sec. II. Thus $T_2(\zeta(z),\eta(z))$ is analytic in z for all $\mathrm{Im}z>0$.

The function $q^2(z)$ given by (3.10) is real for real z. It is spacelike for $x>b>2$, infinite for $x=b$, timelike in the interval $1<x<b$, and again spacelike for $x<1$.

Since we want to avoid timelike q^2 we concentrate our attention on the domain that lies inside the unit semicircle in the upper half z plane. In the real interval, $-1\leq x\leq 1$, $q^2(x)$ is spacelike and varies in the interval

$$-\left(\frac{2}{b+1}\right)m_0^{\,2}\leq q^2(x)\leq 0.\qquad(3.11)$$

By choosing $m_0^{\,2}\cong 10m_p^{\,2}$, we have then a range of q^2 that goes from zero up to and beyond the onset of scaling. The constant b we take to lie in $2<b<3$. The sum rule will be trivial if both on the real interval $-1<x<1$ and on the unit semicircle, $|q^2|$ is everywhere in the scaling region so that we cannot test the low-q^2 resonance region.[6]

Along the unit semicircle, $q^2(e^{i\phi})$ is large except for a region $0\leq\phi\leq\phi_1\cong 2m^2/m_0^{\,2}$. As the phase of z varies in the interval $2m^2/m_0^{\,2}<\phi<\pi$, $|q^2(e^{i\phi})|$ varies between

$$\frac{2}{b-1}m^2\lesssim|q^2(e^{i\phi})|\lesssim\frac{2}{b+1}m_0^{\,2}.$$

Taking $b=2.5$, this puts this range almost completely in the scaling region. For the range $0\leq\phi<2m^2/m_0^{\,2}$, $|q^2|$ is small and vanishes at $\phi=0$. In this region we are going to use the Regge fits to calculate T_2. These fits have been made to the data for large ν and small real q^2.[7] We shall use them also for small complex q^2. All the Regge fits proposed do scale so we have a smooth transition from the Regge region, $0\leq\phi\leq\phi_1$, to the scaling region, $\phi_1\leq\phi\leq\pi$, which is not sensitive to the precise value we choose for ϕ_1 as long as it is roughly $O(m^2/m_0^{\,2})$.

At this stage we are still free to adjust the behavior of $\zeta(x)$ along the real axis. To get a useful sum rule one has to explore the resonance region. Therefore, we must choose $\zeta(x)$ and $\eta(x)$ in such a way that even for $m_0^{\,2}=10m_p^{\,2}$ the variable $\nu(x)$ becomes small for some values of x in the interval $-1<x<1$. One way to do this is to choose $h(x)$ such that $\zeta(0)$ is zero. This can be done by setting

$$h(0)=(1/4b)^{1/2}.\qquad(3.12)$$

One has to satisfy (3.12) but choose $h(x)$ in such a way that $\nu(1)=\zeta(1)m$ is large enough to be close to the Regge region for small q^2. To give a specific example, just for demonstration, we take $b=2.5$, and let $h(z)=(\frac{5}{2})^{1/2}/(5-2z)$, and set $m^2=10m_p^{\,2}$. With this choice the value of $\nu(0)=0$ and $\nu(1)=\frac{5}{3}m^2$. With $q^2\cong 0$ this means that at the end of our interval $s(1)=\frac{13}{3}m^2$ just barely below the Regge region but still in the region where the Regge fit of Ref. 7 is good. With this choice of $h(z)$, b, and m_0, we can easily see that in the neighborhood of $x=\frac{3}{4}$ we are in the midst of the resonance region. A simple calculation gives $\nu(\frac{3}{4})=\frac{33}{28}m^2$ and $q^2(\frac{3}{4})=-\frac{10}{7}m^2$. This gives a value for $s(\frac{3}{4})=(q+p)^2\cong 2m^2$, which is right in the middle of the resonance region. A better choice of $h(z)$, m_0, and b has to await a numerical evaluation of our sum rules which is being carried out. Along the unit circle $|\nu(z)|$, with h given as above, will always be $O(mm_0)$ and hence large.

We define the amplitude $T_2'\equiv T_2-T_2^B$ as in (2.36). The Born term explicitly satisfies all the analyticity properties we use and the sum rule is an identity for it since the best experimental fit we have for $G_E(q^2)$ and $G_M(q^2)$ has only singularities for timelike q^2.

To obtain our sum rule we integrate $T_2'(\zeta(z),\eta(z))$ with ζ and η given by (3.7) along a contour extending along the real axis from $x=-1$ to $x=+1$ and then along the unit semicircle back to the point $z=-1$. Along the real axis $T_2'(\zeta(x),\eta(x))$ is given by the dispersion relation (2.5). On the semicircle,

$z=e^{i\phi}$, for $\phi_1 \leq \phi \leq \pi$, $\phi_1 \cong 2m^2/m_0^2$, T_2' is given by $\mathfrak{F}_2(\omega)$; while in the interval $0 \leq \phi \leq \phi_1$ we use the Regge fits to determine T_2'. The result is

$$\frac{4}{m^2}\int_{-1}^{+1} dx\,\nu(x)\int_{\nu_t(x)}^{\infty} d\nu'\,\frac{\nu' W_2(\nu', q^2=-m_0^2(x-1)/(x-b))}{\nu'^2-\zeta^2(x)m^2}$$
$$\cong -4i\int_{\phi_1}^{\pi} d\phi\, e^{i\phi}\omega(e^{i\phi})\int_1^{\infty} d\omega'\,\frac{F_2(\omega')}{\omega'^2-[\omega(e^{i\phi})]^2} - i\int_0^{\phi_1} d\phi\, e^{i\phi}R(q^2(\phi),\nu(\phi))\,. \tag{3.13}$$

The left-hand side of (3.13) involves only physical values of $W_2(\nu', q^2(x))$ and can be evaluated directly from the data. Note that we have shown above that by a proper choice of $h(z)$, m_0^2, and b we can make the integration over x go right through the region where resonances are important for W_2. The first term on the right in (3.13) also involves only physical values of $F_2(\omega')$, and ω is given by $\omega(z)=-2\zeta(z)m/q^2(z)$. In the second term one has to put in for R one of the Regge fits that have been recently obtained from the data on W_2 for small q^2. These fits have the form[7]

$$R(q^2,\nu)=\frac{\beta_P(q^2)}{\nu}[\nu^{\alpha_P}+(-\nu)^{\alpha_P}]\frac{1}{\sin\pi\alpha_P}+\frac{\beta_{P'}(q^2)}{\nu}[(\nu)^{\alpha_{P'}}+(-\nu)^{\alpha_{P'}}]\frac{1}{\sin\pi\alpha_{P'}}\,, \tag{3.14}$$

with $\alpha_P=1$, $\alpha_{P'}\cong\frac{1}{2}$, and β_P and $\beta_{P'}$ explicitly given in terms of q^2. In all the fits β_P and $\beta_{P'}$ vanish as $q^2 \to 0$ as they should, and also scale for large q^2.

The sum rule (3.13) will probably not be sensitive to the region $q^2\approx 0$ since T_2' vanishes as $O(q^2)$ as $q^2 \to 0$. Thus it is hard for (3.13) to distinguish between the different Regge fits or the model calculation of $\beta_P(q^2)$ by Preparata,[8] since for small ϕ, $q^2(\phi)=O(\phi)$, and hence $R(q^2(\phi),\nu(\phi))$ vanishes as $O(\phi)$ as $\phi\to 0$. The main idea that (3.13) will test is complex scaling and its precocious nature.[9] One can even write a sum rule less sensitive to the Regge region than (3.13) by integrating $(z-1)^{1/2}\nu(z)T_2'(\zeta(z),\eta(z))$ along the same contour as before. This will further deemphasize the $q^2\cong 0$ region near $z=1$.

To get a sum rule that is more sensitive to the input near $q^2=0$, we use the fact that $T_2'(\nu,q^2)=O(q^2)$ as $q^2\to 0$ and divide by q^2/m_0^2 to get a sum rule of the form

$$\frac{4}{m^2}\int_{-1}^{+1} dx\,\nu(x)\left(\frac{x-b}{x-1}\right)\int_{\nu_t(x)}^{\infty} d\nu'\,\frac{\nu' W_2(\nu', q^2=-m_0^2(x-1)/(x-b))}{\nu'^2-\zeta^2(x)m^2}$$
$$\cong -4i\int_{\phi_1}^{\pi} d\phi\, e^{i\phi}\left(\frac{e^{i\phi}-b}{e^{i\phi}-1}\right)\omega(e^{i\phi})\int_1^{\infty} d\omega'\,\frac{F_2(\omega')}{\omega'^2-\omega^2(e^{i\phi})} - i\int_0^{\phi_1} d\phi\, e^{i\phi}\left(\frac{e^{i\phi}-b}{e^{i\phi}-1}\right)R(q^2(\phi),\nu(\phi))\,. \tag{3.15}$$

This sum rule we hope will be sensitive enough to distinguish between the different Regge-type fits for q^2 small and ν large of Refs. 7 and 8.

As $m_0/m\to\infty$, the sum rules (3.13) and (3.15) become identities.[10] This is analogous to what happens to the finite-energy sum rule as the cutoff tends to infinity.

Finally, we remark that the Regge fits are made for νW_2 and there is no direct information about the real part of T_2 in the Regge region. In (3.14) we have chosen the simplest Regge form for the real part consistent with the fit for νW_2. This choice is somewhat arbitrary and one should check in evaluating the sum rules (3.13) and (3.15) whether they are very sensitive to changes in the form of the Regge fits (3.14).

We close this section by discussing several models which both scale in the real and complex directions.

(a) *Regge model.* In this model νT_2 has the form

$$\nu T_2(\nu,q^2)=\sum_i \frac{\beta_i(q^2)}{\nu}[\nu^{\alpha_i}+(-\nu)^{\alpha_i}]\frac{1}{\sin\pi\alpha_i}\,. \tag{3.16}$$

Scaling would occur if $\beta_i(q^2)\to C_i(-q^2)^{1-\alpha_i}$ as $-q^2\to\infty$. To get complex scaling we have to assume that $\beta_i(q^2)$ will have the same power behavior as above for large complex q^2. This is a feature of the ladder-type models that exhibit Regge scaling. One can easily check that in this model given complex scaling it will be path-independent and the identification made in (3.6) is trivially true.

(b) *The DGS model.* We assume that $T_2(\nu,q^2)$

satisfies the DGS representation.[4] This representation is known to hold to all orders in perturbation theory. We write

$$\frac{\nu}{m^2} T_2(\nu, q^2) = -\nu q^2 \int_{-1}^{+1} d\beta \int_0^\infty d\sigma \frac{h(\sigma,\beta)}{q^2+2\beta\nu-\sigma}. \quad (3.17)$$

A retarded $i\epsilon$ in the denominator had been suppressed. In terms of ω and q^2 we get

$$\mathcal{T}_2(\omega, q^2) = +\frac{q^2\omega}{2}\int_{-1}^{+1} d\beta\int_0^\infty d\sigma \frac{h(\sigma,\beta)}{1-\beta\omega-\sigma/q^2}. \quad (3.18)$$

One can formally expand the denominator in (3.18) in powers of $\sigma/q^2(1-\beta\omega)$. It is easy to see then that to get real scaling the DGS spectral function h must satisfy the relation[11]

$$\int d\sigma\, h(\sigma,\beta) = 0. \quad (3.19)$$

In that case we have for the standard Bjorken limit,

$$\lim_{-q^2\to\infty;\ \omega\,\text{fixed}} \mathcal{T}_2(\omega, q^2) \equiv \mathfrak{F}_2(\omega) = +\frac{\omega}{2}\int_{-1}^{+1} d\beta \int d\sigma \frac{h(\sigma,\beta)\sigma}{(1-\beta\omega)^2}. \quad (3.20)$$

The important point to notice here is that for real spacelike q^2 and fixed complex ω the denominator in (3.18) does not vanish and the expansion in powers of $\sigma/q^2(1-\beta\omega)$ does not lead to trouble. For timelike q^2 or complex q^2 this expansion could be quite dangerous. However, if we restrict ourselves to the forward tube, $q=q_r+iq_i$, $q_i^2>0$, the denominator in (3.17) does not vanish. One can easily check this fact by choosing a frame such that $q_i = (q_i^0, 0)$. This can be done since q_i is timelike. The vanishing of the denominator in (3.17),

$$q^2+2\beta q\cdot p - \sigma = 0, \quad (3.21)$$

implies that $q_r^0 = -\beta p^0$. Substituting this relation back into the real part of (3.21) one gets an expression that is negative definite.

Thus we can repeat the same arguments that lead from (3.18) to (3.20) for the case when q^2 approaches infinity along a complex direction inside the tube. The limit will not only exist if the real limit exists but should give the same function of ω as given in (3.20) independent of the path.

(c) *Perturbation-theory models.* The DGS representation is based on perturbation theory. It is clear from the discussion of the DGS model that models in which the sum of a given subset of diagrams exhibits scaling in the usual Bjorken sense will also scale in the complex Bjorken limit as long as that limit is taken inside the tube. After introducing the Feynman parameters, the perturbation-theory denominators one gets will also not vanish inside the tube.

One should add that in general it is possible to construct counterexamples to complex scaling and exhibit functions that scale in the real $-q^2$ limit and fail to scale in some complex directions. An example would be the existence of a nonscaling term in νT_2 which asymptotically behaves like $\{\exp[i(q^2-4\mu^2)^\lambda]\}/q^2$, $\lambda=\frac{2}{3}$. Such a term will vanish for large real q^2 but blows up exponentially in some complex q^2 direction. It is interesting to note that it has exponential growth in q^2.

The above models do not prove complex scaling but they tend to show that it is a reasonable feature to assume. There would be a significant change in one's outlook on scaling if complex scaling turns out not to be true. We hope that sum rules like (3.13) and (3.15) will shed some light on this question.

IV. LARGER DOMAINS OF ANALYTICITY AND SPACELIKE SUM RULES

We return in this section to the problem posed originally in Sec. II as problem A. The question we ask is: Does there exist an enlargement of the analyticity domain D such that given the larger domain one can find a "special" contour along which q^2 is spacelike? We give an example of such a domain. However, unfortunately, the example is only of mathematical interest and has drastically unphysical features.

Consider a function $f(\zeta,\eta)$ analytic in ζ and η regular in domain D', defined by

$$D':\ \text{Im}\zeta > (1-\epsilon)|\text{Im}\eta|,\quad 1>\epsilon>0. \quad (4.1)$$

Comparing this with (2.11) we see that $D'\supset D$. The problem we now have to solve, in analogy to A, is to find functions $\zeta(z)$ and $\eta(z)$ such that:

(i) $\zeta(z)$ and $\eta(z)$ are analytic for $\text{Im}z>0$;

(ii) $\zeta^2(x)-\eta^2(x)\le 0$, real, and nonconstant on real axis;

(iii) $\text{Im}\zeta(z) > (1-\epsilon)|\text{Im}\zeta(z)|$, for all $\text{Im}z>0$.

This problem has many solutions. We shall exhibit one of them:

$$\zeta(z) = m_0\{[h(z)+az]^2 - z^2\}^{1/2},$$
$$\eta(z) = m_0[h(z)+az], \quad (4.2)$$

with $a>1$ and such that $(a^2-1)^{1/2}-(1-\epsilon)a>0$, and

$$h(z) = c + \int_{-\infty}^{+\infty}\frac{\rho(x)}{x-z}dx,\quad \rho(x)\ge 0,\quad c>0. \quad (4.3)$$

The first two properties (i) and (ii) can be easily checked. The bracket defining ζ has no zeros in the upper half-plane and the analyticity is guaranteed. We only have to prove (iii).

On the real axis, we have

$$\mathrm{Im}\{[h(x)+ax]^2-x^2\}^{1/2} \geq \mathrm{Im}[h(x)+ax] > (1-\epsilon)\,\mathrm{Im}[h(x)+ax]\,. \tag{4.4}$$

Let us consider the function $w(z)$ defined by

$$w(z) \equiv \{[h(z)+az]^2-z^2\}^{1/2}-(1-\epsilon)[h(z)+az]\,. \tag{4.5}$$

As $|z|\to\infty$, we have $w(z)\to[(a^2-1)^{1/2}-(1-\epsilon)a]z$. Also $\mathrm{Im}w(z)$ is positive for real z. It is trivial to show that as long as $(a^2-1)^{1/2}-(1-\epsilon)a>0$, as we have chosen it, then $w(z)$ is a Herglotz function. All one has to do is to consider

$$\tilde{w}\equiv w-[(a^2-1)^{1/2}-(1-\epsilon)a]z\,.$$

Then $\tilde{w}(z)$ is a Herglotz function since $\tilde{w}(z)\to$ (real const) as $|z|\to\infty$, and $\mathrm{Im}\tilde{w}(x)>0$ on the real axis. It follows that $w(z)$ is also a Herglotz function when $(a^2-1)^{1/2}-(1-\epsilon)a>0$.

One can write for $f(\zeta,\eta)$ a sum rule of the form

$$\int_{-\infty}^{+\infty}\frac{dx}{(x+i)^N}f(\zeta(x),\eta(x))\,dx=0, \tag{4.6}$$

where ζ and η are given by (4.2). The integration in (4.6) is carried out along a path for which $q^2(x)\equiv\zeta^2(x)-\eta^2(x)=-m_0{}^2x^2$ is spacelike and variable. The factor $(x+i)^N$ is introduced to damp out the asymptotic behavior of f for large $|z|$.

This is all mathematically satisfying, but the region D' cannot be obtained in any reasonable physical theory. Its most disastrous feature is that it contains timelike q^2 points of arbitrary values. One can check this directly from (4.1), but it is easier to see it from the example (4.2). There, $q^2(z)=-m_0{}^2z^2$, and the points on the imaginary z axis are in the analyticity domain but they correspond to positive, real q^2 and complex ν. However, we know that for timelike q^2 we have singularities coming from the class II intermediate states.

There are two main questions which arise from this discussion. First, can one enlarge the tube domain D in the physical case to get a domain for which a spacelike sum rule holds? This seems difficult and most probably not achievable; however, work along this direction might lead to less ambitious but still useful results—namely, our second question: Can one enlarge the domain D in order to get a domain for which better sum rules of the type considered in Sec. III can be found? For example, one can look for a sum rule which does not depend on the Regge region.

A preliminary look at perturbation theory shows that one does get domains larger than D. However, these domains approach D in the crucial region $\mathrm{Im}\eta\to\infty$. They seem to be of no help in the case of the more ambitious first question, but they could still be helpful for our second question. Work along these lines is in progress.

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APPENDIX A

In this Appendix we give a proof due to Dyson of the assertion made in Sec. II that the only solutions to problem A have the form (2.20).

The problem is to find functions $\zeta(z)$ and $\eta(z)$ such that:

(i) $\zeta(z)$ and $\eta(z)$ are analytic in z for $\mathrm{Im}z>0$, and continuous in $\mathrm{Im}z\geq 0$ except at isolated points on the real axis;

(ii) on the real z axis, $\zeta^2(x)-\eta^2(x)$ is real, and $\zeta^2(x)-\eta^2(x)\leq 0$, except at the isolated points mentioned in (i);

(iii) for all $\mathrm{Im}z>0$, $\mathrm{Im}\zeta(z)>|\mathrm{Im}\eta(z)|$.

The assertion is that all pairs of functions ζ and η that satisfy (i), (ii), (iii) must be related such that $\zeta^2(z)-\eta^2(z)=-C$, with C real and $C>0$.

To prove this fact we first note that (i) and (iii) imply that $\zeta(z)$ is a Herglotz function.[12] We consider the two functions $u(z)$ and $v(z)$ defined by

$$u(z)=\zeta(z)+\eta(z),\qquad v(z)=\zeta(z)-\eta(z). \tag{A1}$$

Then it also follows from (iii) that for all $\mathrm{Im}z>0$, $\mathrm{Im}u(z)=\mathrm{Im}\zeta(z)+\mathrm{Im}\eta(z)\geq\mathrm{Im}\zeta-|\mathrm{Im}\eta|>0$, and also $\mathrm{Im}v(z)>0$. Hence, both $u(z)$ and $v(z)$ are Herglotz functions.

Next we consider the function $f(z)$ given by

$$f(z)\equiv\zeta^2(z)-\eta^2(z)\equiv u(z)v(z). \tag{A2}$$

This function is analytic for $\mathrm{Im}z>0$. It is continuous except at isolated points for $\mathrm{Im}z\geq 0$. From (ii), $f(z)$ is real on the real axis except at isolated points. Hence, $f(z)$ is single valued and analytic in the whole z plane except for isolated poles, and isolated essential singularities on the real axis. But (A2) states that f is the product of two Herglotz functions. Using this fact, and the standard representation for Herglotz functions,[12] one can exclude the possibility that any real point x_0 can be an isolated essential singularity of $f(z)$.

Thus one concludes that $f(z)$ is meromorphic in the entire z plane with poles and zeros only on the real z axis. But condition (ii) states that on the real axis $f(z)$ is almost everywhere negative. Therefore, all the poles and zeros of $f(z)$ must be double or of even order. However, $f(z)=u(z)v(z)$ is a product of two Herglotz functions. Herglotz functions can only have simple poles or zeros. Hence, $u(z)$ and $v(z)$ must each have a simple pole at each double pole of f.

Consider such a pole at $z=x_0$. In the neighborhood of such a pole we have

$$u(z)\cong \frac{r}{x_0-z},\quad v(z)\cong\frac{s}{x_0-z}, \tag{A3}$$

with $r>0$ and $s>0$. Hence near x_0 we have

$$f(z)\cong\frac{rs}{(x_0-z)^2},\quad z\approx x_0. \tag{A4}$$

This leads to a contradiction with condition (ii) which restricts $f(x)$ to be negative almost everywhere on the real axis. We conclude that $f(z)$ can have no poles and is an entire function.

Using the well-known upper bound for Herglotz functions, we get from (A2) the bound for large $|z|$,

$$|f(z)|<\text{const }|z|^2. \tag{A5}$$

Thus $f(z)$ is a polynomial of degree less than or equal to two.

Now, the same argument we used to prove that $f(z)$ has no poles can be used to show that f has no zeros. A Herglotz function can have only simple zeros. Hence, $f(z)$ must be a negative real constant and $\zeta^2-\eta^2=-C,\ C>0$.

APPENDIX B

We prove here the assertion made in Sec. III concerning the limit of $\mathcal{T}_2(\omega,q^2)$ defined in (3.2) for $|q^2|\to\infty$, $\text{Re}q^2<0$, and ω fixed and chosen such that $-1<\text{Re}\omega<+1$. The statement we want to prove is: Given a $\mathcal{T}_2(\omega,q^2)$ with analyticity in ω and q^2 in the domain that follows from perturbation theory, and assuming that $|T_2(\omega,q^2)|$ for fixed ω and large $|q^2|$ is bounded by $\exp(|q^2|^{1-\epsilon})$, then if

$$\lim_{\substack{-q^2\to\infty;\ \omega\text{ fixed}\\ -1<\text{Re}\omega<+1}}\mathcal{T}_2(\omega,q^2)=\mathfrak{F}_2(\omega), \tag{B1}$$

and if the limit along complex directions in the left half q^2 plane also exists, i.e.,

$$\lim_{\substack{|q^2|\to\infty;\ \omega\text{ fixed}\\ -1<\text{Re}\omega<+1}}\mathcal{T}_2(\omega,q^2)=\tilde{\mathfrak{F}}_2(\omega),\quad \text{Re}q^2<0, \tag{B2}$$

then $\tilde{\mathfrak{F}}_2(\omega)\equiv\mathfrak{F}_2(\omega)$ in the strip $-1<\text{Re}\omega<+1$.

The first point to notice is that it follows from the DGS representation (3.18) that $\mathcal{T}_2(\omega,q^2)$ is analytic in both ω and q^2 in the domain $-1<\text{Re}\omega<+1$ and $\text{Re}q^2<0$. Now if we fix $\omega=\omega_1$ with ω_1 lying in the strip $-1<\text{Re}\omega_1<+1$, the function $\mathcal{T}_2(\omega_1,q^2)$ for fixed $\omega=\omega_1$ is analytic in q^2 in the half-plane $\text{Re}q^2<0$. If $\lim_{-q^2\to\infty}\mathcal{T}_2(\omega_1,q^2)=\mathfrak{F}_2(\omega_1)$ and if also the limit along any complex direction in the left-plane exists, $\lim_{|q^2|\to\infty}\mathcal{T}_2(\omega_1,q^2)=\tilde{\mathfrak{F}}_2(\omega_1)$, $\text{Re}q^2<0$, then the Phragmén-Lindelöf theorem guarantees that $\mathfrak{F}_2(\omega_1)=\tilde{\mathfrak{F}}_2(\omega_1)$. But this relation must be true for every ω_1 in the strip $-1<\text{Re}\omega_1<+1$. Hence since $\mathfrak{F}_2(\omega)$ is an analytic function we can identify the functions $\mathfrak{F}_2$ and $\tilde{\mathfrak{F}}_2$ in the strip, and in any region where $\tilde{\mathfrak{F}}_2(\omega)$ is defined and not separated from the strip by a natural boundary.

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[1]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).

[2]See, for example, R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, (Benjamin, New York, 1964), Chap. 2.

[3]The author wishes to thank Professor Dyson for permission to reproduce his proof in this paper.

[4]S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. 115, 731 (1959); N. Nakanishi, Progr. Theoret. Phys. (Kyoto) 26, 337 (1961).

[5]We are indebted to Dr. G. Preparata for stressing this point.

[6]If one chooses a contour such that $|q^2|$ and $|\nu|$ are large everywhere on the contour, then the sum rule one gets is a trivial identity. The result would be $\int_C\mathfrak{F}_2(\omega(z))dz=0$, where C is some closed contour. But $\mathfrak{F}_2$ is an analytic function of ω, and ω is analytic in z, and hence we just have an identity for any $F_2(\omega')$.

[7]See, for example, F.E. Close and J. F. Gunion, Phys. Rev. D 4, 743 (1971).

[8]G. Preparata, Phys. Letters 36B, 53 (1971).

[9]Throughout this paper we have assumed that the dispersion relation for $T_2(\nu,q^2)$ is unsubtracted. Indeed the integral over W_2 in (2.5) must converge or else σ_L will violate the Froissart bound. However, as we are dealing with a first-order electromagnetic amplitude, T_2 could have a real contribution which violates the Froissart bound. A fixed pole at $J=2$ will force us to add on an extra term in (2.5) which is a real function of q^2 for spacelike q^2. The sum rule (3.13) will still hold as long as the residue of the fixed pole is analytic for $\text{Re}q^2<0$. For in that case one separates out the fixed-pole contribution in the same way we separated the Born term. The fixed-pole term will satisfy the contour integral by itself.

[10]One should make a remark about the sum rules in the case where $m_0>>m$ and also their behavior as $m_0/m\to\infty$. If one takes $m_0/m>>1$, then the small-q^2 region will shrink to a very small neighborhood of the point $z=1$. Except for the neighborhood of $z=1$ and the point $z=0$, the values of $|\nu|$ and $|q^2|$ will be such that all the points

along our contour correspond to the scaling points. Following the remark in Ref. 6, the sum rules become identities as $m_0/m \to \infty$. For $m_0 >> m$, m_0/m finite, it would be more precise to subdivide the unit semicircle into three regions instead of two: (a) The Regge region, $0 < \phi < O(m^2/m_0^2)$, (b) The Bjorken-limit region, $O(m^2/m_0^2) \lesssim \phi \lesssim O(m/m_0)$, and $O(m_0/m) \gtrsim |\omega| \gtrsim O(1)$, and (c) The Johnson-Low-Bjorken-limit region, $O(m/m_0) \lesssim \phi \lesssim \pi$, and $O(1) \gtrsim |\omega| \gtrsim 0$. The JLB for $\mathcal{T}_2(\omega, q^2)$, $|q^2| \to \infty$, $|\omega| \to 0$, is also given by $\mathcal{F}_2(\omega)$ for $|\omega| \to 0$. In this limit $\mathcal{T}_2$ vanishes as $O(\omega)$ and is given by $\mathcal{T}_2 \cong 4\omega \int_1^\infty F_2(\omega')/\omega'^2 d\omega'$. This last remark does not change the form of the sum rules (3.13) and (3.15).

[11]R. Brandt, Phys. Rev. Letters 22, 1149 (1969).

[12]J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, Providence, R. I., 1943), p. 23.

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Renormalizable Symmetry Model of Weak Interactions*

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A renormalizable model of weak interactions is presented. The leptons and the SU(3) quarks are classified in an SU(4)×SU(2) symmetry scheme in which the quantum numbers are charge, baryon number, lepton number, and weakness – a quantum number that replaces the usual second lepton number. The coupling of a sextet of spinless bosons to the leptons and hadrons gives a theory of weak interactions that has universality, the correct selection rules, renormalizability, conserved vector current, and $V-A$ in the local limit. Elastic $e\nu_\mu$ scattering is predicted. The intermediate scalar boson leads to μe and ee pairs in the conventional high-energy neutrino experiments but no $\mu\mu$ pairs are predicted. Some neutral-current effects are expected and these are consistent with present data.

I. INTRODUCTION

The development of a renormalizable theory of weak interactions has received considerable attention,[1,2] but so far no entirely satisfactory solution has been proposed. Soon after weak interactions were found to proceed by the $V-A$ current-current Hamiltonian,[3] a renormalizable theory using spin-zero bosons was proposed by Tanikawa and Watanabe.[1] Their theory was based on the fact that a $V-A$ interaction can be reexpressed in terms of scalar and pseudoscalar interactions by means of the Fierz transformation[4]

$$\overline{\psi}_a\gamma_\mu(1+\gamma_5)\psi_b\overline{\psi}_c\gamma_\mu(1+\gamma_5)\psi_d = -2\overline{\psi}_a(1-\gamma_5)\psi_{\bar{c}}\overline{\psi}_{\bar{b}}(1+\gamma_5)\psi_d, \quad (1)$$

where $\bar{b}$ and $\bar{c}$ are the antiparticles of b and c. By introducing a semiweak coupling of spin-zero bosons to the various densities of the form $\overline{\psi}_{\bar{b}}(1+\gamma_5)\psi_d$, a renormalizable theory is produced.

The two main criticisms[5] of the Tanikawa-Watanabe approach are that universality is accidental and the conserved vector current plays no role. Of these two faults the former is more serious as the conserved vector current can be introduced in a number of ways. For example, one could couple the pseudoscalar mesons to intermediate spinor particles in such a way that the lepton current in the effective second-order Lagrangian is coupled to the conserved isospin current if the right relations exist between the coupling constants and between the masses of the intermediate particles. This approach makes universality even more accidental. The conserved vector current may also be introduced by coupling the leptons and the SU(3) quarks to intermediate scalar bosons in such a way that the conserved vector current in quark form appears in the effective Lagrangian. This approach again requires certain degeneracies in coupling constants and masses and therefore makes universality accidental unless a reason can be found for the existence of these degeneracies.[6] A possible reason for such degeneracies is that the particles involved belong to irreducible representations of a symmetry group under whose transformations the weak-interaction Hamiltonian is invariant. This point of view is adopted in this paper.

In Sec. II the symmetry of weak interactions is discussed. Section III is devoted to the interesting leptonic processes while Sec. IV outlines the application of the theory to conventional semileptonic processes. The associated production of SU(3) quarks and intermediate scalar bosons is discussed in Sec. V. Sections VI and VII treat neutral-current effects and nonleptonic processes.