PHYSICAL REVIEW D VOLUME 5, NUMBER 2 15 JANUARY 1972

Boundary Dispersion Relations

G. E. Hite* † and R. Jacob‡
Institut für Theoretische Kernphysik, Universität Karlsruhe, Germany
(Received 30 April 1971)

We derive a dispersion relation for direct-channel reactions of the type $a+b\to c+d$ with $m_b=m_d$, e.g., $\gamma N\to\pi N$, which involves contributions from both forward and backward scattering of the direct channel and backward scattering of the crossed channel but no contribution from unphysical regions of the direct channel. In particular, the integral involving the direct-channel contribution is evaluated along the boundary of the physical region. Our result, in the case of elastic scattering, i.e., $m_a=m_c$, reduces to the familiar backward dispersion relation.

I. INTRODUCTION

The value of dispersion relations for the analysis of experimental data in the low- to medium-energy range has been firmly established.[1] The usefulness of a given dispersion relation is, however, dependent on the choice of the dispersion variable x. Dispersion relations essentially relate the real part of the amplitude for a given value of x, via Cauchy contours, to integrals involving the discontinuities of the amplitude in the x plane. In using such relationships to extract model parameters, one is faced with the problem of evaluating these integrals, using data whenever possible. Thus, a choice of the dispersion variable which involves contributions from regions where the discontinuity is unknown is of limited value. Such contributions can arise from the introduction of kinematic singularities in x, dynamical cuts which do not correspond to physical reactions, or to situations where dynamical cuts are not mapped onto the real x axis, forbidding the simplification resulting from the real analytic properties of the amplitudes.

Early recognition of these facts helped lead to the establishment of today's conventional Mandelstam variables[2] in terms of which, it is assumed, one can find amplitudes with simple analytic structure. In a four-body reaction, for example, of the two independent Mandelstam variables, one (usually the squared momentum transfer, t) is held fixed while some analytic function of the other, e.g., $\nu'=(s-u)/4m$ of Chew, Goldberger, Low, and Nambu (CGLN),[3,4] is adopted as the dispersion variable. These "fixed-t" relations have been widely used to study photoproduction of pions[5] and pion-nucleon scattering.[6] Such dispersion relations, except for one value of t, $-\mu^2m/(m+\mu)$ and zero for the respective reactions, involve contributions from unphysical regions. Thus, their usefulness at general values of t depends on the accuracy of analytic-continuation procedures used in estimating such contributions.[7]

In order to avoid this problem of unphysical regions altogether, dispersion relations have been written for fixed values of the (s-channel) scattering angle θ_s. This has been done with considerable success for $\pi N\to\pi N$ at $\theta_s=\pi$ (Ref. 8) and the formalism has been worked out by Białkowski and Jurewicz (BJ)[9] for $\gamma N\to\pi N$ at general angles. For these reactions, the dispersion variable was taken to be the square of the πN center-of-mass momentum, $p_{\pi N}$. In the former case, this is equivalent to dispersing in t, and no serious problem is incurred. In the latter case, however, this selection of dispersion variable introduced, even for $\theta_s=\pi$, three kinematical branch points at $p_{\pi N}{}^2=-m^2$, $-\mu^2$, and 0. To eliminate contributions from the associated kinematical cuts, BJ wrote a dispersion relation involving the amplitude on all four kinematical sheets, which resulted in an involved numerical problem indeed.

Such complications point out the importance of proper selection of fixed and dispersion variables and of amplitudes in using dispersion relations to study the phenomenology of particle scattering.

In this paper, we show that a large simplification occurs, at least for $\cos^2\theta_s=1$, for reactions of the type $ab\to cd$ with $m_b=m_d$ when t, rather than $p_{cd}{}^2$, is chosen as the dispersion variable. In particular, with t as the dispersion variable, it is easy to define amplitudes which are free of kinematic singularities and for which the dynamical cuts map in a simple manner onto the real t axis.

II. NOTATION AND BASIC IDEA

A. Notation

For this paper it will be useful to introduce the following variables.

The s-channel, $ab\to cd$, initial and final center-of-mass momenta p_s and p_s', respectively:

$$p_s{}^2=[s-(m_a+m_b)^2][s-(m_a-m_b)^2]/4s\,,$$
$$p_s'^2=[s-(m_c+m_d)^2][s-(m_c-m_d)^2]/4s\,. \tag{1}$$

Cosine of the scattering angle θ_s in the s channel:

$$Z_s \equiv \cos\theta_s = [s(t-u) + (m_a^2 - m_b^2)(m_c^2 - m_d^2)]/4sp_sp_s' . \tag{2}$$

The t-channel, $a\bar{c} \to \bar{b}d$, initial and final center-of-mass momenta p_t and p_t', respectively:

$$p_t^2 = [t - (m_a + m_c)^2][t - (m_a - m_c)^2]/4t,$$
$$p_t'^2 = [t - (m_b + m_d)^2][t - (m_b - m_d)^2]/4t. \tag{3}$$

Cosine of the scattering angle θ_t in the t channel:

$$Z_t \equiv \cos\theta_t = [t(s-u) + (m_a^2 - m_c^2)(m_b^2 - m_d^2)]/4tp_tp_t' . \tag{4}$$

Since we restrict ourselves to reactions with $m_b = m_d = m$, we can simplify the last two expressions:

$$p_t'^2 = -m^2 + \tfrac{1}{4}t,$$
$$Z_t = \nu/4p_tp_t', \tag{5}$$

where $\nu \equiv s - u$. A final useful quantity is

$$\Sigma \equiv s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 . \tag{6}$$

B. Invariant Amplitudes

Dispersion relations are generally written for the Lorentz-invariant amplitudes, $A(s, t, u)$, since these functions contain only dynamical singularities in s, t, and u. By selecting t as the dispersion variable and fixing $\cos\theta_s$, which is a nonanalytic function of s and t, we introduce into the amplitudes

$$A(t, s(t), Z_s) = A(s, t, u)|_{Z_s = \text{const}}$$

kinematic singularities since the expression $s(t)$ contains kinematic branch points in t. In the following it will become clear how new amplitudes $\tilde{A}$ can be defined which are free of such kinematic singularities and thus amenable to dispersion relations involving only dynamical singularities.

We will, for simplicity, assume dispersion relations can be written without subtractions. In the case of inelastic reactions, this is a quite reasonable assumption since inelastic reaction amplitudes in general vanish more rapidly than elastic amplitudes. For the elastic reactions, e.g., $\pi N \to \pi N$, it is not certain whether subtractions are necessary.[10] In any event, the procedures for introducing subtractions are well known and can be easily applied once the form of the dispersion relation is known.

C. The Reaction $\pi N \to \pi N$

Before proceeding to the general situation, $m_a \neq m_c$, let us review the case of the elastic reaction $\pi N \to \pi N$. For $\theta_s = \pi$, we have

$$s = (E + \omega)^2 = \tfrac{1}{2}(\Sigma + 4p_s^2 + 4E\omega),$$
$$t = -2p_s^2(1 - Z_s) = -4p_s^2, \tag{7}$$
$$u = \Sigma - s - t = (E - \omega)^2 = \tfrac{1}{2}(\Sigma + 4p_s^2 - 4E\omega),$$

where E and ω are the nucleon and pion center-of-mass energies, respectively. Since here, $p_s^2 = -\tfrac{1}{4}t$, p_s^2 and t are analytically equivalent variables. One can also write

$$s = \tfrac{1}{2}(\Sigma - t - 4p_tp_t'),$$
$$u = \tfrac{1}{2}(\Sigma - t + 4p_tp_t'), \tag{8}$$

where p_t and p_t', the initial and final center-of-mass momenta for the t-channel reaction $\pi\pi \to \bar{N}N$, and Σ are defined in Sec. II A. In these equations for s and u the phase of p_tp_t' is chosen to be such that $p_tp_t' < 0$ for $t \leq 0$.

If we express the CGLN invariant amplitudes,[3] defined by[11]

$$T^{\pm} = -A^{\pm}(s, t, u) + \tfrac{1}{2}\gamma \cdot (p_{\pi_1} + p_{\pi_2})B^{\pm}(s, t, u) \tag{9}$$

as functions of the variables t and ν,

$$\nu = -4p_tp_t'$$

(recall we are still at $\theta_s = \pi$), the crossing relations are

$$A^{\pm}(\nu, t) = \pm A^{\pm}(-\nu, t), \quad B^{\pm}(\nu, t) = \mp B^{\pm}(-\nu, t). \tag{10}$$

A^+ and B^-, which are even under $\nu \to -\nu$, must contain only even powers of p_tp_t' and therefore are free of kinematical singularities. Because A^- and B^+ must vanish at $\nu = 0$, the functions A^-/ν and B^+/ν are also even functions of p_tp_t' and thus free of kinematical singularities. Consequently, dispersion relations which involve only dynamical singularities can be written for the amplitudes A^+, A^-/ν, B^+/ν, and B^-.

The mapping of the dynamical s, t, and u cuts onto the two-sheeted t plane, or, equivalently, the p_s^2 plane, has been discussed often in the literature.[12] In particular, while the t dynamical cut maps onto both sheets, the s cut maps only onto the first t sheet, where $p_tp_t' > 0$ (for $t \to \infty$), and the u cut maps only onto the second t sheet, where $p_tp_t' < 0$ (for $t \to \infty$) (see Fig. 1).

Because $\theta_s = 0$ corresponds here to a fixed value of t, i.e., zero, one cannot disperse in t, and the $\theta_s = 0$ dispersion relations are just "fixed-t" dispersion relations. This of course is the unique value of t for which there is no contribution from the dynamical cuts outside of the physical region.

D. Choice of Dispersion Variable

In the more general case, where $m_a \neq m_c$, t and p_s^2 are no longer analytically equivalent variables at $\theta_s = \pi$. One must now decide which is the more

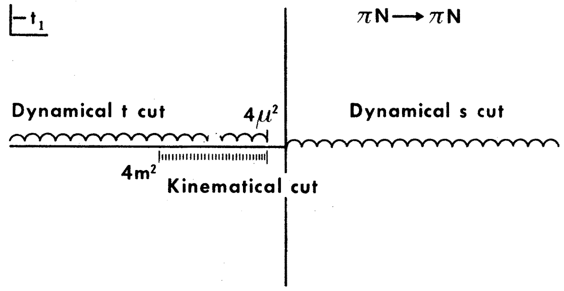

FIG. 1. Dynamical and kinematical cuts for the reaction $\pi N \to \pi N$ mapped onto the t sheet with $p_t p_t' > 0$ as $t \to \infty$. The t sheet with $p_t p_t' < 0$ as $t \to \infty$ is identical except the dynamical s cut is replaced by the dynamical u cut.

practical dispersion variable for fixed θ_s. In considering the reaction $\gamma N \to \pi N$ at fixed angles, BJ selected $p_{\pi N}{}^2$ and found that to eliminate the resultant kinematic singularities they had to write dispersion relations involving values of the amplitude on all four of the kinematical sheets. This complication remained for $\theta_s = 0$ and π.

On the other hand, choosing t as the dispersive variable necessitates calculating s and u as functions of t. For both $\cos\theta_s$ equal to $+1$ and -1 in pion photoproduction, one finds $s(t)$ and $u(t)$ as given in Eq. (8) with Σ, p_t, and p_t' appropriate to pion photoproduction. Because the only branch points in the product $p_t p_t'$ for pion photoproduction are at zero and $4m^2$, one has only two kinematical sheets. One finds, as in the pion-nucleon scattering case, that while the dynamical t cut is on both sheets, the u cut and s cut occur only once and on different sheets. Also similarly, the simple crossing-symmetry properties of the invariant amplitudes[4] under $\nu \to -\nu$, which still result from the equality of masses m_b and m_a, allow one again to define amplitudes free of kinematical singularities (KSF):

$$\tilde{A} \equiv \begin{cases} A\ , & \text{if } A(\nu) = +A(-\nu) \\ A/\nu, & \text{if } A(\nu) = -A(-\nu)\,. \end{cases} \qquad (11)$$

Consequently, one sees that the choice of t as a dispersion variable for $\cos^2\theta_s = 1$ in $\gamma N \to \pi N$ leads to a far simpler representation than the uses of $p_{\pi N}{}^2$.

It is useful to observe that the expressions for $s(t)$ and $u(t)$ in terms of Σ, t, and $p_t p_t'$ are the same for $\pi N \to \pi N$ at $\theta_s = \pi$ and for $\gamma N \to \pi N$ for $\theta_s = 0, \pi$. The origin of this is very obvious when one considers the relationship between the t- and s-channel scattering angles and the Kibble boundary function, $\phi(s, t)$:

$$\phi(s, t) = 4 s p_s{}^2 p_s'^2 \sin^2\theta_s = 4 t p_t{}^2 p_t'^2 \sin^2\theta_t\,. \qquad (12)$$

Except in the forward direction for elastic scattering, $\sin^2\theta_s = 0$ implies $\sin^2\theta_t = 0$. One finds for the boundary of the physical s channel that $\sin^2\theta_s = 0$ implies $\cos\theta_t = -1$. For the reactions being considered, $m_b = m_a$ and one has $\cos\theta_t = \nu/4p_t p_t'$ giving

$$\nu \equiv s - u = -4p_t p_t'\,,$$
$$s + u = \Sigma - t\,, \qquad (13)$$

which demonstrates the generality of the expressions for $s(t)$ and $u(t)$ of Eq. (8).

In addition to selecting a dispersion variable, one must hold some quantity fixed, e.g., θ_s. As mentioned in the Introduction, fixing θ_s avoids the introduction of unphysical contributions inherent in fixed-t dispersion relations. For $\theta_s = \pi$ or 0 (inelastic) one has $Z_t = -1$; thus, one could equivalently fix θ_s or fix $Z_t = -1$. In fact, fixed $Z_t \leqslant -1$ curves do not lead to unphysical s-channel contributions. One advantage of using fixed Z_t instead of Z_s is that crossing-symmetry properties in $\nu = 4p_t p_t' Z_t$ allow one to work with the KSF amplitudes defined in Eq. (11).

III. DISPERSION RELATIONS IN t

A. Dispersion Relations for $\theta_s = 0, \pi$

The arguments used in Sec. II D concerning the definition of the KSF amplitudes $\tilde{A}$, as well as the expressions for $s(t)$ given by Eq. (13), can easily be seen to hold for any reaction, $ab \to cd$, for which $m_b = m_a$. In order to write dispersion relations for the KSF amplitudes $\tilde{A}$ of these reactions, we must find the mapping of the dynamical cuts into the t plane. The dynamical t cut of course maps onto all t sheets, and, with the Born terms considered independently, will begin at the value of t, t_2, corresponding to the least massive two-body intermediate state. In $\pi N \to \pi N$ and $\gamma N \to \pi N$, one has $t_2 = 4\mu^2$.

The equation

$$u(t) = \tfrac{1}{2}(\Sigma - t + 4p_t p_t')$$

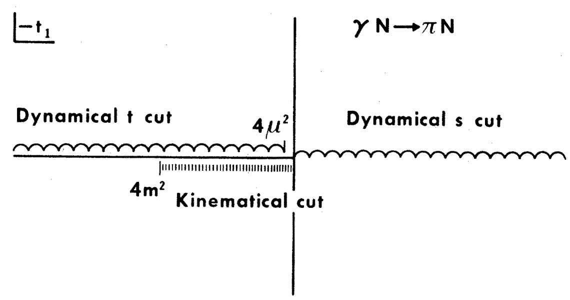

FIG. 2. Dynamical and kinematical cuts for the reaction $\gamma N \to \pi N$ mapped onto the t sheet with $p_t p_t' > 0$ as $t \to \infty$. The t sheet with $p_t p_t' < 0$ as $t \to \infty$ is identical except the dynamical s cut is replaced by the dynamical u cut.

shows that u-channel singularities map only onto sheets for which $p_t p_t' < 0$ as $t \to \infty$. Similarly, s-channel singularities map only onto sheets for which $p_t p_t' > 0$ as $t \to \infty$. The KSF amplitude $\tilde{A}$ is, however, the same on all sheets; we therefore need only consider one sheet, which we take to be the physical t sheet on which both p_t and p_t' are positive as $t \to \infty$. Because Eq. (8) describes the physical boundary of the s channel (except for the $t=0$ part in the elastic case) and the boundary of the physical region in the forward direction either is $t=0$ or approaches $t=0$ asymptotically, it is evident that the s-channel dynamical cut maps onto $-\infty \leq t \leq 0$. The cut structures for $\pi N \to \pi N$ and $\gamma N \to \pi N$ are shown in Figs. 1 and 2.

Since both the forward and backward directions for physical s-channel reactions map onto the region $-\infty \leq t \leq 0$, it is interesting to see how a fixed $Z_s^2 = 1$ dispersion relation involving only physical s-channel regions can be written. Consider for example a dispersion relation for $Z_s = -1$. (In the following, we discuss the inelastic case since in the elastic case $Z_s = 1$ corresponds only to one value of t, i.e., zero.) Designating the Born contribution by $\tilde{A}^B$, we have

$$\tilde{A}(t, s(t))|_{Z_s=-1} = \tilde{A}^B(t, s(t))|_{Z_s=-1} + \frac{1}{\pi}\int_{t_2}^{\infty} dt' \frac{\operatorname{Im}\tilde{A}(t', s(t'))}{t'-t}\bigg|_{Z_s=-1} + \frac{1}{\pi}\int_{-\infty}^{t_0} dt' \frac{\operatorname{Im}\tilde{A}(t', s(t'))}{t'-t}\bigg|_{Z_s=-1} + \frac{1}{\pi}\int_{t_0}^{0} dt' \frac{\operatorname{Im}\tilde{A}(t', s(t'))}{t'-t}\bigg|_{Z_s=-1},$$

where t_0 is the t value for which $s = s_{\text{threshold}}$. Whereas for $Z_s = -1$, $t < t_0$ corresponds to physical scattering, $t_0 < t < 0$ does not. In fact, to maintain $Z_s = -1$ as t increases through t_0, one must slide through the kinematical s cut to the second sheet where $p_s p_s'$ is negative [cf. Eq. (2)]. However, since reversing the sign of both Z_s and $p_s p_s'$ leaves t invariant, and $\tilde{A}$ is the same on all kinematical s sheets,

$$\tilde{A}(s, p_s p_s' < 0, Z_s = -1) = \tilde{A}(s, t) = \tilde{A}(s, p_s p_s' > 0, Z_s = +1),$$

we can replace the $\operatorname{Im}\tilde{A}(t', s(t'))|_{Z_s=-1}$ evaluated on the second s sheet by its value on the first s sheet evaluated at $Z_s = +1$. Thus, the final two integrals can be combined into one involving the discontinuity of $\tilde{A}$ along the boundary of the physical region. The superscript $Z_s = -1$ is unnecessary for the integration over the dynamical t cut since the path of integration is along $Z_t = -1$ for both $Z_s = \pm 1$.

Similarly, if we had begun by writing a forward dispersion relation, we would have found the path of integration again changing kinematical s sheets, but, again by using the fact that the value of the amplitude is the same on both s sheets, we would have obtained the same expression. Consequently, we can write our dispersion relation,

$$\tilde{A}(t, s(t)) = \tilde{A}^B(t, s(t)) + \frac{1}{\pi}\int_{t_2}^{\infty} dt' \frac{\operatorname{Im}\tilde{A}(t', s(t'))}{t'-t} + \frac{1}{\pi}\int_{-\infty}^{0} dt' \frac{\operatorname{Im}\tilde{A}(t', s(t'))}{t'-t}, \tag{15}$$

where

$$s(t) = \tfrac{1}{2}(\Sigma - t - 4p_t p_t'),$$

$\tilde{A}$ is defined by Eq. (11), and the second integral is evaluated along the boundary of the physical s channel. This relationship, which holds for the general reaction $ab \to cd$ with $m_b = m_d$, reduces in the case of $m_a = m_c$ to the familiar backward dispersion relation. Figure 3 illustrates the path of integration for pion photoproduction.

B. Dispersion Relations for $Z_t \neq -1$

The dispersion relation obtained in Sec. III A involved $Z_t = -1$. We saw in Sec. II D that as long as $\nu \equiv s - u$ is proportional to $p_t p_t'$, $\tilde{A}$ is free of kinematical singularities. For $\nu \leq -4p_t p_t'$ the integrations over the dynamical s cut will involve

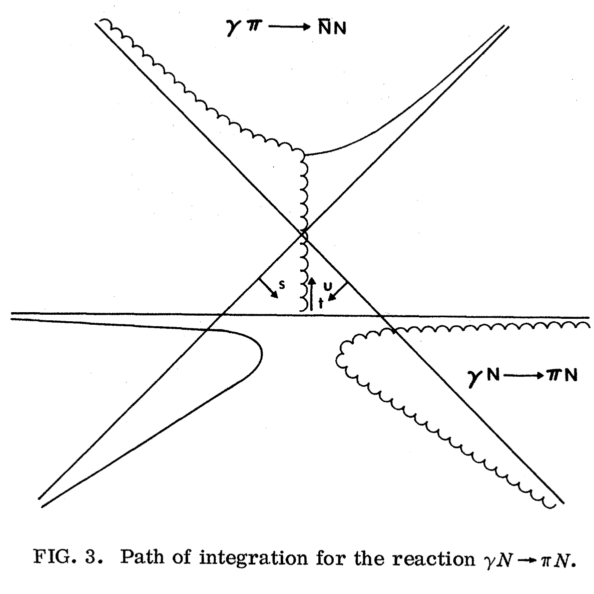

FIG. 3. Path of integration for the reaction $\gamma N \to \pi N$.

only contributions from physical regions. In such a case the dispersion equation would be the same with $s(t)$ given by

$$s(t)=\tfrac{1}{2}(\Sigma-t-4p_t p_t' c) \tag{16}$$

with $c\geq 1$. However, the integration over the dynamical t-cut values of $c=-Z_t\geq 1$ would always be outside of the physical crossed-channel scattering region. The evaluation of the t-channel integrand by, say, partial-wave expansions would therefore be questionable.

IV. CONCLUSION

We have developed a dispersion relation for analysis of reactions of the type $ab\rightarrow cd$ where $m_b=m_d$. The integral involving the direct channel includes no contributions from unphysical regions, but requires a knowledge of the scattering amplitude at physical energies for both forward and backward scattering.

There may now be enough data for $\gamma N\rightarrow \pi N$ at low and medium energies to allow evaluating the integral, assuming of course the existence of at least moderately successful high-energy models for forward and backward production.[13] The advent, in the near future, of meson factories may also provide sufficient data for the evaluation of the integral for pion-induced reactions such as $\pi N\rightarrow \rho N$, $\pi N\rightarrow \omega N$. As is usual in such analysis, the t-cut integral may be approximated by a sum of nearby t channel poles. Consequently, assuming the direct-channel integral can be reasonably approximated by data and high-energy fits, one may determine parameters for the t-channel poles, e.g., ρ in $\gamma N\rightarrow \pi N$. Calculations of this nature are being pursued by the authors and will be reported later.

By judicious choice of the t value, one may also hope to derive sum rules for particular amplitudes from which low-energy parameters can be extracted or high-energy models tested. In particular, sum rules which involve Regge amplitudes in the forward and backward directions may provide interesting insight into mechanisms employed to retain behavior when $Z_t^2=1$.[14] By realizing that for inelastic reactions $t=0$ corresponds to infinite s, one obtains the sum rule

$$\int_{-\infty}^{\infty} dt\,\mathrm{Im}\tilde{A}(t,s(t))/t=0\,, \tag{17}$$

where we have included the Born terms symbolically in the integral. The dispersion relation can also be evaluated at t_0, $s(t_0)=s_{\text{threshold}}$,

$$\tilde{A}(t_0,s_{\text{threshold}})=\tilde{A}^B(t_0,s_{\text{threshold}})+\frac{1}{\pi}\int_{-\infty}^{0}dt'\frac{\mathrm{Im}\tilde{A}(t',s(t'))}{t'-t_0}+\frac{1}{\pi}\int_{t_2}^{\infty}dt'\frac{\mathrm{Im}\tilde{A}(t',s(t'))}{t'-t_0}\,, \tag{18}$$

to give a sum rule for the combination of scattering lengths that remain when the amplitude is evaluated at threshold.

One could subtract a boundary dispersion relation from a fixed-u, $u=0$, dispersion relation, which, having the same contribution for $t\rightarrow\pm\infty$, would reduce the asymptotic t contributions.[15] For each value of u which intersects the forward direction and for which a fixed-u dispersion relation can be written, one has a sum rule relating the more rapidly converging integrals to crossed-channel poles.

Similarly, one could consider comparing the expression for $\tilde{A}$ obtained from a fixed-t dispersion relation evaluated on the boundary to that from our boundary dispersion relation. Probably the most interesting point for such a sum rule would be at $t=t_0$ where the fixed-t dispersion relation would not involve unphysical contributions.

Generalizations of this dispersion relation to fixed values of Z_t other than -1 may also be of some value, although it is not clear how the t-channel contributions can easily be evaluated. In particular, with large $Z_t^2>1$, higher partial waves become important. A Regge-pole model will not suffice for positive t since little is known about the discontinuities of Regge residues and trajectories.

In conclusion, it appears that the new dispersion relation obtained in this paper will become another useful tool for analyzing data.

ACKNOWLEDGMENTS

The authors would like to express their appreciation to Dr. W. Schmidt and Dr. F. Steiner for valuable discussions and to Professor Dr. G. Höhler for his gracious hospitality at the Institute for Theoretical Nuclear Physics, University of Karlsruhe.

---

*Supported by Bundesministerium für Bildung und Wissenschaft.

†Now at the University of Kaiserslautern, Kaiserslautern, Germany.

‡Guest Professor on leave from Arizona State University, USA. Permanent address: Department of Physics, Arizona State University, Tempe, Ariz. 85281.

[1]J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963); D. Amati and S. Fubini, Ann. Rev. Nucl. Sci. 12, 359 (1962).

[2]See, for example, R. J. Eden, *High Energy Collisions of Elementary Particles* (Cambridge Univ. Press, New York, 1967), Sec. 3.1.

[3]G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957).

[4]G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1345 (1957).

[5]J. Engels, A. Müllensiefen, and W. Schmidt, Phys. Rev. 175, 1951 (1968) and references therein.

[6]G. Höhler, J. Baacke, and R. Strauss, Phys. Letters 21, 223 (1966); G. Höhler, H. G. Schlaile, and R. Strauss, Z. Physik 229, 217 (1969); G. Höhler and R. Strauss, *ibid.* 240, 377 (1970).

[7]S. Ciulli, Nuovo Cimento 61A, 787 (1969); 62A, 301 (1969); H. Pfister, Nucl. Phys. B20, 320 (1970).

[8]A. V. Efremov and V. A. Meshcheryakov, Zh. Eksperim. i Teor. Phys. 39, 1099 (1960) [Soviet Phys. JETP 12, 766 (1961)]; D. Atkinson, Phys. Rev. 128, 1908 (1962); C. Lovelace, R. M. Heinz, and A. Donnachie, Phys. Letters 22, 332 (1966); J. Engels, G. Höhler, and B. Petersson, Nucl. Phys. B15, 365 (1970). See also A. V. Efremov, V. A. Meshcheryakov, and O. V. Shirkov, Zh. Eksperim. i Teor. Phys. 39, 438 (1960) [Soviet Phys. JETP 12, 308 (1961)] for work on fixed general θ_s dispersion relations for $\pi N \to \pi N$.

[9]G. Białkowski and A. Jurewicz, Ann. Phys. (N.Y.) 31, 436 (1965); A. Jurewicz, Z. Physik 207, 393 (1967).

[10]See Höhler and Strauss, Ref. 6.

[11]The ± superscripts are isospin labels.

[12]E.g., D. Atkinson, Phys. Rev. 128, 1908 (1962).

[13]See P. Noelle, W. Pfeil, and D. Schwela, Nucl. Phys. B26, 461 (1971), for a recent analysis of and references to photoproduction data.

[14]See P. D. B. Collins and E. J. Squires, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, Berlin, 1968), Vol. 45, for a review of this problem.

[15]The use of fixed-u dispersion relations would reintroduce contributions from unphysical regions.

PHYSICAL REVIEW D VOLUME 5, NUMBER 2 15 JANUARY 1972

Bjorken Scaling For Inclusive and Quasi-Inclusive Processes in the Dual-Resonance Model

M. Chaichian
Mathematics Department, Durham University, Durham, United Kingdom
(Received 14 July 1971)

From a dual-resonance model with currents included through the minimal gauge interaction, the deep-inelastic electron scattering is considered. It gives the Bjorken scaling for νW_2 while W_1 vanishes, a property which is common among the models where the current is coupled to bosons. Scaling occurs because of the existence of a current-algebra fixed pole. Deep-inelastic electron scattering with one detected final-state particle is also considered which, following Mueller, is connected with the discontinuity of a six-point amplitude. In a special kinematic region, three out of four structure functions scale because of a fixed pole, while outside this region the fixed pole cannot be responsible for scaling anymore. Then a speculation of Pomeranchukon assignment does show the scaling. Inelastic Compton scattering is also considered which, in the parton model of Bjorken and Paschos, scales and is proportional to νW_2. This property is satisfied in the present model. Electron-positron annihilation into hadrons is considered without renormalization whose cross section falls off as $s^{-5/2}$. It is suggested that a proper dual renormalization for the self-mass diagram of the photon may change this result.

I. INTRODUCTION

There exists the possibility that the dual-resonance model (DRM)[1,2] may finally provide us with a theory of hadronic processes. The unitarization program to treat the model as a Born term[3] puts it on the same footing as a field-theory expansion. Further, besides some quantitative agreements of the model with the data, it reproduces some qualitative features of the hadronic inclusive reactions[4,5] such as the Feynman scaling law,[6] pionization, limiting fragmentation,[7] small transverse momentum of the produced particles, etc.[8] However, notice that these limiting distributions are obtained when one puts the intercept of the relevant Regge trajectory α_0 equal to unity, i.e., the Pomeranchukon is exchanged, while in the case of the usual Regge trajectories with $\alpha_0 \neq 1$ exchanged, one gets scaling (generalized) only for the ratio of the differential cross section to the total cross section.[5,8] The above-mentioned successes of DRM in purely hadronic exclusive and inclusive reactions are certainly interesting both theoretically and phenomenologically.

In the processes where currents are involved there exists a "similar" kind of scaling behavior, namely the one originally predicted by Bjorken[9] for the deep-inelastic electroproduction structure