K Matrix for Three-Particle Scattering*

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(Received 16 August 1971)

The realization of the three-body K-matrix formalism in terms of Faddeev-type scattering integral equations is established. A prescription is suggested for handling possible resonance-pole singularities in the two-particle K matrices which enter as input into these equations. As an application of the formalism a fully unitary impulse approximation is developed.

I. INTRODUCTION

The usefulness of the K matrix as a unitarization device needs little commentary. What is remarkable is that integral equations of the Faddeev type for determing this quantity in nonrelativistic three-particle scattering theory have apparently not been developed except in special cases.[1,2] We will establish such a set of equations and discuss their possible applications.

In Sec. II we present a general characterization of a variety of K-matrix formalisms. This will illustrate the essential features of these techniques unencumbered by the detailed realizations of the various operators as solutions of well-defined scattering integral equations. The last topic is taken up in Sec. III, where we develop a detailed formalism for constructing fully unitary approximations in three-particle scattering.

Recently Cahill[3] has proposed a unitary formalism which is similar in spirit and in execution to that contained in Sec. III. However, it is not, strictly speaking, a formalism for the K matrix itself. The differences between our approach and results and those of Cahill are also examined in Sec. III.

Section IV contains a detailed exposition of the simplest application of the general formalism of Sec. III. In nucleon-deuteron scattering this example amounts to the full unitarization of the impulse and one-nucleon-exchange graphs. It is possible that this particular case will turn out to be the most useful in practice.

In Sec. V we will comment upon a difficulty which seems to have been overlooked in previous three-particle formalisms[1,3,4] which involve two-particle K matrices as input into three-body equations. We refer here to the interpretation of the possible resonance-type pole singularities which may appear in the two-particle K matrices for positive parametric energies. We will suggest a practical procedure for handling these singularities when exploiting a typical K-matrix formalism.

II. GENERAL K-MATRIX FORMALISM

It will prove very useful to consider first the K-matrix formalism in a general context with no concern about the detailed realizations of the various relevant operators in terms of elementary interactions. We shall do this in the Heisenberg picture. Also, we will have in mind throughout this paper the standard three-particle nonrelativistic scattering problem. Namely, there exists no true particle production; the asymptotic two-particle states consist merely of a free particle plus a (two-body) bound-state configuration of the other two. We have, of course, the possibility of a transition either to another two-particle state or to a three-free-particle state; finally, we have transitions from three-free-particle states to states of the same type or to two-particle states.

The unitarity of the scattering operator S,

$$S = 1 - 2\pi i T ,$$

implies that the transition operator T satisfies the unitary constraints

$$T - T^\dagger = -2\pi i T^\dagger T = -2\pi i T T^\dagger . \tag{2.1}$$

We define the (full) K operator as the solution of the equations

$$K = T + i\pi K T = T + i\pi T K . \tag{2.2}$$

If we were to regard K as known with T determined by (2.2), we see that T will satisfy Eqs. (2.1) if and only if K is Hermitian.

However, more than just the Hermiticity of K is required if it is to yield a correctly structured transition operator. The crucial constraint, besides unitarity, is that T yield a correct disconnected structure for the amplitude corresponding to the 3-3 process. We will investigate later in detail what structure K must have in order to ensure this property of T.

It is often very convenient in applications[2,5-7] to introduce a reduced K operator, K_p, defined by

$$K_p = T + i\pi TPK_p$$
$$= T + i\pi K_p PT, \tag{2.3}$$

where P is a projection operator. Equations (2.1) do not imply that K_p is Hermitian but, in fact, that[7]

$$K_p - K_p^\dagger = -2\pi i K_p Q K_p^\dagger$$
$$= -2\pi i K_p^\dagger Q K_p, \tag{2.4}$$

where

$$Q = 1 - P.$$

The connection between K_p and K is given by

$$K_p = K - i\pi KQK_p$$
$$= K - i\pi K_p QK. \tag{2.5}$$

We now specialize the preceding general formalism to the situation appropriate to the nonrelativistic three-particle problem in the case in which any pair of particles can form a bound state. For the sake of notational simplicity we will assume that there is no more than one bound state per pair of particles. The channels in this problem are then designated by the asymptotic configurations of a noninteracting two-particle state comprised of a particle α (= 1, 2, 3) moving freely and a bound state of the other two or ($\alpha = 0$) a state of three noninteracting particles.

The in (+) and out (−) states corresponding to the asymptotic channel α will be denoted by $|\psi_\alpha^{(\pm)}(\eta_\alpha)\rangle\rangle$ where η_α refers to any other labels which are needed to specify the asymptotic configuration. One choice for P is

$$P = \sum_{\alpha \neq 0} \sum_{\eta_\alpha} |\psi_\alpha^{(\pm)}(\eta_\alpha)\rangle\rangle\langle\langle\psi_\alpha^{(\pm)}(\eta_\alpha)| \tag{2.6a}$$

so that

$$Q = \sum_{\eta} |\psi_0^{(\pm)}(\eta)\rangle\rangle\langle\langle\psi_0^{(\pm)}(\eta)|, \tag{2.6b}$$

where we have omitted all single-particle states (including any possible three-particle bound states) from Q since the T and K operators have a null effect on these states by virtue of the stability condition on S.

The choice (2.6a) seems to be most convenient if one is interested in reactions initiated from a two-particle asymptotic state.[2,7] Another choice in which the roles of P and Q as given by Eqs. (2.6) are interchanged has been proposed in Ref. 1. This latter case does not appear to possess any practical advantages. In what follows we will assume that P and Q are given by Eqs. (2.6).

If we are solely interested in the transitions $\alpha \to \beta$ where $\alpha \neq 0$, then we need only obtain the operators PTP and QTP. If, in addition, K is regarded as known, there are two alternative methods of applying the K-matrix formalism to obtain the latter projections of T. First, one can use Eq. (2.2) directly to obtain

$$PTP = PKP - i\pi(PKP)(PTP) - i\pi(PKQ)(QTP), \tag{2.7a}$$

$$QTP = QKP - i\pi(QKP)(PTP) - i\pi(QKQ)(QTP), \tag{2.7b}$$

which are two coupled integral equations for the desired quantities.

The second method exploits the K_p formalism. In this case we obtain from Eqs. (2.3) and (2.5)

$$PTP = PK_pP - i\pi(PK_pP)(PTP), \tag{2.8a}$$

$$QTP = (QK_pP)(1 - i\pi PTP), \tag{2.8b}$$

where

$$PK_pP = PKP - i\pi(PKQ)(QK_pP) \tag{2.8c}$$

and

$$QK_pP = QKP - i\pi(QKQ)(QK_pP). \tag{2.8d}$$

We observe that the K_p method is in effect a technique for uncoupling Eqs. (2.7). With Eqs. (2.8) one has only one difficult, but uncoupled, equation to solve, namely, (2.8d). Equation (2.8a) reduces to a set of algebraic equations after a partial-wave decomposition. All other quantities are computed by quadrature. The practical K-matrix formalism to be developed in Sec. III is based upon Eqs. (2.8).

Next let us consider the disconnected structure of QTQ and QKQ. Let us imagine, for the sake of simplicity, that there are no two-particle bound states so that the only possible physical process is the 3-3 (so $Q = 1$). Let us write

$$T = T_d + T_c, \tag{2.9}$$

where T_d (T_c) is that portion of T which consists of disconnected (connected) diagrams when one considers matrix elements of T with respect to states $|\psi_0^{(\pm)}(\eta)\rangle\rangle$ labeled by the three free-particle momenta. T_d, of course, decomposes into the sum

$$T_d = \sum_{\alpha=1}^{3} T_d^\alpha, \tag{2.10}$$

where T_d^α gives rise to the disconnected diagram with particle α noninteracting. Products such as $T_d^\alpha T_d^\alpha$ etc. will still be disconnected. The T_d^α satisfy the two-body unitarity constraints

$$T_d^\alpha - T_d^{\alpha\dagger} = -2\pi i T_d^\alpha T_d^{\alpha\dagger}$$
$$= -2\pi i T_d^{\alpha\dagger} T_d^\alpha. \tag{2.11}$$

Let us suppose that we decompose K in a manner similar to Eqs. (2.9) and (2.10). Substituting these decompositions into Eq. (2.2) and equating the connected and disconnected parts, we find

$$T_c = K_c(1 - i\pi T_d) - i\pi[K_d T_d]_c - i\pi K T_c , \tag{2.12a}$$

$$T_d = K_d - i\pi[K_d T_d]_d . \tag{2.12b}$$

Here

$$[K_d T_d]_d = \sum_\alpha K_d^\alpha T_d^\alpha$$

and

$$[K_d T_d]_c = \sum_{\alpha,\beta} K_d^\alpha \bar{\delta}_{\alpha\beta} T_d^\beta ,$$

where

$$\bar{\delta}_{\alpha\beta} = 1 - \delta_{\alpha\beta} .$$

Equation (2.12b) obviously implies that

$$T_d^\alpha = K_d^\alpha - i\pi K_d^\alpha T_d^\alpha \tag{2.13}$$

so that the disconnected parts of K must be related to those of T by a two-particle Heitler equation. Equations (2.11) and (2.13) imply that

$$K_d^\alpha = K_d^{\alpha\dagger} ,$$

so that

$$K_c = K_c^\dagger$$

as well.

The part of K that is arbitrary, apart from the Hermiticity requirement, is evidently K_c. We can imagine a model for the 3-3 process which is obtained by taking $K_c \equiv 0$. The connected part of T in this instance is determined from

$$T_c = -i\pi \sum_{\alpha,\beta} K_d^\alpha \bar{\delta}_{\alpha\beta} T_d^\beta - \left(\sum_\alpha K_d^\alpha\right) T_c .$$

It is clear from (2.3) that the disconnected parts of T and K_p must be identical. This has been pointed out previously.[7]

For the discussion of Sec. III it will be convenient to introduce the counterparts of T, K, and K_p in the interaction picture. The connection between the two types of operators is defined, for example, by

$$\langle \psi_\beta^{(-)}(\eta_\beta, E_\beta) | T | \psi_\alpha^{(+)}(\eta_\alpha, E_\alpha)\rangle = \delta(E_\beta - E_\alpha)\langle \phi_\beta(\eta_\beta, E_\beta) | T_{\beta\alpha} | \phi_\alpha(\eta_\alpha, E_\alpha)\rangle , \tag{2.14}$$

for states of well-defined energies. The channel states $|\phi_\alpha\rangle$ for $\alpha = 1, 2, 3$ refer to a noninteracting two-particle state comprised of a particle α moving freely and a bound state of the other two; $|\phi_0\rangle$ corresponds to a three-particle plane-wave state.

We can rewrite Eq. (2.2) in terms of interaction-picture operators as

$$T_{\beta\alpha} = K_{\beta\alpha} - i\sum_\gamma K_{\beta\gamma} D_\gamma T_{\gamma\alpha} , \tag{2.15}$$

where

$$D_\gamma \equiv \pi \sum_{E'_\gamma, \eta_\gamma} |\phi_\gamma(\eta_\gamma, E'_\gamma)\rangle\rangle \delta(E - E') \langle\langle \phi_\gamma(\eta_\gamma, E'_\gamma)| .$$

Similarly,

$$T_{\beta\alpha} = (K_p)_{\beta\alpha} - i \sum_{\gamma\neq 0} (K_p)_{\beta\gamma} D_\gamma T_{\gamma\alpha} . \tag{2.16}$$

Equations (2.15) and (2.16) should be regarded as on-shell equations sandwiched between the states $\langle\langle \phi_\beta(\eta_\beta, E)|$ and $|\phi_\alpha(\eta_\alpha, E)\rangle\rangle$. Finally, we remark that the Hermiticity of K implies that

$$K_{\beta\alpha} = K_{\alpha\beta}^\dagger \tag{2.17}$$

for the interaction-picture operators.

III. K-MATRIX INTEGRAL EQUATIONS

We will now determine off-shell extensions of the operators $T_{\beta\alpha}$, $K_{\beta\alpha}$, and $(K_p)_{\beta\alpha}$ which are determined via Faddeev-type scattering integral equations. Our starting point will be the Alt, Grassberger, and Sandhas[8] formalism in which the scattering operators $U(z)$ satisfy

$$U(z) = \bar{\delta} G_0(z)^{-1} + \bar{\delta} t(z) G_0(z) U(z) = \bar{\delta} G_0(z)^{-1} + U(z) G_0(z) t(z) \bar{\delta} , \tag{3.1}$$

where we have employed the usual matrix notation with respect to the channel indices.[9] That is, $U(z)$ represents the 4×4 matrix whose elements are the operators $U_{\beta\alpha}(z)$, $t(z)$ is a diagonal matrix whose elements are the two-particle operators (defined in the three-particle Hilbert space) $t_\alpha(z)$, for $\alpha \neq 0$ and $t_0(z) \equiv 0$, and $\bar{\delta}$ is the matrix with elements $1 - \delta_{\beta\alpha}$. The free three-particle propagator is denoted by $G_0(z)$ where z is the (complex) parametric energy. For $E_\beta = E_\alpha$ and $\epsilon \to 0+$,

$$\langle \phi_\beta(\eta_\beta, E_\beta) | T_{\beta\alpha} | \phi_\alpha(\eta_\alpha, E_\alpha)\rangle = \langle \phi_\beta(\eta_\beta, E_\beta) | U_{\beta\alpha}(E_\alpha + i\epsilon) | \phi_\alpha(\eta_\alpha, E_\alpha)\rangle . \tag{3.2}$$

Now $U(z)$ is an operator analytic in the z plane cut from the lowest scattering threshold to positive infinity except for poles corresponding to three-body bound states at energies below any scattering threshold. The discontinuity relations for $U(z)$ across the cut are related to the physical unitarity constraints (2.1) as we will show below and is also well known. The unitary cut has a composite structure consisting of the cuts generated by the two-particle bound-state poles and the three-particle scattering cut. The rationale of the development to follow is to construct scattering operators which are continuous across one or both of these two classes of subcuts.

We recall[9] that

$$t(z)^\dagger = t(z^*)$$

and

$$G_0(z)^\dagger = G_0(z^*) ,$$

so

$$U(z)^\dagger = U(z^*). \tag{3.3}$$

In connection with the last relation we point out that the adjoint operation includes a transposition with respect to the channel indices; thus for the individual components of $U(z)$ (3.3) implies

$$U_{\beta\alpha}(z)^\dagger = U_{\alpha\beta}(z^*).$$

If we define

$$U(\pm) = [U(z)]_{z=E\pm i\epsilon}, \tag{3.4}$$

where E is real and above the threshold for scattering, the discontinuity ΔU of $U(z)$ across the unitary cut in the z plane is given by

$$\Delta U \equiv U(+) - U(-) = U(+) - U(+)^\dagger. \tag{3.5}$$

The notation introduced in Eqs. (3.4) and (3.5) will be employed for operators other than $U(z)$. Actually it will, perhaps, be more consistent to use the adjoint form of (3.5) because some of the auxiliary operators we will introduce are not necessarily the boundary values of operator-analytic quantities.

The unitarity constraints on the two-particle transition operators can be written in the form

$$\Delta t = -2it(\pm)D_0 t(\mp) + \Delta t_b, \tag{3.6}$$

where Δt_b represents the contribution to Δt arising from the two-particle bound-state poles. If one uses (3.6) and the fact that

$$G_0(\pm)(\Delta t_b)_\alpha G_0(\mp) = G_0(\pm)(\Delta t_b)_\alpha G_0(\pm) = -2iD_\alpha,$$

it follows from Eqs. (3.4) that the on-shell discontinuity equation for U is[9]

$$\Delta U = -2iU(\pm)DU(\mp), \tag{3.7}$$

where D is, of course, the diagonal matrix with elements $D_\alpha\delta_{\alpha\beta}$. Equations (3.7) are entirely equivalent to the physical unitarity constraints (2.1) as follows by use of Eqs. (2.14) and (3.2).[9]

Next we will express $U(z)$ in terms of an operator $\overline{U}(z)$ which is continuous across the portion of the unitarity cut generated by the two-particle bound-state poles.[2,7,9] In order to do this we decompose $t(z)$ into the sum

$$t(z) = \overline{t}(z) + t_b(z). \tag{3.8}$$

Here $[t_b(z)]_\alpha$ is that part of the bound-state pole contained in $t_\alpha(z)$ which gives rise to a Dirac δ function when $z = E + i\epsilon$; the form of $[t_b(z)]_\alpha$ for arbitrary z is given in Ref. 9. Then using the technique of Alt *et al.*[8,9] we find that

$$\begin{aligned} U(z) &= \overline{U}(z) + \overline{U}(z)G_0(z)t_b(z)G_0(z)U(z) \\ &= \overline{U}(z) + U(z)G_0(z)t_b(z)G_0(z)\overline{U}(z), \end{aligned} \tag{3.9}$$

where $\overline{U}(z)$ satisfies

$$\begin{aligned} \overline{U}(z) &= \overline{\delta}G_0(z)^{-1} + \overline{\delta}\,\overline{t}(z)G_0(z)\overline{U}(z) \\ &= \overline{\delta}G_0(z)^{-1} + \overline{U}(z)G_0(z)\overline{t}(z)\overline{\delta}. \end{aligned} \tag{3.10}$$

For $z = E \pm i\epsilon$ we have[9]

$$[G_0(\pm)t_b(\pm)G_0(\pm)]_\alpha = \mp iD_\alpha, \quad \alpha \neq 0$$

so that

$$\begin{aligned} U_{\beta\alpha}(\pm) &= \overline{U}_{\beta\alpha}(\pm) \mp i \sum_{\gamma\neq 0} \overline{U}_{\beta\gamma}(\pm)D_\gamma U_{\gamma\alpha}(\pm) \\ &= \overline{U}_{\beta\alpha}(\pm) \mp i \sum_{\gamma\neq 0} U_{\beta\gamma}(\pm)D_\gamma \overline{U}_{\gamma\alpha}(\pm). \end{aligned} \tag{3.11}$$

If we compare Eqs. (2.16) with Eqs. (3.11) we see that $\overline{U}_{\beta\alpha}(+)$ is an off-shell extension of $(K_\rho)_{\beta\alpha}$.

Now $\overline{t}(z)$ satisfies[9]

$$\Delta\overline{t} = -2i\overline{t}(\pm)D_0\overline{t}(\mp), \tag{3.12}$$

which is the usual two-particle off-shell unitarity relation with no bound-state pole contributions. Using (3.12) one finds from Eqs. (3.10) that on shell[9]

$$(\Delta\overline{U})_{\beta\alpha} = -2i\overline{U}_{\beta 0}(\pm)D_0\overline{U}_{0\alpha}(\mp) \tag{3.13}$$

which demonstrates explicitly that $\overline{U}(z)$ has a discontinuity only across the three-particle cut. The $\overline{U}(z)$ equations correspond to a three-body scattering theory in which the only physical process is the 3-3 transition.

Henceforth, we will require only the strictly on-shell forms of Eqs. (3.7), (3.11), and (3.13), for example,

$$\begin{aligned} &\langle\phi_\beta(\eta_\beta, E)|[U_{\beta\alpha}(\pm) - \overline{U}_{\beta\alpha}(\pm)]|\phi_\alpha(\eta_\alpha, E)\rangle \\ &\quad = \mp i\pi \sum_{\gamma\neq 0} \sum_{\eta_\gamma, E'_\gamma} \langle\phi_\beta(\eta_\beta, E)|\overline{U}_{\beta\gamma}(\pm)|\phi_\gamma(\eta_\gamma, E'_\gamma)\rangle\delta(E'_\gamma - E)\langle\phi_\gamma(\eta_\gamma, E'_\gamma)|U_{\gamma\alpha}(\pm)|\phi_\alpha(\eta_\alpha, E)\rangle. \end{aligned} \tag{3.14}$$

It is evident then that if the $U_{\beta\alpha}$ (on shell) are defined in terms of the $\overline{U}_{\beta\alpha}$ (on shell) by (3.14) and if the $\overline{U}_{\beta\alpha}$ satisfy (3.13), then the $U_{\beta\alpha}$ satisfy Eqs. (3.7) and one has a fully unitary theory. This is, of course, already obvious from Sec. II once we have made the identifications of $T_{\beta\alpha}$ with $U_{\beta\alpha}(+)$ and $(K_\rho)_{\beta\alpha}$ with $\overline{U}_{\beta\alpha}(+)$ on shell.

Our next step is to define $\overline{U}(z)$, or equivalently

K_p, in terms of operators which are continuous across the entire unitary cut and which are defined in terms of the solutions of Faddeev-type integral equations. These operators will be shown to be, essentially, off-shell extensions of the full K matrix, $K_{\beta\alpha}$.

If we note that

$$\bar{t}(z)G_0(z)\bar{U}(z)=\Gamma(z)\bar{\delta},$$

where

$$\begin{aligned}\Gamma(z)&=\bar{t}(z)+\bar{t}(z)\bar{\delta}G_0(z)\Gamma(z)\\&=\bar{t}(z)+\Gamma(z)\bar{\delta}G_0(z)\bar{t}(z),\end{aligned}\tag{3.15}$$

then we can express $\bar{U}(z)$ as

$$\bar{U}(z)=\bar{\delta}G_0(z)^{-1}+\bar{\delta}\Gamma(z)\bar{\delta}.\tag{3.16}$$

We point out that $\Gamma_{0\beta}=\Gamma_{\beta 0}=0$ for all β. The discontinuity of the auxiliary operator Γ across the three-particle cut is

$$\Delta\Gamma=-2i\Gamma(\pm)D_0(1+\bar{\delta})\Gamma(\mp)\tag{3.17}$$

or, in component form,

$$(\Delta\Gamma)_{\beta\alpha}=-2i\Big(\sum_{\sigma}\Gamma_{\beta\sigma}(\pm)\Big)D_0\Big(\sum_{\lambda}\Gamma_{\lambda\alpha}(\mp)\Big).\tag{3.17'}$$

Let us suppose that $\bar{U}(z)$ is defined, on shell, by (3.16) and that $\Gamma(z)$ satisfies Eqs. (3.17) on shell. Thus, again on shell,

$$\Delta\bar{U}=\bar{\delta}\Delta\Gamma\bar{\delta},$$

which in component form becomes

$$(\Delta\bar{U})_{\beta\alpha}=-2i[\bar{\delta}\Gamma(\pm)\bar{\delta}]_{\beta 0}D_0[\bar{\delta}\Gamma(\mp)\bar{\delta}]_{0\alpha}.\tag{3.18}$$

Since $G_0(\pm)^{-1}D_0$ vanishes on shell, (3.18) is equivalent to Eqs. (3.13). We have made no progress thus far in obtaining an operator continuous across the unitary cut. The operator $\Gamma(z)$ will, however, prove to be a very convenient redefinition of $\bar{U}(z)$.

We next define two-particle K matrices as solutions of the Heitler equations

$$\begin{aligned}k&=\bar{t}(\pm)+i\bar{t}(\pm)D_0k\\&=\bar{t}(\pm)+ikD_0\bar{t}(\pm).\end{aligned}\tag{3.19}$$

As a consequence of (3.12) we find that

$$k^{\dagger}=k.\tag{3.20}$$

We reserve until Sec. V the discussion of the interpretation of Eqs. (3.19) and k itself when there exist resonance-type poles in the two-body K matrix; for the present discussion we will assume that the use of k is entirely well defined.

If we decompose $G_0(\pm)$ into its principal-value (G) and Dirac-δ-function ($\mp iD_0$) parts,

$$G_0(\pm)=G\mp iD_0,$$

we find using the Heitler expression (3.19) for $\bar{t}(\pm)$ that

$$\Gamma(\pm)=k[1\mp i(1+\bar{\delta})D_0\Gamma(\pm)]+k\bar{\delta}G\Gamma(\pm).$$

Therefore

$$\Gamma(\pm)=\kappa\mp i\kappa D_0(1+\bar{\delta})\Gamma(\pm)\tag{3.21a}$$

and similarly

$$\Gamma(\pm)=\kappa\mp i\Gamma(\pm)(1+\bar{\delta})D_0\kappa,\tag{3.21b}$$

where κ is defined as the solution of

$$\begin{aligned}\kappa&=k+k\bar{\delta}G\kappa\\&=k+\kappa G\bar{\delta}k.\end{aligned}\tag{3.22}$$

If Eq. (3.20) is satisfied, then κ is Hermitian or

$$\kappa^{\dagger}_{\beta\alpha}=\kappa_{\alpha\beta}.\tag{3.23}$$

Equations (3.21) and (3.22) are the central results of this paper.

Henceforth we regard Eqs. (3.21) as on-shell equations although not quite in the same sense as (3.14), for example. That is, Eqs. (3.21) are to be considered as involving matrix elements of $\Gamma_{\beta\alpha}$ and $\kappa_{\beta\alpha}$ with respect to states of energy E but not necessarily of the form $\langle\phi_\beta|\Gamma_{\beta\alpha}|\phi_\alpha\rangle$ for instance. The relationship (3.16) between $\bar{U}$ and Γ should be kept in mind in connection with these remarks.

Thus, if we are given, on shell, any Hermitian κ, we find from (3.21) that Eqs. (3.17) are satisfied. This completes the chain. Namely, if we specify the required κ, we can generate via Eqs. (3.21), (3.16), and (3.11) [or (3.14)], in that order, a set of physical transition amplitudes which satisfy full three-particle unitarity.

We observe that

$$\bar{\delta}\Gamma(+)\bar{\delta}=\bar{\delta}\kappa\bar{\delta}-i\bar{\delta}\kappa(1+\bar{\delta})D_0\Gamma(+)\bar{\delta}$$

or, in component form,

$$[\bar{\delta}\Gamma(+)\bar{\delta}]_{\beta\alpha}=(\bar{\delta}\kappa\bar{\delta})_{\beta\alpha}-i(\bar{\delta}\kappa\bar{\delta})_{\beta 0}D_0[\bar{\delta}\Gamma(+)\bar{\delta}]_{0\alpha}.\tag{3.24}$$

Since (3.24) is to be interpreted as an on-shell equation we can rewrite it as

$$\bar{U}_{\beta\alpha}(+)=K_{\beta\alpha}-iK_{\beta 0}D_0\bar{U}_{0\alpha}(+),\tag{3.25a}$$

where

$$K_{\beta\alpha}\equiv\bar{\delta}_{\beta\alpha}G_0(+)^{-1}+(\bar{\delta}\kappa\bar{\delta})_{\beta\alpha}.\tag{3.26}$$

Similarly

$$\bar{U}_{\beta\alpha}(+)=K_{\beta\alpha}-i\bar{U}_{\beta 0}(+)D_0K_{0\alpha}.\tag{3.25b}$$

If we compare Eqs. (3.25) with Eqs. (2.5), we see that $K_{\beta\alpha}$ can be identified as the complete K matrix (in the interaction picture). Equation (3.23) implies Eqs. (2.17).

Now

$$K_{00}=\sum_{\beta,\alpha=1}^{3}\kappa_{\beta\alpha}.$$

If we write

$$\kappa_{\beta\alpha} = k_\beta \delta_{\beta\alpha} + \kappa^c_{\beta\alpha},$$

where, if κ is determined by Eqs. (3.22), κ^c is given by

$$\kappa^c = k\bar{\delta}G\kappa = \kappa G\bar{\delta}k$$

and is a connected operator, then

$$\boldsymbol{K}_{00} = \sum_{\alpha=1}^{3}\kappa_\alpha + \sum_{\beta,\alpha=1}^{3}\kappa^c_{\beta\alpha}.$$

By virtue of the analysis of Sec. II we recognize $\kappa^c_{\beta\alpha}$ as the arbitrary portion of the $\boldsymbol{K}$ matrix. It is constrained only to satisfy

$$\kappa^{c\dagger}_{\beta\alpha} = \kappa^c_{\alpha\beta}$$

and to be connected.

It is easily shown that if κ is determined by (3.22), then

$$\kappa^c = kCk, \tag{3.27}$$

where C is defined as the solution of

$$C = \bar{\delta}G + \bar{\delta}GkC$$
$$= \bar{\delta}G + CkG\bar{\delta}.$$

C is the primordial operator in Cahill's[3] formalism. We note that C must satisfy

$$C^\dagger_{\beta\alpha} = C_{\alpha\beta}$$

in order to generate a unitary theory.

We will not explore in any detail the practical utilization of the set of equations (3.11), (3.16), and (3.21) once one is given κ except for the special case to be discussed in Sec. IV. Cahill[3] has shown that after a partial-wave analysis the most complicated case [in the present formalism Eq. (3.21)] reduces to a one-dimensional integral equation.

We close this section with a discussion of Cahill's[3] work as compared to that developed here. First of all, our general formalism essentially consists of a hierarchy of only three basic equations involving U, $\overline{U}$, Γ, and κ whereas his involves four. We point out in this connection that the relationship between $\overline{U}$ and Γ is quite trivial. Also as a consequence of employing the reduced K-matrix technique we have far less coupling of integral equations than is encountered in Ref. 3.

The basic dissimilarities between the formalism of this section and that of Ref. 3 appear to have their origin in the following circumstance. The development in Ref. 3 is committed at the outset to the structure

$$\sum_{\beta,\alpha} t_\beta(z) F_{\beta\alpha}(z) t_\alpha(z) \tag{3.28}$$

for the connected part of the 3-3 amplitude, where

$$F_{\beta\alpha}(z) = G_0(z) U_{\beta\alpha}(z) G_0(z).$$

Thereafter, the formulation is based upon the operators $F_{\beta\alpha}(z)$ and their discontinuity relations and this appears to complicate matters, comparatively speaking.

Since the present theory is entirely general, the results of Ref. 3 should follow as a special case upon imposing a further constraint on κ^c which will guarantee the structure (3.28) for the connected part of the 3-3 amplitude. We conjecture that this constraint is that κ^c have the form (3.27), but we have been unable to establish this. It seems likely that under the assumption of only two-body forces, in which case the structure (3.28) is valid, a unitary approximation procedure which imposes such a constraint will be superior to one that does not.

IV. FULLY UNITARY IMPULSE APPROXIMATION

Let us consider the simplest application of the formalism of Sec. III. Namely, we set $\kappa^c = 0$. Then Eq. (3.21) becomes

$$\Gamma = k - ikD_0(1+\bar{\delta})\Gamma, \tag{4.1}$$

where we have omitted the (+) notation with the understanding that $z = E + i\epsilon$ throughout this section. If we multiply (4.1) from the left by $1 - i\bar{t}D_0$, we obtain an equation

$$\Gamma = \bar{t} - i\bar{t}D_0\bar{\delta}\Gamma \tag{4.2}$$

entirely free of the two-particle K matrices.

With Eq. (4.2) we can derive an on-shell integral equation for determining $\overline{U}$ in an especially simple manner. If we define

$$\zeta \equiv \bar{\delta}\Gamma\bar{\delta},$$

then

$$\overline{U} = \bar{\delta}G_0^{-1} + \zeta.$$

From (4.2) we see that ζ satisfies

$$\zeta = \bar{\delta}\bar{t}\bar{\delta} - i\bar{\delta}\bar{t}D_0\zeta. \tag{4.3}$$

It is now possible to recognize the $\kappa^c = 0$ case as the complete unitarization of the impulse approximation ($\bar{\delta}\bar{t}\bar{\delta}$) plus the exchange term $\bar{\delta}G_0^{-1}$. This is the natural generalization of the Sloan approximation[2] which corresponds to taking the zeroth-order iteration of Eq. (4.3).

If one is only interested in scattering processes initiated from an initial two-particle channel, then one requires only the on-shell matrix elements of the form

$$\langle \phi_0 | \zeta_{\beta\alpha} | \phi_\alpha \rangle, \quad \alpha \neq 0$$

and

$$\langle \phi_\beta | \zeta_{\beta\alpha} | \phi_\alpha \rangle, \quad \beta, \alpha \neq 0.$$

The principal numerical problem involves the solution of (4.3) for the matrix elements of the first type with $\beta \neq 0$. These elements satisfy the integral equations

$$\langle \phi_0(\eta, E) | \zeta_{\beta\alpha} | \phi_\alpha(\eta_\alpha, E) \rangle = \langle \phi_0(\eta, E) | (\bar{\delta}\, \bar{t}\, \bar{\delta})_{\beta\alpha} | \phi_\alpha(\eta_\alpha, E) \rangle - i\pi \sum_\gamma \sum_{E', \eta'} \langle \phi_0(\eta, E) | (\bar{\delta}\, \bar{t})_{\beta\gamma} | \phi_0(\eta', E') \rangle \delta(E - E') \langle \phi_0(\eta', E') | \zeta_{\gamma\alpha} | \phi_\alpha(\eta_\alpha, E) \rangle . \tag{4.4}$$

Cahill[3] has shown that an on-shell integral equation of the form (4.4) reduces to a one-dimensional integral equation after a partial-wave decomposition.

The matrix elements $\langle \phi_\beta | \zeta_{\beta\alpha} | \phi_\alpha \rangle$ for $\alpha \neq 0$ which are needed to obtain the desired $\bar{U}$ amplitudes follow by quadrature:

$$\langle \phi_\beta(\eta_\beta, E) | \zeta_{\beta\alpha} | \phi_\alpha(\eta_\alpha, E) \rangle = \langle \phi_\beta(\eta_\beta, E) | (\bar{\delta}\, \bar{t}\, \bar{\delta})_{\beta\alpha} | \phi_\alpha(\eta_\alpha, E) \rangle - i\pi \sum_\gamma \sum_{E', \eta'} \langle \phi_\beta(\eta_\beta, E) | (\bar{\delta}\, \bar{t})_{\beta\gamma} | \phi_0(\eta', E') \rangle \delta(E - E') \langle \phi_0(\eta', E') | \zeta_{\gamma\alpha} | \phi_\alpha(\eta_\alpha, E) \rangle . \tag{4.5}$$

Using the amplitudes determined by (4.5) all physical processes except the 3-3 transition can be computed by solving Eqs. (3.14). The latter for the submatrix of nonbreakup transitions reduce to algebraic equations after a partial-wave analysis. Given these amplitudes the breakup amplitudes follow using (3.14) as a quadrature rule.

It is interesting to point out that the approximation described in this section is independent of any ambiguities in interpretation of the possible singularities in the two-particle K matrices. One can easily show that both Γ and ζ as defined by Eqs. (4.2) and (4.3), respectively, satisfy the correct discontinuity relations across the three-particle cut.

We also remark that a computational program based upon the approximation described in this section will possess no essential limitations arising from the complexity of the two-particle interactions.

V. SINGULARITIES IN THE TWO-PARTICLE K MATRICES

Let us return to the problem of the definition of the two-particle K matrices by Eq. (3.19). The definition of any two-particle operator such as $\bar{t}(z)$ or k in the three-particle Hilbert space necessarily involves an integration of the actual two-body operator with respect to its parametric energy. That is, if $|\Phi\rangle$ and $|\Psi\rangle$ are two three-body states corresponding to zero total linear momentum, then

$$\langle \Psi | \bar{t}_\alpha(z) | \Phi \rangle = \int d\vec{p}'_\alpha \int d\vec{p}_\alpha \int d\vec{q}_\alpha \times \langle \Psi | \vec{p}'_\alpha, \vec{q}_\alpha \rangle \langle \vec{p}'_\alpha | \bar{t}^{(2)}_\alpha(z - \omega_\alpha) | \vec{p}_\alpha \rangle \langle \vec{p}_\alpha, \vec{q}_\alpha | \Phi \rangle , \tag{5.1}$$

where the superscript (2) on $\bar{t}^{(2)}_\alpha$ refers to the fact that it is defined in the appropriate two-particle subspace. Also, $\vec{p}_\alpha$ is the relative momentum of the interacting pair and $\vec{q}_\alpha$ and ω_α are the momentum and energy, respectively, of the free particle in the three-particle c.m. frame. We recall that ω_α is proportional to $\vec{q}_\alpha^{\,2}$. We see then from Eq. (3.19) that an integration similar to (5.1) will be involved in the definition of k.

For negative parametric energies $k^{(2)}_\alpha$ and $\bar{t}^{(2)}_\alpha$ are identical. For both then the integration (5.1) over the bound-state pole singularity is carried out according to a principal-value prescription. However, for positive parametric energies $k^{(2)}_\alpha$ is identical with the ordinary off-shell two-particle K matrix and therefore may possess poles corresponding to those two-particle energies for which the phase shift is an odd-integer multiple of $\frac{1}{2}\pi$. Our essential problem is how to interpret the integration (5.1) over these resonance-type singularities.

It is evident that the most convenient interpretation of such a singularity relative to the integration (5.1) would be in the sense of a principal value. This would preserve the reality properties of the matrix elements of k_α as well as ensuring their definition. We will attempt to justify this interpretation.

The analytic properties of $k^{(2)}_\alpha$ are somewhat obscured primarily as a consequence of the usual identification of this operator, for positive parametric energies, with the solution of a singular integral equation with a principal-value prescription.[10] If, however, unlike the case of $t^{(2)}_\alpha(z)$, we do not necessarily identify $k^{(2)}_\alpha(z)$ with the solution of a scattering integral equation except for positive energies (i.e., on the cut) it may be possible to define a continuation of $k^{(2)}_\alpha$ to complex z. A continuation based on off-shell unitarity has been pro-

posed by Mishima.[11] Such considerations may enable us to interpret the singularities of $k_\alpha^{(2)}$ on the cut.

In order to obtain an idea of what is involved let us consider the case of the *on-shell* partial-wave amplitudes of $t^{(2)}(z)$ which we denote by $\tau_l(z)$. The presence or absence of bound-state poles will be irrelevant to the following discussion so that there will be no need to distinguish between t and $\bar{t}$. In the on-shell case it is possible to define an on-shell K matrix $k_l(z)$ which is meromorphic in a region enclosing the unitary cut and which is real analytic,

$$k_l(z)^* = k_l(z^*),$$

by

$$k_l(z) \equiv \tau_l(z)[1 - i\rho(z)\tau_l(z)]^{-1}. \tag{5.2}$$

Here $\rho(z)$ is a phase-space factor analytic in the z plane cut from 0 to $+\infty$ and such that

$$\rho(-) = -\rho(+),$$

with $\rho(+)$ real.

The most important feature of (5.2) for the purposes of the present discussion is that the $i\epsilon$ prescription for $\tau_l(z)$ is contained entirely in the phase-space factor. For example, suppose that $k_l(z)$ has the structure

$$k_l(z) = \theta_l(z)(z - E_R)^{-1}, \tag{5.3}$$

where $E_R > 0$ and $\theta_l(z)$ is a real analytic function which is continuous across the cut. In the limit $z = E + i\epsilon$, $\epsilon \to 0+$, we obtain the same $\tau_l(+)$ independently of whether we employ $k_l(z)$ or $k_l(z^*)$, for instance, in (5.2).

Also, if we were to integrate the Heitler equation

$$k_l(+) = \tau_l(+) + i\rho(+)\tau_l(+)k_l(+)$$

over an energy interval which includes the point E_R, we find using (5.3) that

$$\mathcal{P}\int dE \frac{f(E)\theta_l(E)}{E - E_R} = \int dE\, f(E)\tau_l(+) + i\mathcal{P}\int dE \frac{f(E)\rho(+)\tau_l(+)\theta_l(E)}{E - E_R},$$

where $\mathcal{P}$ denotes the Cauchy principal value and $f(E)$ is some smooth function of E. Thus, at least in the on-shell case we can regard the integrated Heitler equation as well defined with a principal-value prescription at the poles of the K matrix. Stated another way, the integrated Heitler equation is satisfied identically independently of whether or not we include the Dirac-δ-function part of these poles. These remarks indicate that in any application of $k_l(E)$ which involves an integration over E it is consistent to treat the resonance-type singularities in the sense of the principal value.

If we had an off-shell $k^{(2)}$ with similar analytic properties and relationship to $t^{(2)}(z)$, we could extend the preceding argument to this case and assert that a resonance-type pole in $k^{(2)}$ be regarded in the sense of a principal value in the context of a definition of the operator in the three-particle space by an integral such as (5.1). The work in Ref. 11 indicates that such a continuation exists.

Let us illustrate this by the following heuristic argument. We denote the partial-wave amplitudes of $k^{(2)}$ and $t^{(2)}(+)$ by $k_l(p', p; E)$ and $t_l(p', p; +)$, respectively. The full content of off-shell unitarity, time-reversal invariance, and the off-shell Heitler equation is the statement that these two partial-wave amplitudes have the structure[12]

$$t_l(p', p; +) = f_l(p', E)\tau_l(+)f_l(p, E) + R_l(p', p; E) \tag{5.4a}$$

and

$$k_l(p', p; E) = f_l(p', E)k_l(E)f_l(p, E) + R_l(p', p; E), \tag{5.4b}$$

where f_l and R_l are real and satisfy, for $E > 0$,

$$f_l(\sqrt{E}, E) = 1,$$

$$R_l(p', \sqrt{E}; E) = R_l(\sqrt{E}, p; E) = 0.$$

$\tau_l(+)$ and $k_l(E)$ are related by (5.2).[12] Equation (5.4b) implies that the entire resonance-type pole content of $k_l(p', p; E)$ is contained in the on-shell part.

For a rather general class of interactions, continuations of f_l and R_l exist to complex E which are analytic in a domain enclosing a resonance-type pole of $k_l(p', p; E)$.[12-14] In these cases we can define the continuations

$$t_l(p', p; z) = f_l(p', z)\tau_l(z)f_l(p, z) + R_l(p', p; z), \tag{5.5a}$$

$$k_l(p', p; z) = f_l(p', z)k_l(z)f_l(p, z) + R_l(p', p; z), \tag{5.5b}$$

where again $\tau_l(z)$ and $k_l(z)$ are related by (5.2). Equations (5.5) imply an extension of the off-shell Heitler equation to z not on the cut. We can, evidently, proceed through the same reasoning as in the on-shell case to reach the conclusion that a consistent definition of the two-particle K matrix in the three-particle Hilbert space incorporates a principal-value prescription in the integration over any possible resonance-type poles.

The essence of the not exactly incisive discussion of this section can be summarized as follows. The existence of an analytic continuation in the parametric energy z of the K matrix allows the interpretation of this quantity as a real analytic

meromorphic function in a domain enclosing the unitary cut with possible poles on the cut. The main point of the argument is that the discontinuity across these poles is irrelevant as far as the definition of $t^{(2)}(+)$ is concerned. As a consequence the two-particle Heitler equation (3.19) in the three-particle space is satisfied identically whether or not we take into account the Dirac-δ-function piece of these poles. Therefore, it appears to be entirely consistent to retain only the principal-value part of these poles when defining the two-body K matrices on the three-particle space.

ACKNOWLEDGMENTS

I have benefited from several discussions on the subject of this paper with T. A. Osborn, S. C. Pieper, C. M. Shakin, and W. Tobocman.

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*This work was supported in part by the U. S. Atomic Energy Commission.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 2 15 JANUARY 1972

Some Aspects of Field Symmetries. III

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(Received 21 April 1971)

Some aspects of the connection between trilinear and bilinear equal-time commutation relations are clarified. Locality requirements satisfied by the generalized fields under discussion are examined in some detail. Using algebraic arguments it is shown that for representations of operators satisfying the generalized trilinear equal-time commutation relations either the half-integral- or the integral-spin fields or both satisfy conventional statistics. In connection with a question raised previously it is shown that the hadronic schemes considered suggest which fields should in each scheme be associated with the (hypothetical) intermediate bosons. It is shown that the discrepancy in physical content between the various particle classification schemes discussed can be substantially reduced, if not eliminated, by the introduction of a new selection rule in addition to the selection rules derived from locality and self-adjointness of the Lagrangian. The new selection rule is not unrelated to locality considerations and is also related to the metric. The question of the metric in the context of the generalized fields is briefly considered.

I. INTRODUCTION

It is well known that the bilinear equal-time commutation relations between distinct fields have implications bearing on the interactions of the fields concerned, on the selection rules they satisfy, and on their vacuum expectation values.[1-15] It is also known that *TCP* invariance requirements do not uniquely determine the bilinear equal-time commutation relations between distinct fields.[10, 16, 17] It is therefore of interest to inquire whether from first principles[18] it is possible to derive a set of fields which has the property that the bilinear equal-time commutation or anticommutation re-