Damped One- and Two-Pion-Exchange Contributions to Relativistic Nucleon-Nucleon Scattering

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Employing Gupta's unitary expansion of the scattering operator, the exact one-pion-exchange (OPE) and two-pion-exchange (TPE) nucleon-nucleon scattering phase parameters are evaluated at 18 energies in the elastic scattering range. These phase parameters are damped relative to the OPE and TPE results previously presented. The damping effect is large for values of $L \leq 3$, and it improves the undamped results in a comparison with the phenomenological values. The core phase parameters are drastically reduced and greatly improved by the damping effect.

I. INTRODUCTION

The exact form of the two-pion-exchange (TPE) contribution to the elastic nucleon-nucleon scattering matrix[1] has been analyzed in terms of partial waves,[2-4] and it has been established that the TPE contribution is a large correction to the one-pion-exchange (OPE) contribution. Indeed, it has been shown recently that TPE brings about reasonably good agreement with the phenomenological phase parameters for values of $L \geq 2$ when the D-state phase shift that is coupled to the S-state is excluded.[5] It is apparent from these analyses, however, that where the TPE contribution is large, it is too large compared with the phenomenological values.

This suggests that some mechanism for reducing the large TPE contribution might bring about better agreement with experiment. It has been known for some time that damping should play a significant role in the meson theory of nuclear forces,[6] and a covariant form of the damping theory was given by Gupta.[7] The exact relativistic first- and second-order terms in the unitary expansion of the scattering operator were obtained in Ref. 1. It is the intent of this work to determine the exact damping effects of these terms on the OPE + TPE contributions to the nucleon-nucleon scattering phase parameters.

II. DAMPED PION CONTRIBUTION TO THE M MATRIX

It is shown in Ref. 7 that if the scattering operator S is expressed in terms of the Hermitian operator K as

$$S = (1 - \tfrac{1}{2}iK)/(1 + \tfrac{1}{2}iK), \tag{1}$$

and K is expanded in powers of the pion-nucleon coupling constant as

$$K = \sum_n K_n, \tag{2}$$

S is unitary even if the expansion is terminated. The term K_n is shown to contain all the nth-order nucleon-nucleon scattering transitions which are not made up of two or more real transitions. In terms of the operator $U = (S-1)/2i$, Eq. (1) can be written in the form

$$2U = -K - iKU, \tag{3}$$

which is the covariant form of the Heitler integral equation.

The M and W matrices are related to U and K as[8]

$$\begin{aligned} U &= [(2\pi)^5/p_0]\delta(p - p' + q - q') \\ &\quad \times :\psi_L^{*-}(\vec{p}')\psi_L^{*-}(\vec{q})M\psi_L^{+}(\vec{q})\psi_L^{+}(\vec{p}):, \\ K &= [(2\pi)^4/c\hbar]\delta(p - p' + q - q') \\ &\quad \times :\psi_L^{*-}(\vec{p}')\psi_L^{*-}(\vec{q}')W\psi_L^{+}(\vec{q})\psi_L^{+}(\vec{p}):, \end{aligned} \tag{4}$$

for elastic nucleon-nucleon scattering, where p and q are the incident and p' and q' the final propagation four-vectors. The ψ_L are large-component Pauli spinors. Center-of-mass coordinates are used, where $\vec{p} = -\vec{q}$, $\vec{p}' = -\vec{q}'$, and $p_0 = p_0' = q_0 = q_0'$.

The y and z directions are taken along $\vec{p}\times\vec{p}'/|\vec{p}\times\vec{p}'|$ and $\vec{p}/|\vec{p}|$, respectively. Then, a real state Ψ_j may be characterized by the isotopic spin of the system, p_0, θ_j, ϕ_j, $\lambda_j^{(1)}$, $\lambda_j^{(2)}$, where θ_j and ϕ_j are the respective polar and azimuthal angles of the momentum of nucleon 1 in this state, and $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$ are the respective helicities of nucleons 1 and 2. The matrix elements of U and K between normalized real states Ψ_j and Ψ_k may then be expressed as

$$\begin{aligned} \Psi_j^* U \Psi_k &= [(2\pi)^5/p_0]\delta(p - p' + q - q')M_{jk}, \\ \Psi_j^* K \Psi_k &= [(2\pi)^4/p_0]\delta(p - p' + q - q')W_{jk}, \end{aligned} \tag{5}$$

with

$$M_{jk}=\langle p_0,\theta_j,\phi_j,\lambda_j^{(1)},\lambda_j^{(2)}|M|p_0,\theta_k,\phi_k,\lambda_k^{(1)},\lambda_k^{(2)}\rangle,$$
$$W_{jk}=\langle p_0,\theta_j,\phi_j,\lambda_j^{(1)},\lambda_j^{(2)}|W|p_0,\theta_k,\phi_k,\lambda_k^{(1)},\lambda_k^{(2)}\rangle. \quad (6)$$

The matrix element of Eq. (3) between the initial Ψ_i and final Ψ_f scattering states is

$$-2\Psi_f^*U\Psi_i=\Psi_f^*K\Psi_i+i\sum_n\Psi_f^*K\Psi_n\Psi_nU\Psi_i, \quad (7)$$

where Ψ_n is a normalized real intermediate two-nucleon scattering state. The sum is over all such states, taking into account the identity of the nucleons. On substituting Eqs. (5) into (7) and making use of the δ functions, Eq. (7) reduces to the integral equation

$$M_{fi}=-\frac{p_0}{4\pi c\hbar}\left(W_{fi}+\frac{i|\vec{p}|}{4\pi}\sum_n\int d\Omega_nW_{fn}M_{ni}\right), \quad (8)$$

where $d\Omega_n=\sin\theta_n\,d\theta_n d\phi_n$, and the sum is over the isospin and helicities of the intermediate states.[9]

III. DAMPED OPE+TPE PHASE PARAMETERS

The Heitler equation has been treated extensively in the literature,[10] and both approximate and exact solutions have been obtained for either specialized or first-order particle interactions. Despite the extremely complicated form of W for the relativistic OPE + TPE contribution to the nucleon-nucleon scattering interaction, Eq. (8) can be solved exactly for the partial-wave amplitudes of M in terms of the partial-wave amplitudes of W by using the method of Jacob and Wick.[11] Numerical values of these latter amplitudes are available at many energies from Ref. 3.

The real particle states are expressed in terms of the eigenstates of the total angular momentum $\vec{J}$, its z component J_z, and the particle helicities $\lambda^{(1)}$ and $\lambda^{(2)}$ as

$$|p_0,\theta,\phi,\lambda^{(1)},\lambda^{(2)}\rangle=\sum_{J,M}|p_0,J,M,\lambda^{(1)},\lambda^{(2)}\rangle\left(\frac{2J+1}{4\pi}\right)^{1/2}D^J_{M\lambda}(\phi,\theta,-\phi),$$

where $D^J_{M\lambda}(\phi,\theta,-\phi)$ is the rotation-matrix element and $\lambda=\lambda^{(2)}-\lambda^{(1)}$. Using the orthogonality relations for the rotation-matrix elements and the fact that M and W are diagonal in $\vec{J}$ and J_z, Eq. (8) reduces to

$$\langle p_0,J,M,\lambda_f^{(1)},\lambda_f^{(2)}|M+(p_0/4\pi c\hbar)W)|p_0,J,M,\lambda_i^{(1)},\lambda_i^{(2)}\rangle$$
$$=-\frac{|\vec{p}|}{4\pi}\sum_{\lambda_n^{(1)},\lambda_n^{(2)}}\langle p_0,J,M,\lambda_f^{(1)},\lambda_f^{(2)}|(p_0/4\pi c\hbar)W)|p_0,J,M,\lambda_n^{(1)},\lambda_n^{(2)}\rangle\langle p_0,J,M,\lambda_n^{(1)},\lambda_n^{(2)}|M|p_0,J,M,\lambda_i^{(1)},\lambda_i^{(2)}\rangle.$$

If the states of definite helicity are next expanded in terms of the eigenstates of $\vec{J}$, J_z, $\vec{L}$, and $\vec{S}$, where the latter two operators are the total orbital angular momentum and spin, respectively, the above equation further reduces to[12]

$${}^s\alpha_J^{L,L'}={}^s\bar{\alpha}_J^{L,L'}+i\sum_{\bar{L}=|J-S|}^{\bar{L}=J+S}{}^s\bar{\alpha}_J^{L,\bar{L}}\,{}^s\alpha_J^{\bar{L},L'}, \quad (9)$$

where

$${}^s\alpha_J^{L,L'}=\frac{|\vec{p}|}{4\pi}\langle J,M,L,S|M|J,M,L',S\rangle,$$
$${}^s\bar{\alpha}_J^{L,L'}=-\frac{p_0|\vec{p}|}{16\pi^2c\hbar}\langle J,M,L,S|W|J,M,L',S\rangle. \quad (10)$$

The matrix elements of M and W are independent of the quantum number M since the operators U and K are rotationally invariant. Moreover, as U and K are also invariant under the parity transformation, for a fixed value of J, the only nonvanishing matrix elements are

$${}^0\alpha_J^{L=L'=J}\equiv{}^0\alpha_J,\quad {}^0\bar{\alpha}_J^{L=L'=J}\equiv{}^0\bar{\alpha}_J,\quad {}^1\alpha_J^{L=L'=J}\equiv{}^1\alpha_J^J,$$
$${}^1\bar{\alpha}_J^{L=L'=J}\equiv{}^1\bar{\alpha}_J^J,\quad {}^1\alpha_J^{L=J\pm1,L'=J\mp1}\equiv{}^1\alpha_J,\quad {}^1\bar{\alpha}_J^{L=J\pm1,L'=J\mp1}\equiv{}^1\bar{\alpha}_J,$$
$${}^1\alpha_J^{L=J\pm1,L'=J\pm1}\equiv{}^1\alpha_J^{J\pm1},\quad {}^1\bar{\alpha}_J^{L=J\pm1,L'=J\pm1}\equiv{}^1\bar{\alpha}_J^{J\pm1}.$$

Any approximation to the K operator leads to a unitarity S, and hence the damped partial-wave amplitudes α exactly satisfy the unitarity conditions

$$
\begin{aligned}
{}^0\alpha_J &= \frac{1}{2i}\left[\exp(2iK_J) - 1\right], \\
{}^1\alpha_J^J &= \frac{1}{2i}\left[\exp(2i\,{}^3\delta^J{}_J) - 1\right], \\
{}^1\alpha_J^{J\pm1} &= \frac{1}{2i}\left\{\left[1 - \rho_J{}^2\right]^{1/2}\left[\exp(2i\,{}^3\theta^{J\pm1}{}_J)\right] - 1\right\}, \\
{}^1\alpha_J &= \tfrac{1}{2}\,\rho_J \exp\left[i({}^3\theta^{J+1}{}_J + {}^3\theta^{J-1}{}_J)\right],
\end{aligned}
\tag{11}
$$

where K_J, ${}^3\delta^J{}_J$, ${}^3\theta^{J\pm1}{}_J$, and ρ_J are the damped singlet, uncoupled triplet, coupled triplet phase shifts, and the coupling parameter, respectively, in the Yale notation. If the operator K contains the OPE and TPE contributions, the undamped partial-wave amplitudes are given by

$$
\begin{aligned}
{}^0\bar{\alpha}_J &= K_J(\mathrm{OPE}) + K_J(\mathrm{TPE}), \\
{}^1\bar{\alpha}_J^J &= {}^3\delta^J{}_J(\mathrm{OPE}) + {}^3\delta^J{}_J(\mathrm{TPE}), \\
{}^1\bar{\alpha}_J^{J\pm1} &= {}^3\theta^{J\pm1}{}_J(\mathrm{OPE}) + {}^3\theta^{J\pm1}{}_J(\mathrm{TPE}), \\
{}^1\bar{\alpha}_J &= \tfrac{1}{2}\left[\rho_J(\mathrm{OPE}) + \rho_J(\mathrm{TPE})\right],
\end{aligned}
\tag{12}
$$

where the quantities on the right-hand side are the undamped OPE and TPE phase parameters given in Tables I and II of Ref. 3. If follows from Eq. (9) that the damped and undamped amplitudes are related by

$$
\mathrm{Re}\,{}^0\alpha_J = \frac{{}^0\bar{\alpha}_J}{1 + ({}^0\bar{\alpha}_J)^2}, \qquad \mathrm{Re}\,{}^1\alpha_J^J = \frac{{}^1\bar{\alpha}_J^J}{1 + ({}^1\bar{\alpha}_J^J)^2},
\tag{13}
$$

for the singlet and uncoupled triplet amplitudes,[13] respectively, and

$$
\begin{aligned}
\mathrm{Re}\,{}^1\alpha_J^{J\pm1} &= \frac{{}^1\bar{\alpha}_J^{J\pm1} - ({}^1\bar{\alpha}_J)^2\,{}^1\bar{\alpha}_J^{J\mp1}/\left[1 + ({}^1\bar{\alpha}_J^{J\mp1})^2\right]}{1 + ({}^1\bar{\alpha}_J^{J\pm1})^2 + ({}^1\bar{\alpha}_J)^2\left\{\left[2 + ({}^1\bar{\alpha}_J)^2 - 2\,{}^1\bar{\alpha}_J^{J\pm1}\,{}^1\bar{\alpha}_J^{J\mp1}\right]/\left[1 + ({}^1\bar{\alpha}_J^{J\mp1})^2\right]\right\}}, \\
\mathrm{Re}\,{}^1\alpha_J &= \frac{{}^1\bar{\alpha}_J}{\left[1 + ({}^1\bar{\alpha}_J)^2 - {}^1\bar{\alpha}_J^{J-1}\,{}^1\bar{\alpha}_J^{J+1}\right] + ({}^1\bar{\alpha}_J^{J-1} + {}^1\bar{\alpha}_J^{J+1})^2/\left[1 + ({}^1\bar{\alpha}_J)^2 - {}^1\bar{\alpha}_J^{J-1}\,{}^1\bar{\alpha}_J^{J+1}\right]}
\end{aligned}
\tag{14}
$$

relate the coupled amplitudes. In these equations, "Re" means "the real part of."

IV. COMPARISON OF THE DAMPED AND UNDAMPED PHASE PARAMETERS

The undamped partial-wave amplitudes $\bar{\alpha}$ are available from Ref. 3 at 18 energies in the elastic scattering range, and these are used in Eqs. (13) and (14) to evaluate the damped amplitudes α. The damped phase parameters follow by inverting Eqs. (11), and these phases corresponding to OPE and OPE + TPE are presented in Table I for the isotriplet states and in Table II for the isosinglet state. The phase shifts are in radians. The corresponding undamped OPE + TPE phases, denoted OPE′ + TPE′ in the tables, are also given for comparison. All theoretical phase parameters are evaluated using an average pion mass of 138 MeV, an average nucleon mass of 938.903 MeV, and a nominal value of 14.0 for the pion-nucleon coupling constant. The Yale phenomenological phase parameters are included, with their parallel-shift uncertainties given in parentheses above the related group of phases.[14]

It is first noted that, for the S-state phase shifts K_0 and ${}^3\theta^S{}_1$, the damping effect reduces the OPE + TPE contribution by as much as an order of magnitude at many energies.

The damping effect of the P-state phase shifts ${}^3\delta^P{}_0$, ${}^3\delta^P{}_1$, K_1, and ${}^3\theta^P{}_2$ is as large as 16% at the higher energies, and there is definite improvement at these energies. However, the P state is not well represented by the OPE + TPE contribution, whether damped or undamped. Additional effects, such as the vector and scalar resonances, should play an important role here.

While the damping effect for the phase shift ${}^3\theta^D{}_3$ is negligible, the D-state triplet phase shifts ${}^3\delta^D{}_2$ and ${}^3\theta^D{}_1$ are strongly damped, and the damping improves these values relative to the Yale phases. The coupled phase shift ${}^3\theta^D{}_1$ is reduced by more than a factor of 1.5 at the higher energies, and, while there is still a large discrepancy with the experimental values at the intermediate energies, this phase shift is much more reasonable. The singlet phase shift K_2 is slightly damped, and the reduction is an improvement.

The damping effect becomes negligible for the

TABLE I. Damped OPE and TPE contributions to the phase parameters for the isotriplet ($T=1$) states. The undamped OPE′ +TPE′ phase parameters and the Yale phenomenological values with their parallel-shift uncertainties are included for comparison. The phase shifts are in radians. The Yale parallel-shift uncertainties are given in parentheses above the related group of phases.

E (MeV)	OPE	OPE+TPE	OPE′ +TPE′	Yale	OPE	OPE+TPE	OPE′ +TPE′	Yale
		K_0				$^3\delta P_0$		
				(0.0002)				(0.0030)
10	−0.0773	1.0752	1.8498	1.0476	0.0773	0.0976	0.0979	0.0665
30	−0.2306	1.2443	2.9531	0.8322	0.2306	0.2967	0.3057	0.1660
50	−0.3457	1.3080	3.5748	0.6875	0.3457	0.4300	0.4586	0.2051
				(0.0059)				(0.0067)
70	−0.4351	1.3255	3.9948	0.5813	0.4351	0.5151	0.5661	0.2032
90	−0.5070	1.3421	4.2964	0.4912	0.5070	0.5680	0.6381	0.1808
110	−0.5662	1.3530	4.5185	0.4157	0.5662	0.5983	0.6816	0.1500
130	−0.6162	1.3604	4.6830	0.3460	0.6162	0.6119	0.7018	0.1181
150	−0.6589	1.3655	4.8035	0.2792	0.6589	0.6124	0.7026	0.0862
				(0.0080)				(0.0134)
170	−0.6959	1.3691	4.8893	0.2157	0.6959	0.6018	0.6868	0.0538
190	−0.7284	1.3713	4.9471	0.1588	0.7284	0.5813	0.6570	0.0195
210	−0.7571	1.3727	4.9816	0.1027	0.7572	0.5514	0.6150	−0.0138
230	−0.7828	1.3733	4.9966	0.0473	0.7828	0.5122	0.5623	−0.0464
250	−0.8059	1.3732	4.9951	−0.0009	0.8059	0.4639	0.5003	−0.0820
270	−0.8267	1.3726	4.9793	−0.0459	0.8367	0.4062	0.4301	−0.1202
				(0.0229)				(0.0246)
290	−0.8457	1.3705	4.9512	−0.0863	0.8457	0.3391	0.3527	−0.1584
310	−0.8630	1.3700	4.9122	−0.1262	0.8630	0.2626	0.2688	−0.1987
330	−0.8789	1.3680	4.8637	−0.1641	0.8789	0.1773	0.1792	−0.2381
350	−0.8936	1.3657	4.8069	−0.1990	0.8936	0.0842	0.0844	−0.2753
			$^3\delta P_1$				$^3\theta P_2$	
				(0.0010)				(0.0009)
10	−0.0488	−0.0310	−0.0310	−0.0441	0.0017	0.0177	0.0177	0.0147
30	−0.1392	−0.0702	−0.0703	−0.0993	0.0099	0.0784	0.0786	0.0577
50	−0.2058	−0.0917	−0.0919	−0.1436	0.0188	0.1439	0.1449	0.1065
				(0.0021)				(0.0016)
70	−0.2585	−0.1091	−0.1096	−0.1840	0.0272	0.2060	0.2090	0.1487
90	−0.3021	−0.1271	−0.1278	−0.2201	0.0351	0.2628	0.2691	0.1834
110	−0.3394	−0.1471	−0.1482	−0.2527	0.0423	0.3141	0.3249	0.2111
130	−0.3719	−0.1698	−0.1714	−0.2835	0.0491	0.3600	0.3765	0.2330
150	−0.4007	−0.1951	−0.1956	−0.3107	0.0554	0.4011	0.4242	0.2492
				(0.0067)				(0.0040)
170	−0.4265	−0.2228	−0.2266	−0.3372	0.0612	0.4378	0.4682	0.2616
190	−0.4498	−0.2529	−0.2583	−0.3619	0.0668	0.4706	0.5089	0.2695
210	−0.4711	−0.2846	−0.2925	−0.3851	0.0720	0.5001	0.5466	0.2742
230	−0.4906	−0.3179	−0.3291	−0.4066	0.0769	0.5265	0.5815	0.2765
250	−0.5087	−0.3524	−0.3677	−0.4256	0.0816	0.5503	0.6138	0.2771
270	−0.5253	−0.3876	−0.4083	−0.4431	0.0860	0.5717	0.6437	0.2770
				(0.0150)				(0.0098)
290	−0.5409	−0.4234	−0.4507	−0.4598	0.0903	0.5911	0.6715	0.2767
310	−0.5554	−0.4593	−0.4946	−0.4756	0.0943	0.6086	0.6972	0.2764
330	−0.5689	−0.4951	−0.5400	−0.4902	0.0982	0.6244	0.7210	0.2754
350	−0.5817	−0.5306	−0.5867	−0.5040	0.1019	0.6388	0.7431	0.2737

TABLE I *(Continued)*

E (MeV)	OPE	OPE+TPE	OPE′+TPE′	Yale	OPE	OPE+TPE	OPE′+TPE′	Yale
		ρ_2				K_2		
				(0.0014)				(0.0003)
10	−0.0073	−0.0073	−0.0073	−0.0091	0.0027	0.0031	0.0031	0.0035
30	−0.0358	−0.0360	−0.0362	−0.0511	0.0121	0.0161	0.0161	0.0205
50	−0.0621	−0.0619	−0.0626	−0.0746	0.0193	0.0301	0.0301	0.0354
				(0.0022)				(0.0019)
70	−0.0844	−0.0825	−0.0844	−0.0845	0.0245	0.0441	0.0441	0.0486
90	−0.1034	−0.0987	−0.1025	−0.0917	0.0284	0.0580	0.0581	0.0616
110	−0.1199	−0.1115	−0.1176	−0.0963	0.0313	0.0717	0.0718	0.0739
130	−0.1344	−0.1216	−0.1305	−0.0987	0.0335	0.0851	0.0853	0.0852
150	−0.1473	−0.1296	−0.1414	−0.0997	0.0353	0.0983	0.0986	0.0964
				(0.0047)				(0.0031)
170	−0.1590	−0.1358	−0.1508	−0.0997	0.0367	0.1111	0.1115	0.1069
190	−0.1697	−0.1408	−0.1588	−0.0993	0.0378	0.1235	0.1242	0.1175
210	−0.1794	−0.1447	−0.1659	−0.0986	0.0387	0.1356	0.1365	0.1277
230	−0.1884	−0.1477	−0.1719	−0.0973	0.0394	0.1473	0.1484	0.1374
250	−0.1968	−0.1501	−0.1772	−0.0956	0.0400	0.1587	0.1600	0.1473
270	−0.2046	−0.1519	−0.1818	−0.0933	0.0405	0.1696	0.1712	0.1569
				(0.0141)				(0.0101)
290	−0.2119	−0.1532	−0.1858	−0.0908	0.0409	0.1801	0.1821	0.1659
310	−0.2187	−0.1542	−0.1892	−0.0883	0.0412	0.1903	0.1926	0.1749
330	−0.2252	−0.1548	−0.1922	−0.0858	0.0414	0.2001	0.2028	0.1840
350	−0.2313	−0.1552	−0.1948	−0.0834	0.0416	0.2096	0.2127	0.1927

		$^3\theta F_2$		
				(0.0017)
10	0.0002	0.0002	0.0002	0.0002
30	0.0026	0.0027	0.0027	0.0024
50	0.0061	0.0064	0.0065	0.0059
				(0.0027)
70	0.0100	0.0104	0.0107	0.0087
90	0.0139	0.0143	0.0150	0.0102
110	0.0177	0.0181	0.0191	0.0108
130	0.0214	0.0217	0.0231	0.0112
150	0.0245	0.0251	0.0269	0.0114
				(0.0061)
170	0.0284	0.0282	0.0304	0.0113
190	0.0316	0.0311	0.0336	0.0110
210	0.0347	0.0337	0.0366	0.0105
230	0.0377	0.0361	0.0393	0.0097
250	0.0406	0.0382	0.0417	0.0088
270	0.0433	0.0401	0.0439	0.0078
				(0.0099)
290	0.0460	0.0417	0.0457	0.0066
310	0.0485	0.0431	0.0474	0.0056
330	0.0510	0.0443	0.0487	0.0047
350	0.0533	0.0453	0.0498	0.0037

TABLE II. Damped OPE and TPE contributions to the phase parameters for the isosinglet ($T=0$) state. The undamped OPE′+TPE′ phase parameters and the Yale phenomenological values with their parallel-shift uncertainties are included for comparison. The Yale parallel-shift uncertainties are given in parentheses above the related group of phases.

E (MeV)	OPE	OPE+TPE	OPE′+TPE′	Yale	OPE	OPE+TPE	OPE′+TPE′	Yale
			$^3\theta S_1$				ρ_1	
				(0.0024)				(0.0171)
10	−0.0772	1.3821	5.2359	1.7984	0.1925	0.0196	0.1266	0.0404
30	−0.2283	1.4562	8.6797	1.3034	0.4792	0.0279	0.2442	0.0682
50	−0.3395	1.4785	10.7950	1.0458	0.6288	0.0243	0.2684	0.0792
				(0.0064)				(0.0220)
70	−0.4252	1.4901	12.3516	0.8750	0.7147	0.0205	0.2619	0.0862
90	−0.4946	1.5073	13.5809	0.7499	0.7686	0.0169	0.2444	0.0904
110	−0.5527	1.5024	14.5904	0.6496	0.8047	0.0139	0.2237	0.0927
130	−0.5984	1.5061	15.4404	0.5632	0.8299	0.0115	0.2034	0.0947
150	−0.6452	1.5090	16.1688	0.4859	0.8472	0.0095	0.1853	0.0975
				(0.0710)				(0.1151)
170	−0.6836	1.5114	16.8009	0.4203	0.8625	0.0079	0.1702	0.1021
190	−0.7176	1.5133	17.3548	0.3620	0.8733	0.0069	0.1586	0.1088
210	−0.7511	1.5148	17.8439	0.3109	0.8814	0.0060	0.1506	0.1179
230	−0.7752	1.5162	18.2781	0.2662	0.8886	0.0054	0.1462	0.1289
250	−0.8017	1.5173	18.6656	0.2259	0.8944	0.0050	0.1455	0.1418
270	−0.8226	1.5183	19.0125	0.1883	0.8990	0.0047	0.1482	0.1559
				(0.0872)				(0.1321)
290	−0.8430	1.5191	19.3241	0.1525	0.9029	0.0046	0.1542	0.1707
310	−0.8619	1.5198	19.6044	0.1180	0.9062	0.0046	0.1636	0.1859
330	−0.8796	1.5205	19.8572	0.0859	0.9090	0.0046	0.1759	0.2009
350	−0.8957	1.5211	20.0853	0.0561	0.9116	0.0047	0.1912	0.2155
			K_1				$^3\theta D_1$	
				(0.0085)				(0.0074)
10	−0.0603	−0.0468	−0.0468	−0.0450	−0.0078	−0.0067	−0.0059	−0.0122
30	−0.1353	−0.0943	−0.0946	−0.0956	−0.0341	−0.0631	−0.0615	−0.0662
50	−0.1706	−0.1121	−0.1126	−0.1224	−0.0495	−0.1472	−0.1466	−0.1189
				(0.0194)				(0.0044)
70	−0.1898	−0.1219	−0.1225	−0.1515	−0.0556	−0.2405	−0.2439	−0.1651
90	−0.2012	−0.1303	−0.1310	−0.1860	−0.0560	−0.3349	−0.3470	−0.2016
110	−0.2082	−0.1398	−0.1407	−0.2254	−0.0532	−0.4260	−0.4530	−0.2285
130	−0.2126	−0.1513	−0.1525	−0.2689	−0.0524	−0.5114	−0.5606	−0.2478
150	−0.2153	−0.1652	−0.1668	−0.3145	−0.0436	−0.5900	−0.6691	−0.2638
				(0.1045)				(0.0970)
170	−0.2168	−0.1815	−0.1836	−0.3599	−0.0372	−0.6614	−0.7779	−0.2802
190	−0.2175	−0.2001	−0.2028	−0.4050	−0.0314	−0.7256	−0.8868	−0.2983
210	−0.2176	−0.2208	−0.2244	−0.4505	−0.0227	−0.7833	−0.9959	−0.3174
230	−0.2173	−0.2433	−0.2482	−0.4950	−0.0204	−0.8349	−1.1040	−0.3397
250	−0.2167	−0.2674	−0.2740	−0.5408	−0.0166	−0.8811	−1.2121	−0.3629
270	−0.2158	−0.2929	−0.3016	−0.5866	−0.0105	−0.9225	−1.3197	−0.3881
				(0.0910)				(0.0584)
290	−0.2148	−0.3195	−0.3308	−0.6318	−0.0062	−0.9596	−1.4268	−0.4152
310	−0.2137	−0.3470	−0.3616	−0.6798	−0.0021	−0.9930	−1.5334	−0.4432
330	−0.2124	−0.3752	−0.3939	−0.7278	0.0018	−1.0232	−1.6394	−0.4720
350	−0.2111	−0.4039	−0.4273	−0.7745	0.0053	−1.0505	−1.7448	−0.5015

TABLE II *(Continued)*

E (MeV)	OPE	OPE+TPE	OPE′+TPE′	Yale	OPE	OPE+TPE	OPE′+TPE′	Yale
		$^3\delta^D_2$				$^3\theta^D_3$		
				(0.0073)				(0.0035)
10	0.0140	0.0153	0.0153	0.0150	−0.0005	0.0001	0.0001	0.0004
30	0.0733	0.0856	0.0858	0.0911	−0.0055	0.0016	0.0016	0.0033
50	0.1318	0.1604	0.1618	0.1858	−0.0131	0.0062	0.0062	0.0072
				(0.0078)				(0.0037)
70	0.1833	0.2295	0.2336	0.2552	−0.0214	0.0136	0.0136	0.0116
90	0.2284	0.2916	0.3001	0.3053	−0.0299	0.0232	0.0231	0.0168
110	0.2682	0.3471	0.3617	0.3443	−0.0382	0.0342	0.0341	0.0230
130	0.3036	0.3967	0.4189	0.3776	−0.0462	0.0462	0.0461	0.0301
150	0.3353	0.4411	0.4722	0.4056	−0.0539	0.0589	0.0587	0.0379
				(0.0784)				(0.0405)
170	0.3640	0.4811	0.5220	0.4274	−0.0613	0.0719	0.0717	0.0460
190	0.3901	0.5170	0.5686	0.4436	−0.0683	0.0852	0.0848	0.0541
210	0.4140	0.5496	0.6125	0.4553	−0.0750	0.0985	0.0981	0.0620
230	0.4360	0.5791	0.6539	0.4627	−0.0815	0.1119	0.1113	0.0698
250	0.4564	0.6060	0.6929	0.4661	−0.0877	0.1250	0.1244	0.0768
270	0.4752	0.6305	0.7298	0.4665	−0.0936	0.1380	0.1374	0.0834
				(0.0860)				(0.0248)
290	0.4928	0.6529	0.7648	0.4644	−0.0943	0.1501	0.1502	0.0895
310	0.5091	0.6736	0.7981	0.4602	−0.1047	0.1633	0.1628	0.0951
330	0.5245	0.6925	0.8296	0.4546	−0.1100	0.1756	0.1751	0.1002
350	0.5389	0.7100	0.8596	0.4478	−0.1150	0.1876	0.1871	0.1048

		ρ_3		
				(0.0266)
10	0.0029	0.0029	0.0029	0.0031
30	0.0273	0.0271	0.0271	0.0292
50	0.0586	0.0577	0.0577	0.0625
				(0.0085)
70	0.0891	0.0871	0.0873	0.0899
90	0.1171	0.1139	0.1143	0.1080
110	0.1425	0.1380	0.1388	0.1237
130	0.1656	0.1597	0.1610	0.1408
150	0.1866	0.1794	0.1813	0.1584
				(0.0826)
170	0.2057	0.1973	0.1999	0.1755
190	0.2232	0.2136	0.2171	0.1915
210	0.2394	0.2286	0.2331	0.2057
230	0.2543	0.2424	0.2480	0.2182
250	0.2682	0.2552	0.2621	0.2287
270	0.2810	0.2670	0.2753	0.2376
				(0.0530)
290	0.2931	0.2780	0.2878	0.2449
310	0.3043	0.2883	0.2996	0.2506
330	0.3149	0.2979	0.3109	0.2553
350	0.3249	0.3069	0.3217	0.2591

phase parameters corresponding to $L>2$ with the exception of the coupled phase ${}^3\theta^F{}_2$. It is damped by 10% at the higher energies and thereby improved.

As seen from Eqs. (14), the damping effect on the coupled phase shifts and the coupling parameters is rather complicated in form, and the results of damping are not evident without making the calculation. For example, the coupling parameter ρ_1 given in Table II is drastically reduced relative to the undamped OPE + TPE values, and this reduction provides better agreement with the Yale values, especially at the lower energies. Both ρ_2 and ρ_3 are slightly improved and reduced by damping.

A comparison of the damped OPE and OPE + TPE contributions reveals that TPE significantly improves upon OPE in a comparison with most of the phenomenological values. This is in marked contrast with the undamped calculation of the core-dependent phase parameters K_0, ${}^3\theta^S{}_1$, ${}^3\theta^D{}_1$, and ρ_1. Here, TPE so strongly overcorrects OPE that the comparison with the phenomenological values is worsened considerably. A rough estimate of the improvement of OPE + TPE over OPE in the damped calculation can be achieved by evaluating

$$\chi^2 = \sum_E \frac{[\delta - \delta(E)]^2}{\Delta\delta(E)^2}$$

for each core phase parameter, where δ is the theoretical value, $\delta(E)$ the Yale value, and $\Delta\delta(E)$ the parallel-shift uncertainty in the Yale value. The ratio $\chi^2(\mathrm{OPE})/\chi^2(\mathrm{OPE}+\mathrm{TPE})$ is 20 for K_0, 40 for ${}^3\theta^S{}_1$, 50 for ρ_1, and 0.9 for ${}^3\theta^D{}_1$. The last phase shift still suffers from a TPE contribution that is too large at the intermediate energies. Excluding the core-dependent phase parameters, the damped and undamped results are close enough so that the conclusions of Ref. 5 apply. That is, OPE + TPE is in reasonable agreement with the phenomenological values for $L \geq 2$.

V. CONCLUSION

The covariant form of the Heitler equation is solved exactly for the OPE and TPE contributions to elastic nucleon-nucleon scattering. This leads to a damping effect on the previously determined OPE and TPE phase parameters that is found to be large for the lower values of L. Although the damping effect is negligible in the region $L>5$ where OPE dominantes the interaction, it is as large as 10% for the phase shift ${}^3\theta^F{}_2$ and 20% for ${}^3\delta^D{}_2$. In the range $L \geq 2$, excluding ${}^3\theta^D{}_1$, damping improves the already reasonably good agreement with the phenomenological values. The damping effect on the P-state phase parameters is as large as 16%, and it improves the undamped values in a comparison with experiment. For the core parameters K_0, ${}^3\theta^S{}_1$, ${}^3\theta^D{}_1$, and ρ_1, the damping effect is as large as an order of magnitude of the undamped values, and it produces a TPE contribution that is reasonable and a definite improvement over the OPE contribution, in contrast with the undamped results.

Certainly, damping does not compensate for a more complete determination of the scattering matrix. The omission of the higher-order pion-nucleon and other boson-exchange effects remains a serious error. However, it is evident that the OPE + TPE approximation is substantially improved by requiring it to satisfy unitarity through the addition of the real intermediate processes.

It is further noted that since such additional effects as the one-vector-meson-exchange contributions are most important for the lower values of L, these effects should be strongly influenced by damping. Therefore, the parameters of pion and resonance models of nucleon-nucleon scattering may be expected to be significantly affected by damping.

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Single π^+ Electroproduction at $w\approx 2$ GeV and the Pion Form Factor

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Recently published data on $ep \to en\pi^+$ near $w = 2$ GeV and $0.0 < |k^2| < 1.0$ GeV^2 are investigated using (i) the model-independent Chew-Low extrapolation technique and (ii) fixed-t dispersion relations. The data are not accurate enough to allow the Chew-Low fit to give an unambiguous determination of F_π. Using fixed-t dispersion relations we obtain $F_\pi(k^2) = 1/(1 - k^2/0.5\ \text{GeV}^2)$. The effect of resonances other than the P_{33} is small.

I. INTRODUCTION

It is well known that single-π^+ electroproduction $ep \to en\pi^+$ could in principle determine the pion form factor for spacelike values of the photon mass squared ($k^2 < 0$).[1,2] However, attempts to determine F_π from electroproduction data in the first-resonance region[3,4] have not produced results sufficiently accurate to distinguish between F_π given by the ρ-dominance model or $F_\pi = G_p^e$ (the dipole form factor).

Recently, data have been published at $w = 2$ GeV (Refs. 5–7) and it has been shown[8] that such data could give a much better determination of F_π, in particular ruling out the choice $F_\pi = G_p^e$.

Frazer[1] suggested in 1959 that the Chew-Low extrapolation technique could be used on $ep \to en\pi^+$ data to give a model-independent determination of F_π. In this paper we have tried to apply this technique to the data.

All the attempts to determine F_π from electroproduction data have used fixed-t dispersion relations. We examine this approach and explain why the region $w \approx 2$ GeV is a good place to look at F_π. We also give a critical discussion of the uncertainties in the dispersion relations, in particular those arising from resonances other than the first resonance and from a subtraction term.

II. CHEW-LOW EXTRAPOLATION[1]

The assumption required for this approach is that the pion pole will dominate the cross sections for small t. If the pole denominator is multiplied out of the experimental cross sections then the results should lie on a smooth curve which can be extrapolated to the pole position in t and hence give the residue of the pole. If the coupling constants involved are known, the pion form factor can be determined in principle.

Figure 1 defines our kinematic variables and detailed formulas are given in Appendix A.

From these formulas we can calculate the residue of the pion pole in the differential cross section [formulas (A9), (A11), and (A12)]. Notice that as written the differential cross section has a term involving $\sin\theta$ (the scalar-transverse interference term). This must be removed in some way before the extrapolation is attempted as it has a square-root branch point in t.