PHYSICAL REVIEW D VOLUME 5, NUMBER 2 15 JANUARY 1972

Scale and Conformal Transformations in Galilean-Covariant Field Theory*

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(Received 26 March 1971)

It is demonstrated that there exists the possibility of defining scale and conformal transformations in such a way that these constitute exact invariance operations of the Schrödinger equation. Unlike the relativistic case there is only a single conformal transformation and the usual eleven-parameter extended Galilei group is consequently enlarged to a thirteen-parameter group. The generalization to the case of fields of arbitrary spin is carried out within the framework of minimal-component theories whose interactions respect scale and conformal invariance. One finds that the bare-internal-energy term can be used to break these additional invariance operations in much the same way as the mass term in special relativity. The generators and conservation laws associated with all space-time symmetries of minimal-component Galilean-invariant field theories are derived, it being shown that, in analogy to the relativistic case, the operators which appear in these equations can be redefined so as to allow the formulation of scale and conformal invariance entirely in terms of those operator densities relevant to the transformations of the Galilei group.

I. INTRODUCTION

One of the most actively pursued areas of research in particle physics at the present time consists of the investigation of the possible relevance of scale invariance in high-energy interactions. Thus one asks that, given the absence of any intrinsic mass or dimensional coupling constant in a given physical system, what inferences can one make concerning fundamental processes and, secondly, what are the possible consequences of terms which explicitly break dilatation invariance. In consequence of this considerable emphasis upon the absence of dimensional parameters in order that one be able to discuss scale invariance, the prospect of studying this particular symmetry in the nonrelativistic limit might at first sight appear to be virtually nil. More specifically the explicit appearance of the mass m in the Schrödinger equation for the field operator (or wave function) ψ,

$$(E - p^2/2m)\psi = 0\,, \tag{1.1}$$

would seem to preclude any possible relevance of scale transformations to systems described by (1.1). On the other hand, one can note that the absence of the velocity of light as an available parameter in the Galilean limit means that $1/m$, unlike the relativistic case, is not dimensionally equivalent to a length. Consequently one can imagine the possibility of scaling in the space and time coordinates while at the same time retaining quantities such as the mass which have inequivalent dimensions and no scaling properties. In order to implement this suggestion one observes that if x_i is scaled such that

$$x_i' = e^{\tau} x_i\,, \tag{1.2}$$

then the invariance of (1.1) under (1.2) can be assured by the additional scale transformation of the time coordinate

$$t' = e^{2\tau} t\,, \tag{1.3}$$

i.e., by scaling t "twice" as much as x_i. This together with the demonstration of the existence of a conformal transformation which preserves (1.1) comprise in the simplest possible terms the essential observation which underlies this paper.

Inasmuch as Galilean relativity has largely been superseded in particle physics by special relativity, it is perhaps appropriate to pause before proceeding with any further development of this topic in order to assess the possible value of such a study. There appear in fact to be at least three good reasons for pursuing this investigation.

(i) The question of the full invariance group compatible with Galilean relativity is of interest in its own right.

(ii) The Galilei group is an entirely consistent group of space-time transformations which in some respects possesses a more complex structure than the Poincaré group. Thus there emerges the possibility of examining whether the intimate connection between the usual space-time transformations and scale and conformal transformations as found by Callan *et al.*[1] is invariant with respect to the choice of the underlying kinematical group or is largely accidental.

(iii) The recent widespread use of perturbative methods in discussions of scale invariance could quite profitably be incorporated within the framework of Galilean relativity for the very good reason that the perturbation series always terminates in the Galilean case.

With the foregoing brief defense of the present study one can now resume the main development of this paper. In Sec. II the properties of the Galilei group essential to this discussion are presented and explicit calculation made of the generators of the transformations of the extended Galilei group including dilatation and conformal transformations, in the case of a spin-zero field. Section III briefly reviews the structure of theories of arbitrary spin before proceeding to the construction of the eleven generators and associated conservation laws which follow from Galilean invariance. Section IV presents the corresponding results for scale and conformal transformations.

II. THE SPIN-ZERO CASE

In order to properly discuss the dilatation and conformal transformations within the context of Galilean relativity it is essential to summarize briefly some well-known properties of that group.[2] To this end one recalls that the proper Galilei group consists of all space-time transformations of the form

$$\vec{x}' = R\vec{x} + \vec{v}t + \vec{a}\,,$$

$$t' = t + b\,,$$

where R is a 3×3 orthogonal matrix. The ten-parameter element which describes a given transformation can be denoted by $g = (b, \vec{a}, \vec{v}, R)$ and is readily seen to satisfy the group law

$$(b', \vec{a}', \vec{v}', R')(b, \vec{a}, \vec{v}, R) = (b + b', \vec{a}' + R'\vec{a} + b\vec{v}', \vec{v} + R'\vec{v}, R'R)\,. \tag{2.1}$$

Making use of this result one can infer that the Lie algebra is described by the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k\,,$$
$$[J_i, P_j] = i\epsilon_{ijk}P_k, \quad [K_i, H] = iP_i\,, \tag{2.2}$$
$$[J_i, H] = [K_i, K_j] = [K_i, P_j] = [P_i, P_j] = [P_i, H] = 0\,,$$

where J_i, P_i, H, and K_i are, respectively, the generators of rotations, space translations, time translations, and pure Galilean transformations.

The physical representations (and the ones with which we shall be concerned here) are those which are representations of a central extension of the Galilei group by a one-dimensional Abelian group. This extension can be obtained by making the replacement of the $[K_i, P_j]$ commutator in (2.2) by

$$[K_i, P_j] = i\delta_{ij}M\,,$$

where M commutes with all operators of the group, and retaining without modification all the remaining commutators in (2.2). Using these structure relations the eleven-parameter element $\tilde{g} = (\theta, g)$ of the extended group can be shown to have the multiplication law

$$\tilde{g}'\tilde{g} = (\theta' + \theta + \xi_m(g', g), g'g),$$

where

$$\xi_m(g', g) = m(\tfrac{1}{2}bv'^2 + \vec{v}'\cdot R'\vec{a})$$

is an exponent of the group, and we have written $M = mI$, with I being the identity element. It is to be noted that the enveloping algebra has a three-dimensional center including, in addition to M, the invariants $U \equiv H - (1/2M)\vec{P}^2$ (the internal energy) and $\vec{S}^2 \equiv [\vec{J} - (1/M)\vec{K}\times\vec{P}]^2$ (the square of the intrinsic spin).

In order to avoid at least temporarily the not inconsiderable complications associated with spin, this section will restrict consideration exclusively to physical representations of the Galilei group which correspond to zero spin. Thus one writes the Lagrangian of a spin-zero field[3] which has the customary form[4]

$$\mathcal{L} = \tfrac{1}{2}[\phi^\dagger E\phi - (E\phi^\dagger)\phi] - U_0\phi^\dagger\phi + \frac{1}{2m}(\vec{p}\phi^\dagger)\cdot(\vec{p}\phi) + \mathcal{L}_I\,, \tag{2.3}$$

where the term proportional to U_0 is a bare-internal-energy term. Inasmuch as we allow for the possibility of (nonderivative) local couplings $\mathcal{L}_I$, it is to be understood that, in general, the free Lagrangian consists of a sum over a number of fields of different masses and internal energies. The interaction is assumed furthermore to be invariant under all operations of the extended Galilei group which requires among other things that cognizance be taken of Bargmann's superselection rule on the mass. This stipulates that for an interaction of the form

$$A + B + \cdots \to A' + B' + \cdots\,,$$

one must have the condition

$$m_A + m_B + \cdots = m_{A'} + m_{B'} + \cdots\,.$$

It is now straightforward to show that for the transformation law

$$\psi'(\vec{x}', t') = e^{if(\vec{x}, t)}\psi(\vec{x}, t)\,, \tag{2.4}$$

where

$$f(\vec{x}, t) = m\vec{v}\cdot R\vec{x} + \tfrac{1}{2}mv^2 t\,,$$

the total Lagrangian (2.3) is invariant under all Galilean transformations.

In order to display explicitly the conserved quantities in this theory, one considers separately the various transformations which comprise the extended Galilei group. Simplest of these is the phase transformation

$$\phi \to e^{im\theta}\phi,$$

the introduction of which leads for infinitesimal $\delta\theta$ to the form

$$\delta\mathcal{L} = -m\phi^\dagger\phi\frac{\partial}{\partial t}\delta\theta - \tfrac{1}{2}[\phi^\dagger\vec{p}\phi - (\vec{p}\phi^\dagger)\phi]\cdot\vec{\nabla}\delta\theta.$$

From this, one infers the conservation law

$$\frac{\partial \mathfrak{m}}{\partial t} + \vec{\nabla}\cdot\vec{\mathfrak{p}} = 0 \qquad (2.5)$$

and the generator

$$M = m\int d^3x\,\phi^\dagger\phi,$$

where we have defined

$$\mathfrak{m} = m\phi^\dagger\phi,$$

$$\vec{\mathfrak{p}} = \tfrac{1}{2}[\phi^\dagger\vec{p}\phi - (\vec{p}\phi^\dagger)\phi].$$

Equation (2.5) is, of course, more commonly written in nonrelativistic quantum mechanics without the factor of m and referred to as the conservation of probability.[5]

Spatial translations and rotations are conveniently handled by using the rules[6]

$$\delta(d^3x\,dt) = d^3x\,dt\,\vec{\nabla}\cdot\delta\vec{x},$$

$$\delta\left(\frac{\partial}{\partial t}\right) = -\left(\frac{\partial}{\partial t}\delta\vec{x}\right)\cdot\vec{\nabla}, \qquad (2.6)$$

$$\delta\left(\frac{\partial}{\partial x_i}\right) = -\left(\frac{\partial}{\partial x_i}\delta x_j\right)\nabla_j,$$

and the fact that for a local variation

$$\delta\phi = 0. \qquad (2.7)$$

This leads to

$$\delta W = \int d^3x\,dt\Big(\mathcal{L}\nabla_i\delta x_i + \tfrac{1}{2}[\phi^\dagger p_i\phi - (p_i\phi^\dagger)\phi]\frac{\partial}{\partial t}\delta x_i - \frac{1}{2m}[(p_i\phi^\dagger)(p_j\phi) + (p_j\phi^\dagger)(p_i\phi)]\nabla_j\delta x_i\Big),$$

where

$$W \equiv \int d^3x\,dt\,\mathcal{L}.$$

The resulting conservation law is thus

$$\frac{\partial}{\partial t}\mathfrak{p}^i + \nabla_j T^{ij} = 0, \qquad (2.8)$$

with the symmetrical tensor T^{ij} being given by

$$T^{ij} = \mathcal{L}\delta^{ij} - \frac{1}{2m}[(p_i\phi^\dagger)p_j\phi + (p_j\phi^\dagger)p_i\phi].$$

The expressions for P_i and J_i now follow immediately as one obtains

$$P_i = \int d^3x\,\mathfrak{p}_i$$

and

$$J_i = \int d^3x\,\epsilon_{ijk}x_j\mathfrak{p}_k.$$

Time displacements are handled in an analogous fashion and are seen to imply

$$\frac{\partial}{\partial t}\mathfrak{h} + \nabla_i\mathfrak{h}^i = 0,$$

where

$$\mathfrak{h} = \tfrac{1}{2}[\phi^\dagger E\phi - (E\phi^\dagger)\phi] - \mathcal{L}$$

and

$$\mathfrak{h}^i = -\frac{1}{2m}[(p_i\phi^\dagger)E\phi + (E\phi^\dagger)p_i\phi],$$

with the total energy being given by

$$H = \int d^3x\,\mathfrak{h}.$$

As a consequence of (2.4) pure Galilean transformations are slightly less trivial. In this case (2.6) and (2.7) become

$$\delta(d^3x\,dt) = d^3x\,dt\,t\vec{\nabla}\cdot\delta\vec{v},$$

$$\delta\left(\frac{\partial}{\partial t}\right) = -\delta\vec{v}\cdot\vec{\nabla} - t\left(\frac{\partial}{\partial t}\delta\vec{v}\right)\cdot\vec{\nabla},$$

$$\delta\left(\frac{\partial}{\partial x_i}\right) = -t\left(\frac{\partial}{\partial x_i}\delta v_j\right)\nabla_j,$$

and

$$\delta\phi = im\vec{x}\cdot\delta\vec{v}\phi.$$

Using these relations one finds after straightforward calculation

$$\frac{\partial}{\partial t}(\mathfrak{m}x^i - t\mathfrak{p}^i) + \nabla_j(x_i\mathfrak{p}_j - tT^{ij}) = 0, \qquad (2.9)$$

the consistency of which may be directly verified using (2.5) and (2.8). The conserved operator K_i is thus of the form

$$K_i = \int d^3x\,\mathfrak{m}x_i - tP_i.$$

It may be noted that upon using the equal-time canonical commutation relation

$$[\phi(\vec{x},t),\phi^\dagger(\vec{x}',t)] = \delta(\vec{x}-\vec{x}'),$$

the global operators M, P_i, J_i, H, and K_i can be easily shown to provide a representation of the Lie algebra of the extended Galilei group.

Having thus displayed the explicit Galilean invariance of the Lagrangian (2.3) one can now turn one's attention to the dilatation operation. The transformation law for ϕ corresponding to the scale transformation described by (1.2) and (1.3) is

$$\phi'(\vec{x}', t') = e^{-d\tau}\phi(\vec{x}, t)$$

or

$$\delta\phi = -d\phi\,\delta\tau, \tag{2.10}$$

where d, the dimension matrix, will be found to be $\frac{3}{2}$ times the unit matrix independent of the spin.[7] Using (2.10) and the asserted value of d, one can easily demonstrate the invariance of W for constant $\delta\tau$ provided that the scale-breaking term involving U_0 is discarded and $\mathfrak{L}_I$ assumed to be dilatationally invariant. Upon generalizing to the case in which $\delta\tau$ is allowed to have space-time dependence one finds

$$\delta W = \int d^3x\,dt\left(T^{ij}x_i\nabla_j + \vec{x}\cdot\vec{\mathfrak{p}}\,\frac{\partial}{\partial t} - 2t\mathfrak{h}\frac{\partial}{\partial t} + \frac{3}{4m}[\vec{\nabla}(\phi^\dagger\phi)]\cdot\vec{\nabla} - 2t\vec{\mathfrak{h}}\cdot\vec{\nabla}\right)\delta\tau$$

and the conservation law

$$\frac{\partial}{\partial t}(\vec{x}\cdot\vec{\mathfrak{p}} - 2t\mathfrak{h}) + \nabla_j\left(x_iT^{ij} - 2t\mathfrak{h}^j + \frac{3}{4m^2}\nabla_j\mathfrak{m}\right) = 0.$$

The generator of scale transformations

$$D = \int d^3x\,\vec{x}\cdot\vec{\mathfrak{p}} - 2tH$$

is consequently seen to satisfy the commutation relation

$$[D, \phi(\vec{x}, t)] = i\left(\vec{x}\cdot\vec{\nabla} + 2t\frac{\partial}{\partial t} + d\right)\phi(\vec{x}, t).$$

It is noteworthy that in much the same way as in Ref. 1 one can define a new tensor

$$\Theta_{ij} = T_{ij} + \frac{3}{8m^2}(\delta_{ij}\nabla^2 - \nabla_i\nabla_j)\mathfrak{m} \tag{2.11}$$

such that

$$\frac{\partial}{\partial t}(\vec{x}\cdot\vec{\mathfrak{p}} - 2t\mathfrak{h}) + \nabla_j(x_i\Theta^{ij} - 2t\mathfrak{h}^j) = 0 \tag{2.12}$$

and in terms of which (2.8) and (2.9), respectively, become

$$\frac{\partial}{\partial t}\mathfrak{p}^i + \nabla_j\Theta^{ij} = 0$$

and

$$\frac{\partial}{\partial t}(\mathfrak{m}x_i - t\mathfrak{p}_i) + \nabla^j(x_i\mathfrak{p}_j - t\Theta_{ij}) = 0.$$

The consistency of (2.12) is clearly seen to imply the "tracelessness" condition

$$\Theta_{ii} - 2t\mathfrak{h} = 0,$$

the validity of which can be verified by direct calculation.

In addition to the invariance of (2.3) under dilatations, one finds that of the four conformal transformations of special relativity there is one which remains a symmetry in the nonrelativistic limit. This is of the form

$$x_i' = x_i(1-ct)^{-1}, \quad t' = t(1-ct)^{-1}, \tag{2.13}$$

where c is an arbitrary real parameter.[8] The infinitesimal version of this,

$$\delta x_i = x_i t\delta c, \quad \delta t = t^2\delta c,$$

is a symmetry of (2.3) provided that (i) ϕ has the transformation law

$$\delta\phi = (\tfrac{1}{2}imx^2 - dt)\phi\,\delta c$$

and (ii) the interaction Lagrangian is local, scale-invariant, and of the nonderivative type. The form of the finite transformation is readily inferred upon writing

$$\phi'(\vec{x}', t') = (1-ct)^d\exp[i\eta(\vec{x}, t)]\phi(\vec{x}, t)$$

and demanding invariance of W. This condition leads to the set of equations

$$\frac{\partial\eta}{\partial x_i} = \frac{mcx_i}{1-ct},$$

$$\frac{\partial\eta}{\partial t} = (1-ct)^{-1}c\vec{x}\cdot\vec{\nabla}\eta - \frac{1}{2m}\left(\frac{\partial\eta}{\partial x_i}\right)^2,$$

which may be solved to yield

$$\eta(\vec{x}, t) = \frac{1}{2}\frac{mx^2c}{1-ct}.$$

Proceeding now in the standard way one infers from the action principle

$$\frac{\partial}{\partial t}(t\vec{x}\cdot\vec{\mathfrak{p}} - \mathfrak{h}t^2 - \tfrac{1}{2}\mathfrak{m}x^2) + \nabla_i\left(x_j\,tT^{ij} - \tfrac{1}{2}x^2\mathfrak{p}^i + \frac{3t}{4m^2}\nabla_i\mathfrak{m} - t^2\mathfrak{h}^i\right) = 0$$

which, in terms of Θ_{ij}, reads

$$\frac{\partial}{\partial t}(t\vec{x}\cdot\vec{\mathfrak{p}} - \mathfrak{h}t^2 - \tfrac{1}{2}\mathfrak{m}x^2) + \nabla_i(x_jt\Theta_{ij} - \tfrac{1}{2}x^2\mathfrak{p}^i - t^2\mathfrak{h}^i) = 0.$$

The generator of conformal transformations is seen to be of the form

$$C = \int d^3x(t\vec{x}\cdot\vec{\mathfrak{p}} - \tfrac{1}{2}\mathfrak{m}x^2) - t^2H = tD + t^2H - \tfrac{1}{2}\int d^3x\,\mathfrak{m}\,x^2$$

and satisfies the commutation relation

$$[C, \phi(\vec{x}, t)] = i\left(t\vec{x}\cdot\vec{\nabla} + t^2\frac{\partial}{\partial t} - \tfrac{1}{2}imx^2 + dt\right)\phi(\vec{x}, t).$$

With this result one is now in a position to write down the complete set of commutation relations of C and D with the generators of the extended Galilei group. These are given by

$$[J_i, C] = [J_i, D] = [M, C] = [M, D] = [K, C] = 0,$$

$$[P_i, D] = iP_i, \quad [K_i, D] = -iK_i, \quad [H, D] = 2iH,$$

$$[P_i, C] = -iK_i, \quad [H, C] = iD, \quad [C, D] = -2iC.$$

Upon making the identification

$$S_1 = \tfrac{1}{2}(H + C), \quad S_2 = \tfrac{1}{2}(H - C), \quad S_3 = \tfrac{1}{2}D,$$

the commutation relations of H, C, and D can be brought to the form

$$[S_1, S_2] = -iS_3,$$

$$[S_1, S_3] = iS_2,$$

$$[S_2, S_3] = iS_1.$$

This set comprises an O(2, 1) algebra and serves to illustrate the fact that because of the commutativity of J_i and S_i the conformal Galilei group has $O(3)\times O(2,1)$ as one of its subgroups. Such a result is, of course, to be expected from the existence of O(4, 2) as a symmetry of the conformally invariant Poincaré group.[9]

III. EXTENSION TO ARBITRARY SPIN

In order to determine the extent to which the results of Sec. II constitute general properties of conformally invariant Galilean field theories, it is essential to remove the restriction to scalar fields. The basic ingredient for the construction of higher-spin theories is spin $\frac{1}{2}$ and we consequently review here some properties of Lévy-Leblond's Galilean formulation of spin $\frac{1}{2}$.[10] He has shown that the equation for the four-component wave function (or field operator) ψ has the form

$$G\psi = 0, \tag{3.1}$$

where

$$G = E\tfrac{1}{2}(1+\rho_3) + \rho_1\vec{\sigma}\cdot\vec{p} + m(1-\rho_3),$$

the two independent sets of Pauli matrices ρ_i and σ_i having been introduced to span the 4×4-dimensional spinor space. The equation (3.1) is covariant with respect to the Galilei group provided that the transformation law for ψ is

$$\psi'(\vec{x}', t') = e^{if(\vec{x}, t)}\Delta^{1/2}(\vec{v}, R)\psi(\vec{x}, t),$$

where

$$\Delta^{1/2}(\vec{v}, R) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\vec{\sigma}\cdot\vec{v} & 1 \end{pmatrix} D^{1/2}(R),$$

with $D^{1/2}(R)$ being the usual two-dimensional representation of spin $\frac{1}{2}$. The decomposition

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

into two-component spinors ϕ and χ yields the equations

$$E\phi + \vec{\sigma}\cdot\vec{p}\chi = 0,$$

$$\vec{\sigma}\cdot\vec{p}\phi + 2m\chi = 0,$$

thereby displaying the ϕ_i as independent components with true equations of motion and the χ_i as dependent variables defined in terms of the ϕ_i.

The problem of explicitly writing down a Galilean-invariant theory of spin S which is of first order in all derivatives can now be approached by constructing the most general Galilean-invariant Lagrangian for a totally symmetrized $2S$-rank spinor $\psi_{a_1\cdots a_{2S}}$. An important shortcoming of the multispinor formalism, however, consists in the fact that the equations of motion involve only a small subset of the total number of components of $\psi_{a_1\cdots a_{2S}}$. In order to deal with this problem it is convenient to project out of $\psi_{a_1\cdots a_{2S}}$ a basis of spherical-tensor fields ϕ_S^m and $\chi_{S-1/2,1/2}^{m,m'}$ where ϕ_S^m transforms as the $(2S+1)$-dimensional irreducible representation of O(3) and $\chi_{S-1/2,1/2}^{m,m'}$ transforms as the Kronecker product of the $2S$-dimensional and 2-dimensional irreducible representations of that group. This provides a $(6S+1)$-component representation and defines what will be referred to here as a minimal theory[11] (i.e., it is not possible to write down Galilean-invariant theories describing spin-S particles which are of first order in all derivatives with fewer than $6S+1$ components). It has been shown by the author[12] that in terms of these spherical-tensor fields the most general Lagrangian is[13]

$$\mathcal{L} = \tfrac{1}{2}\Bigg\{\phi_S^{m\dagger}E\phi_S^m - U_0\phi_S^{m\dagger}\phi_S^m + 2m\chi_{S-1/2,1/2}^{m,m'\dagger}\chi_{S-1/2,1/2}^{m,m'} + 6^{1/2}(2S+1)^{1/2}\left[\chi_{S-1/2,1/2}^{m,m'\dagger}p^\nu\begin{pmatrix} m & \frac{1}{2} & S \\ S-\frac{1}{2} & \mu & m'' \end{pmatrix}\begin{pmatrix} m' & 1 & \mu \\ \frac{1}{2} & \nu & \frac{1}{2} \end{pmatrix}\phi_S^{m''} + \phi_S^{m\dagger}p^\nu\begin{pmatrix} m & \mu & S-\frac{1}{2} \\ S & \frac{1}{2} & m' \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \mu & \nu & m'' \end{pmatrix}\chi_{S-1/2,1/2}^{m',m''}\right] + \text{H.c.}\Bigg\} + \mathcal{L}_I, \tag{3.2}$$

where we have made use of the three-j symbol and Wigner's[14] tensor notation for the O(3) group. Thus repeated upper and lower indices are summed while two repeated upper indices are summed, provided that one is the adjoint of a contravariant field. We note that the spherical components of the covariant vector x_μ are given by

$$x_{\pm 1} = \mp 2^{-1/2}(x \pm iy), \quad x_0 = z,$$

while for the contravariant vector p^μ one has

$$p^{\pm 1} = \frac{1}{i}\, 2^{-1/2}\left(\mp\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad p^0 = \frac{1}{i}\frac{\partial}{\partial z}.$$

Contravariant and covariant indices can, of course, be lowered and raised by means of the metric tensors

$$g^S_{mm'} = (-1)^{S+m}\delta_{-m,m'} \quad \text{and} \quad g_S^{mm'} = (-1)^{S-m}\delta_{-m,m'}.$$

As a consequence of the tensor notation one has the important simplification that the Lagrangian (3.2) is manifestly invariant under rotations and the only nontrivial operation is to verify the asserted invariance under Galilean boosts. A proof of this result making use of the transformation law

$$(\phi^m_S)'(\vec{x}', t') = e^{if(\vec{x},t)}\phi^m_S(\vec{x}, t),$$

$$(\chi^{m,m'}_{S-1/2,1/2})'(\vec{x}', t') = e^{if(\vec{x},t)}\left[\chi^{m,m'}_{S-1/2,1/2}(\vec{x}, t) - (\tfrac{3}{2})^{1/2}(2S+1)^{1/2}\begin{pmatrix} m & \frac{1}{2} & S \\ S-\frac{1}{2} & \mu & m'' \end{pmatrix}\begin{pmatrix} m' & 1 & \mu \\ \frac{1}{2} & \nu & \frac{1}{2} \end{pmatrix} v^\nu \phi^{m''}_S(\vec{x}, t)\right],$$

is given in Ref. 12.

It is to be noted that in analogy to the spinless situation one expects to have the free Lagrangian consist, in general, of a sum over fields of different masses, spins, and bare internal energies. The interaction Lagrangian $\mathfrak{L}_I$ which, of course, is taken to be Galilean-invariant and local, is assumed furthermore to not explicitly involve derivatives even though the χ field, if it appears in the interaction term, does give rise to derivative-coupling effects. A detailed consideration of trilinear interactions including the case in which the χ fields are coupled is to be found in Ref. 12.

One can now carry through the derivation of the generators of the extended Galilei group in the nonzero-spin case. The phase transformation

$$\phi \to e^{im\theta}\phi, \quad \chi \to e^{im\theta}\chi$$

may be seen by inspection to yield

$$\frac{\partial}{\partial t}\mathfrak{m} + \nabla^\nu \mathfrak{m}_\nu = 0, \tag{3.3}$$

where

$$\mathfrak{m} = m\phi^{m\dagger}_S\phi^m_S$$

and

$$\mathfrak{m}_\nu = 6^{1/2}(2S+1)^{1/2} m\left[\phi^{m\dagger}_S \begin{pmatrix} m & \frac{1}{2} & S-\frac{1}{2} \\ S & \mu & m' \end{pmatrix}\begin{pmatrix} \mu & 1 & \frac{1}{2} \\ \frac{1}{2} & \nu & m'' \end{pmatrix}\chi^{m',m''}_{S-1/2,1/2} + \chi^{m,m'\dagger}_{S-1/2,1/2}\begin{pmatrix} m & \mu & S \\ S-\frac{1}{2} & \frac{1}{2} & m'' \end{pmatrix}\begin{pmatrix} m' & 1 & \frac{1}{2} \\ \frac{1}{2} & \nu & \mu \end{pmatrix}\phi^{m''}_S\right],$$

with the generator being given by

$$M = m\int d^3x\, \phi^{m\dagger}_S \phi^m_S.$$

We draw attention to the fact that the three-dimensional vector in (3.3) is denoted here by $\mathfrak{m}_\nu$ rather than $\mathfrak{p}_\nu$. That this vector is not identifiable with the momentum density for nonzero spin is a result which will be displayed momentarily.

The conservation law associated with invariance under translations and rotations follows upon application of (2.6) (rewritten in a spherical basis) and the local variations

$$\delta\phi^m_S = -\tfrac{1}{2}(S\|S\|S)\begin{pmatrix} m & 1 & S \\ S & \nu & m' \end{pmatrix}\phi^{m'}_S \epsilon^{\nu\alpha\beta}\nabla_\alpha \delta x_\beta,$$

$$\delta\chi^{m,m'}_{S-1/2,1/2} = -\tfrac{1}{2}\left[(S-\tfrac{1}{2}\|S\|S-\tfrac{1}{2})\begin{pmatrix} m & 1 & S-\frac{1}{2} \\ S-\frac{1}{2} & \nu & m'' \end{pmatrix}\chi^{m'',m'}_{S-1/2,1/2} + (\tfrac{1}{2}\|S\|\tfrac{1}{2})\begin{pmatrix} m' & 1 & \frac{1}{2} \\ \frac{1}{2} & \nu & m'' \end{pmatrix}\chi^{m,m''}_{S-1/2,1/2}\right]\epsilon^{\nu\alpha\beta}\nabla_\alpha\delta x_\beta,$$

where we have introduced the conventional notation $(S\|S\|S)$ for the reduced matrix element of the spin. In much the same way as in Schwinger's derivation[6] of the symmetrical energy-momentum tensor in special relativity, one finds

$$\delta W = \int d^3x\, dt\left(\mathfrak{p}^\mu \frac{\partial}{\partial t}\delta x_\mu + T^{\mu\nu}\partial_\mu \delta x_\nu\right) \tag{3.4}$$

so that

$$\frac{\partial}{\partial t}\mathfrak{p}^\mu + \nabla_\nu T^{\mu\nu} = 0, \tag{3.5}$$

where

$$\mathfrak{p}^\mu = \tfrac{1}{2}\left[\phi^{m\dagger}_S \frac{1}{i}\nabla^\mu \phi^m_S + \frac{i}{2}\epsilon^{\mu\alpha\beta}\nabla_\beta (S\|S\|S)\phi^{m\dagger}_S\begin{pmatrix} m & 1 & S \\ S & \alpha & m' \end{pmatrix}\phi^{m'}_S + \text{H.c.}\right] \tag{3.6}$$

and the symmetrical tensor $T^{\mu\nu}$ is given by

$$\begin{aligned} T^{\mu\nu} = \mathcal{L}g^{\mu\nu} + \tfrac{1}{2}6^{1/2}(2S+1)^{1/2}\Bigg\{ & \phi^{m\dagger}_S p^{(\mu}\begin{pmatrix} m & \frac{1}{2} & S-\frac{1}{2} \\ S & \alpha & m' \end{pmatrix}\begin{pmatrix} \alpha & \nu) & \frac{1}{2} \\ \frac{1}{2} & 1 & m'' \end{pmatrix}\chi^{m',m''}_{S-1/2,1/2} \\ & + \chi^{m,m'\dagger}_{S-1/2,1/2} p^{(\mu}\begin{pmatrix} m & \alpha & S \\ S-\frac{1}{2} & \frac{1}{2} & m'' \end{pmatrix}\begin{pmatrix} m' & \nu) & \frac{1}{2} \\ \frac{1}{2} & 1 & \alpha \end{pmatrix}\phi^{m''}_S \\ & - (S\|S\|S)\epsilon^{\alpha\beta(\mu}p_\alpha\left[\chi^{m,m'\dagger}_{S-1/2,1/2}\begin{pmatrix} m & \frac{1}{2} & S \\ S-\frac{1}{2} & \lambda & m' \end{pmatrix}\begin{pmatrix} m' & \nu) & \lambda \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} m'' & 1 & S \\ S & \beta & m''' \end{pmatrix}\phi^{m'''}_S\right] \\ & - (\tfrac{1}{2}\|S\|\tfrac{1}{2})\epsilon^{\alpha\beta(\mu}p_\alpha\left[\phi^{m\dagger}_S\begin{pmatrix} m & \lambda & S-\frac{1}{2} \\ S & \frac{1}{2} & m' \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \nu) & \frac{1}{2} \\ \lambda & 1 & m'' \end{pmatrix}\begin{pmatrix} m'' & 1 & \frac{1}{2} \\ \frac{1}{2} & \beta & m''' \end{pmatrix}\chi^{m',m'''}_{S-1/2,1/2}\right] \\ & - (S-\tfrac{1}{2}\|S\|S-\tfrac{1}{2})\epsilon^{\alpha\beta(\mu}p_\alpha\left[\phi^{m\dagger}_S\begin{pmatrix} m & \lambda & S-\frac{1}{2} \\ S & \frac{1}{2} & m' \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \nu) & \frac{1}{2} \\ \lambda & 1 & m'' \end{pmatrix}\begin{pmatrix} m' & 1 & S-\frac{1}{2} \\ S-\frac{1}{2} & \beta & m''' \end{pmatrix}\chi^{m''',m''}_{S-1/2,1/2}\right] + \text{H.c.}\Bigg\}. \end{aligned}$$

In writing the above we have used the notation

$$A^{(\mu\nu)} \equiv \tfrac{1}{2}(A^{\mu\nu} + A^{\nu\mu})$$

to denote the symmetrical part of a tensor as well as the customary definition

$$(-1)^{\rho+\sigma+\tau}[A^{-\rho,-\sigma,-\tau,\dots}]^\dagger$$

for the Hermitian conjugate of spherical tensor $A^{\rho\sigma\tau\cdots}$. The expressions for the total momentum and angular momentum are now immediate consequences of (3.4) and (3.6), the results being

$$P^\mu = \int d^3x\, \frac{1}{2}\left[\phi^{m\dagger}_S \frac{1}{i}\nabla^\mu \phi^m_S - \frac{1}{i}(\nabla^\mu \phi^{m\dagger}_S)\phi^m_S\right]$$

and

$$J^{\mu\nu} = \int d^3x\left\{\tfrac{1}{2}\phi^{m\dagger}_S\left(x^\mu \frac{1}{i}\nabla^\nu - x^\nu \frac{1}{i}\nabla^\mu\right)\phi^m_S - \tfrac{1}{2}\left[\left(x^\mu\frac{1}{i}\nabla^\nu - x^\nu\frac{1}{i}\nabla^\mu\right)\phi^{m\dagger}_S\right]\phi^m_S - ie^{\mu\nu\alpha}(S\|S\|S)\phi^{m\dagger}_S\begin{pmatrix} m & 1 & S \\ S & \alpha & m' \end{pmatrix}\phi^{m'}_S\right\}.$$

As in the spinless case time translations are handled in exceedingly straightforward fashion and we are consequently content merely to state that one finds the energy-conservation equation

$$\frac{\partial}{\partial t}\mathfrak{h} + \nabla^\nu \mathfrak{h}_\nu = 0, \tag{3.7}$$

with

$$\mathfrak{h} = \tfrac{1}{2}[\phi^{m\dagger}_S E\phi^m_S - (E\phi^{m\dagger}_S)\phi^m_S] - \mathcal{L}$$

and

$$\mathfrak{h}_\nu = \tfrac{1}{2}6^{1/2}(2S+1)^{1/2}\left[\phi^{m\dagger}_S E\begin{pmatrix} m & \frac{1}{2} & S-\frac{1}{2} \\ S & \mu & m' \end{pmatrix}\begin{pmatrix} \mu & 1 & \frac{1}{2} \\ \frac{1}{2} & \nu & m'' \end{pmatrix}\chi^{m',m''}_{S-1/2,1/2} + \chi^{m,m'\dagger}_{S-1/2,1/2}E\begin{pmatrix} m & \mu & S \\ S-\frac{1}{2} & \frac{1}{2} & m'' \end{pmatrix}\begin{pmatrix} m' & 1 & \frac{1}{2} \\ \frac{1}{2} & \nu & \mu \end{pmatrix}\phi^{m''}_S + \text{H.c.}\right].$$

Thus one has for the total energy the result

$$H=\int d^3x\,\{\tfrac{1}{2}[\phi_S^{m\dagger}E\phi_S^m-(E\phi_S^{m\dagger})\phi_S^m]-\mathfrak{L}\}\,.$$

The treatment of Galilean boosts is the most technically difficult of the Galilean group operations largely because of the somewhat intricate form of the field variations

$$\delta\phi_S^m=imx_\nu\delta v^\nu\phi_S^m\,,$$

$$\delta\chi_{S-1/2,\,1/2}^{m,\,m'}=imx_\nu\delta v^\nu\chi_{S-1/2,\,1/2}^{m,\,m'}-(\tfrac{3}{2})^{1/2}(2S+1)^{1/2}\begin{pmatrix} m & \frac{1}{2} & S\\ S-\frac{1}{2} & \mu & m''\end{pmatrix}\begin{pmatrix} m' & 1 & \mu\\ \frac{1}{2} & \nu & \frac{1}{2}\end{pmatrix}v^\nu\phi_S^{m''}\,.$$

However, simplification ensues upon recalling that the asserted invariance of $\mathfrak{L}$ means that one requires only those contributions to δW which involve a derivative of the boost parameter. This leads to the conservation law

$$\frac{\partial}{\partial t}\left[-mx^\nu\phi_S^{m\dagger}\phi_S^m-\tfrac{1}{2}t\left(\phi_S^{m\dagger}\frac{1}{i}\nabla^\nu\phi_S^m-\frac{1}{i}(\nabla^\nu\phi_S^{m\dagger})\phi_S^m\right)\right]$$
$$+\nabla_\mu\Bigg\{-g^{\mu\nu}t\mathfrak{L}-\mathfrak{m}^\mu x^\nu+\tfrac{1}{2}t6^{1/2}(2S+1)^{1/2}\Bigg[\chi_{S-1/2,\,1/2}^{m,\,m'\dagger}p^\nu\begin{pmatrix} m & \frac{1}{2} & S\\ S-\frac{1}{2} & \mu & m''\end{pmatrix}\begin{pmatrix} m' & 1 & \mu'\\ \frac{1}{2} & \mu & \frac{1}{2}\end{pmatrix}\phi_S^{m''}$$
$$+\phi_S^{m\dagger}p^\nu\begin{pmatrix} m & \mu' & S-\frac{1}{2}\\ S & \frac{1}{2} & m'\end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2}\\ \mu' & \mu & m''\end{pmatrix}\chi_{S-1/2,\,1/2}^{m',\,m''}+\text{H.c.}\Bigg]$$
$$-\frac{i}{2S}\epsilon^{\mu\nu\lambda}\phi_S^{m\dagger}\begin{pmatrix} m & 1 & S\\ S & \lambda & m'\end{pmatrix}\phi_S^{m'}(S\|S\|S)\Bigg\}=0\,. \tag{3.8}$$

Upon adding and subtracting the term

$$\left(1+t\frac{\partial}{\partial t}\right)\frac{i}{2}\epsilon^{\nu\alpha\beta}\nabla_\beta(S\|S\|S)\phi_S^{m\dagger}\begin{pmatrix} m & 1 & S\\ S & \alpha & m'\end{pmatrix}\phi_S^{m'}\,,$$

(3.8) can be brought to the form

$$\frac{\partial}{\partial t}(-\mathfrak{m}x^\nu-t\mathfrak{p}^\nu)+\nabla_\mu\left[-\mathfrak{m}^\mu x^\nu-t\,\tilde{T}^{\mu\nu}+\frac{i}{2}\left(1-\frac{1}{S}\right)(S\|S\|S)\epsilon^{\mu\nu\lambda}\phi_S^{m\dagger}\begin{pmatrix} m & 1 & S\\ S & \lambda & m'\end{pmatrix}\phi_S^{m'}\right]=0\,, \tag{3.9}$$

where the (nonsymmetrical) tensor $\tilde{T}^{\mu\nu}$ is given by

$$\tilde{T}^{\mu\nu}=g^{\mu\nu}\mathfrak{L}-\tfrac{1}{2}6^{1/2}(2S+1)^{1/2}\Bigg[\chi_{S-1/2,\,1/2}^{m,\,m'\dagger}p^\nu\begin{pmatrix} m & \frac{1}{2} & S\\ S-\frac{1}{2} & \mu' & m''\end{pmatrix}\begin{pmatrix} m' & \mu & \mu'\\ \frac{1}{2} & 1 & \frac{1}{2}\end{pmatrix}\phi_S^{m''}$$
$$+\phi_S^{m\dagger}p^\nu\begin{pmatrix} m & \mu' & S-\frac{1}{2}\\ S & \frac{1}{2} & m'\end{pmatrix}\begin{pmatrix} \frac{1}{2} & \mu & \frac{1}{2}\\ \mu' & 1 & m''\end{pmatrix}\chi_{S-1/2,\,1/2}^{m',\,m''}+\text{H.c.}\Bigg]-\frac{i}{2}\frac{\partial}{\partial t}\epsilon^{\mu\nu\lambda}(S\|S\|S)\phi_S^{m\dagger}\begin{pmatrix} m & 1 & S\\ S & \lambda & m'\end{pmatrix}\phi_S^{m'}\,.$$

On the other hand, an examination of the coefficient of t in (3.9) clearly implies the consistency condition

$$\nabla_\mu\tilde{T}^{\mu\nu}=\nabla_\mu T^{\mu\nu}\,,$$

a result which can alternatively be derived by direct calculation using the equations of motion. Thus (3.9) assumes the form

$$\frac{\partial}{\partial t}(-\mathfrak{m}x^\nu-t\mathfrak{p}^\nu)+\nabla_\mu\left[-\mathfrak{m}^\mu x^\nu-t\,T^{\mu\nu}+\frac{i}{2}\left(1-\frac{1}{S}\right)(S\|S\|S)\epsilon^{\mu\nu\lambda}\phi_S^{m\dagger}\begin{pmatrix} m & 1 & S\\ S & \lambda & m'\end{pmatrix}\phi_S^{m'}\right]=0 \tag{3.10}$$

and one has for the generator the result

$$K^\nu=-\int d^3x\,\mathfrak{m}x^\nu-tP^\nu\,.$$

It is to be noted that the consistency of (3.10) with (3.3) and (3.5) leads to the condition

$$\mathfrak{p}^\nu=-\mathfrak{m}^\nu+\frac{i}{2}\left(1-\frac{1}{S}\right)\epsilon^{\mu\nu\lambda}\nabla_\mu(S\|S\|S)\phi_S^{m\dagger}\begin{pmatrix} m & 1 & S\\ S & \lambda & m'\end{pmatrix}\phi_S^{m'}\,, \tag{3.11}$$

a result which can easily be verified by use of the field equations. A number of comments are in order concerning Eq. (3.11). A somewhat trivial one has to do with the fact that since $\mathfrak{p}^i$ coincides with $\mathfrak{m}^i$ in the spinless case, the minus sign on the right-hand side of (3.11) might be considered suspect. This, however, is merely a consequence of the form of the metric tensor and the fact that $\mathfrak{m}_\nu$ was originally defined in covariant form while $\mathfrak{p}^\nu$ was defined in contravariant form. Of greater interest perhaps is the significance of the spin terms in (3.11). Since without the $1/S$ term one has only the spin part which appears in the definition of $\mathfrak{p}^\nu$, the $1/S$ contribution is obviously identifiable with the spin content of the operator $\mathfrak{m}^\nu$. On the other hand, $\mathfrak{m}^\nu$ is essentially the electromagnetic current (as is easily seen upon introduction of minimal coupling) and the usual g factor describing the magnetic-moment interaction with the magnetic field is thus inferred to have the value $1/S$, a result which is in complete agreement with an earlier calculation of this same quantity.[15] However, inasmuch as this latter work also displayed the fact that the g factor can differ from $1/S$ for nonminimal theories, one sees here a definite indication of model dependence in the conservation law (3.10).

IV. SCALE AND CONFORMAL TRANSFORMATIONS

With the derivation of the complete set of conservation laws implied by Galilean invariance in the general-spin case one can now examine the situation which ensues upon imposing the further condition that (3.2) be scale-invariant and consequently also conformally invariant. Inasmuch as the transformations (1.2) and (1.3) imply that the dimensionality of a scale-invariant term in $\mathfrak{L}$ be minus five, the bare-internal-energy term must be set equal to zero as in the spin-zero case. Although scale-breaking terms present no conceptual difficulty and one could readily derive scale-breaking corrections to the conservation laws, we assume here the simplest situation in which the scale and conformal operations are exact.

Using the rules

$$\delta(d^3x\,dt)=d^3x\,dt\left(5+x_\nu\nabla^\nu+2t\frac{\partial}{\partial t}\right)\delta\tau,$$

$$\delta\left(\frac{\partial}{\partial t}\right)=-2\delta\tau\frac{\partial}{\partial t}-2\left(\frac{\partial}{\partial t}\delta\tau\right)t\frac{\partial}{\partial t}-\left(\frac{\partial}{\partial t}\delta\tau\right)x_\nu\nabla^\nu,$$

$$\delta(\nabla^\nu)=-\delta\tau\nabla^\nu-(\nabla^\nu\delta\tau)x_\mu\nabla^\mu-2t(\nabla^\nu\delta\tau)\frac{\partial}{\partial t},$$

and the corresponding scale transformations on the fields

$$\delta\phi_S^m=-d\phi_S^m\,\delta\tau,\qquad \delta\chi_{S-1/2,\,1/2}^{m,\,m'}=-(d+1)\chi_{S-1/2,\,1/2}^{m,\,m'}\delta\tau,$$

one readily verifies that for constant $\delta\tau$ the Lagrangian (3.2) is invariant for the choice $d=\frac{3}{2}$ if $\mathfrak{L}_I$ is.[16] Thus one finds from the coefficients of $(\partial/\partial t)\delta\tau$ and $\nabla_\nu\delta\tau$ in the expression for the variation of W that

$$\begin{aligned}\frac{\partial}{\partial t}&\left[x_\nu\frac{1}{2}\left(\phi_S^{m+}\frac{1}{i}\nabla^\nu\phi_S^m-\frac{1}{i}(\nabla^\nu\phi_S^{m+})\phi_S^m\right)-2t\mathfrak{h}\right]\\
&+\nabla^\nu\Bigg\{-2t\mathfrak{h}_\nu-x^\alpha 6^{1/2}(2S+1)^{1/2}\frac{1}{2}\left[\chi_{S-1/2,\,1/2}^{m,\,m'}p_\alpha\begin{pmatrix}m&\frac{1}{2}&S\\ S-\frac{1}{2}&\mu&m''\end{pmatrix}\begin{pmatrix}m'&1&\mu\\ \frac{1}{2}&\nu&\frac{1}{2}\end{pmatrix}\phi_S^{m''}\right.\\
&\qquad\left.+\phi_S^{m+}p^\alpha\begin{pmatrix}m&\mu&S-\frac{1}{2}\\ S-\frac{1}{2}&\frac{1}{2}&m'\end{pmatrix}\begin{pmatrix}\frac{1}{2}&1&\frac{1}{2}\\ \mu&\nu&m''\end{pmatrix}\chi_{S-1/2,\,1/2}^{m',\,m''}+\text{H.c.}\right]\\
&\qquad+x^\nu\mathfrak{L}-\frac{i}{2}6^{1/2}(2S+1)^{1/2}\left[\chi_{S-1/2,\,1/2}^{m,\,m'+}\begin{pmatrix}m&\frac{1}{2}&S\\ S-\frac{1}{2}&\mu&m''\end{pmatrix}\begin{pmatrix}m'&1&\mu\\ \frac{1}{2}&\nu&\frac{1}{2}\end{pmatrix}\phi_S^{m''}\right.\\
&\qquad\left.-\phi_S^{m+}\begin{pmatrix}m&\mu&S-\frac{1}{2}\\ S&\frac{1}{2}&m'\end{pmatrix}\begin{pmatrix}\frac{1}{2}&1&\frac{1}{2}\\ \mu&\nu&m''\end{pmatrix}\chi_{S-1/2,\,1/2}^{m',\,m''}\right]\Bigg\}=0.\end{aligned}\tag{4.1}$$

In much the same way as was done in the case of pure boosts one can manipulate the spin-type terms so as to bring (4.1) to

$$\begin{aligned}\frac{\partial}{\partial t}(x_\nu\mathfrak{p}^\nu-2t\mathfrak{h})+\nabla_\nu\Bigg\{&-2t\mathfrak{h}^\nu+x_\mu\tilde{T}^{\nu\mu}-\frac{i}{2}\left[\chi_{S-1/2,\,1/2}^{m,\,m'+}\begin{pmatrix}m&\frac{1}{2}&S\\ S-\frac{1}{2}&\mu&m''\end{pmatrix}\begin{pmatrix}m'&\nu&\mu\\ \frac{1}{2}&1&\frac{1}{2}\end{pmatrix}\phi_S^{m''}\right.\\
&\left.-\phi_S^{m+}\begin{pmatrix}m&\mu&S-\frac{1}{2}\\ S&\frac{1}{2}&m'\end{pmatrix}\begin{pmatrix}\frac{1}{2}&\nu&\frac{1}{2}\\ \mu&1&m''\end{pmatrix}\chi_{S-1/2,\,1/2}^{m',\,m''}\right]6^{1/2}(2S+1)^{1/2}\Bigg\}=0.\end{aligned}$$

Inasmuch as $\tilde{T}^{\nu\mu}$ is multiplied by x_μ the replacement of $\tilde{T}^{\nu\mu}$ by the symmetric $T^{\mu\nu}$ is nontrivial in the present case and requires the introduction of a number of counter terms. Doing this one finds the form

$$\frac{\partial}{\partial t}(x_\nu \mathfrak{p}^\nu - 2t\mathfrak{h}) + \nabla_\nu(-2t\mathfrak{h}^\nu + x_\mu T^{\mu\nu} + \sigma^\nu)\,, \tag{4.2}$$

where σ_ν is the complex structure

$$\begin{aligned}\sigma_\nu = -\frac{i}{2}6^{1/2}(2S+1)^{1/2}\Big[&\chi^{m,\,m'+}_{S-1/2,\,1/2}\begin{pmatrix} m & \frac{1}{2} & S \\ S-\frac{1}{2} & \mu & m'' \end{pmatrix}\begin{pmatrix} m' & 1 & \mu \\ \frac{1}{2} & \nu & \frac{1}{2} \end{pmatrix}\phi_S^{m''} - \phi_S^{m+}\begin{pmatrix} m & \mu & S-\frac{1}{2} \\ S & \frac{1}{2} & m' \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \mu & \nu & m'' \end{pmatrix}\chi^{m',\,m''}_{S-1/2,\,1/2}\Big] \\
+6^{1/2}(2S+1)^{1/2}\frac{1}{2}\Big[&i(S\|S\|S)\epsilon_{\nu\alpha\beta}\chi^{m,\,m'+}_{S-1/2,\,1/2}\begin{pmatrix} m & \frac{1}{2} & S \\ S-\frac{1}{2} & \lambda & m'' \end{pmatrix}\begin{pmatrix} m' & \alpha & \lambda \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} m'' & \beta & S \\ S & 1 & m''' \end{pmatrix}\phi_S^{m'''} \\
&+i(\tfrac{1}{2}\|S\|\tfrac{1}{2})\epsilon_{\nu\alpha\beta}\,\phi_S^{m+}\begin{pmatrix} m & \lambda & S-\frac{1}{2} \\ S & \frac{1}{2} & m' \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \alpha & \frac{1}{2} \\ \lambda & 1 & m'' \end{pmatrix}\begin{pmatrix} m'' & \beta & \frac{1}{2} \\ \frac{1}{2} & 1 & m''' \end{pmatrix}\chi^{m',\,m'''}_{S-1/2,\,1/2} \\
&+i(S-\tfrac{1}{2}\|S\|S-\tfrac{1}{2})\epsilon_{\nu\alpha\beta}\phi_S^{m+}\begin{pmatrix} m & \lambda & S-\frac{1}{2} \\ S & \frac{1}{2} & m' \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \alpha & \frac{1}{2} \\ \lambda & 1 & m'' \end{pmatrix}\begin{pmatrix} m' & \beta & S-\frac{1}{2} \\ S-\frac{1}{2} & 1 & m''' \end{pmatrix}\chi^{m''',\,m''}_{S-1/2,\,1/2} + \text{H.c.}\Big]\,,\end{aligned}$$

while the generator of scale transformations is seen to be

$$D = \int d^3x\, x_\nu \mathfrak{p}^\nu - 2tH\,.$$

The consistency of (4.2) with (3.5) and (3.7) obviously implies the condition

$$T^\mu{}_\mu - 2\mathfrak{h} + \nabla^\nu \sigma_\nu = 0\,. \tag{4.3}$$

It is of interest to note that one can make contact with the discussion of Ref. 1 concerning scale transformations by redefining the energy density as

$$\tilde{\mathfrak{h}} = \mathfrak{h} - \tfrac{1}{2}\nabla^\nu \sigma_\nu\,,$$

a step which clearly involves no corresponding modification of the total energy H. In order to retain (3.7), however, one requires a compensating change in the energy flux $\mathfrak{h}_\nu$. To this end one writes

$$\tilde{\mathfrak{h}}_\nu = \mathfrak{h}_\nu + \frac{1}{2}\frac{\partial}{\partial t}\sigma_\nu$$

so that (3.7) becomes

$$\frac{\partial}{\partial t}\tilde{\mathfrak{h}} + \nabla^\nu \tilde{\mathfrak{h}}_\nu = 0\,.$$

Using this result (4.2) assumes the form

$$\frac{\partial}{\partial t}(x_\nu \mathfrak{p}^\nu - 2t\tilde{\mathfrak{h}}) + \nabla_\nu(x_\mu T^{\mu\nu} - 2t\tilde{\mathfrak{h}}^\nu) = 0\,,$$

which implies the replacement of (4.3) by the "traceless" condition

$$T^\mu{}_\mu - 2\tilde{\mathfrak{h}} = 0\,.$$

Thus the main point of contrast with the result of Callan *et al.* in this case consists in the fact that they found no such redefinition to be required in the nonzero-spin case in order to obtain the vanishing-trace condition on the energy-momentum tensor.

Conformal transformations as described by Eq. (2.13) can be shown from the transformation laws

$$\begin{aligned}\delta\phi_S^m &= (\tfrac{1}{2}imx^2 - dt)\phi_S^m \delta c\,, \\
\delta\chi^{m,\,m'}_{S-1/2,\,1/2} &= [\tfrac{1}{2}imx^2 - (d+1)t]\chi^{m,\,m'}_{S-1/2,\,1/2}\delta c + (\tfrac{3}{2})^{1/2}(2S+1)^{1/2}\begin{pmatrix} m & \frac{1}{2} & S \\ S-\frac{1}{2} & \mu & m'' \end{pmatrix}\begin{pmatrix} m' & 1 & \mu \\ \frac{1}{2} & \nu & \frac{1}{2} \end{pmatrix}x^\nu \phi_S^{m''}\delta c\,,\end{aligned} \tag{4.4}$$

to be an invariance of the Lagrangian provided that $\mathfrak{L}$ is local as well as dilatationally invariant and the coupling term does not explicitly contain derivatives. This calculation may be facilitated by making use of the close resemblance between (4.4) and the corresponding expressions appropriate to Galilean boosts. Upon performing the usual variation of W one is led in the standard way to the conservation law

$$\frac{\partial}{\partial t}\left[x_\nu t\,\frac{1}{2}\left(\phi_S^{m+}\frac{1}{i}\nabla^\nu\phi_S^m-\frac{1}{i}(\nabla^\nu\phi_S^{m+})\phi_S^m\right)-t^2\mathfrak{h}-\tfrac{1}{2}\mathfrak{m}\,x^2\right]$$
$$+\nabla^\nu\Bigg\{-t^2\mathfrak{h}_\nu-\tfrac{1}{2}x^2\mathfrak{m}_\nu+x_\nu t\mathcal{L}-6^{1/2}(2S+1)^{1/2}t\,\frac{1}{2}\Bigg[\chi^{m,m'}_{S-1/2,1/2}x_\alpha p^\alpha\begin{pmatrix} m & \frac{1}{2} & S\\ S-\frac{1}{2} & \mu & m''\end{pmatrix}\begin{pmatrix} m' & 1 & \mu\\ \frac{1}{2} & \nu & \frac{1}{2}\end{pmatrix}\phi_S^{m''}$$
$$+\phi_S^{m+}x_\alpha p^\alpha\begin{pmatrix} m & \mu & S-\frac{1}{2}\\ S & \frac{1}{2} & m'\end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2}\\ \mu & \nu & m''\end{pmatrix}\chi^{m',m''}_{S-1/2,1/2}+\text{H.c.}\Bigg]$$
$$-6^{1/2}(2S+1)^{1/2}it\,\frac{1}{2}\Bigg[\chi^{m,m'}_{S-1/2,1/2}\begin{pmatrix} m & \frac{1}{2} & S\\ S-\frac{1}{2} & \mu & m''\end{pmatrix}\begin{pmatrix} m' & 1 & \mu\\ \frac{1}{2} & \nu & \frac{1}{2}\end{pmatrix}\phi_S^{m''}$$
$$+\phi_S^{m+}\begin{pmatrix} m & \mu & S-\frac{1}{2}\\ S & \frac{1}{2} & m'\end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2}\\ \mu & \nu & m''\end{pmatrix}\chi^{m',m''}_{S-1/2,1/2}\Bigg]-x^\mu(S\|S\|S)\frac{i}{2S}\epsilon_{\mu\nu\lambda}\phi_S^{m+}\begin{pmatrix} m & 1 & S\\ S & \lambda & m'\end{pmatrix}\phi_S^{m'}\Bigg\}=0\,,$$

which by means of now familiar manipulations may be rewritten as

$$\frac{\partial}{\partial t}(x_\nu t\mathfrak{p}^\nu-t^2\mathfrak{h}-\tfrac{1}{2}\mathfrak{m}\,x^2)+\nabla_\nu\Bigg\{-t^2\mathfrak{h}^\nu-\tfrac{1}{2}x^2\mathfrak{m}^\nu+x_\mu t\,\tilde{T}^{\nu\mu}+\tfrac{1}{2}it6^{1/2}(2S+1)^{1/2}\Bigg[\chi^{m,m'}_{S-1/2,1/2}\begin{pmatrix} m & \frac{1}{2} & S\\ S-\frac{1}{2} & \mu & m''\end{pmatrix}\begin{pmatrix} m' & 1 & \mu\\ \frac{1}{2} & \nu & \frac{1}{2}\end{pmatrix}\phi_S^{m''}$$
$$-\phi_S^{m+}\begin{pmatrix} m & \mu & S-\frac{1}{2}\\ S & \frac{1}{2} & m'\end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2}\\ \mu & \nu & m''\end{pmatrix}\chi^{m',m''}_{S-1/2,1/2}\Bigg]$$
$$+\frac{i}{2}\epsilon^{\nu\alpha\beta}x_\beta(S\|S\|S)\phi_S^{m+}\begin{pmatrix} m & 1 & S\\ S & \alpha & m'\end{pmatrix}\phi_S^{m'}\left(1-\frac{1}{S}\right)\Bigg\}=0\,,\tag{4.5}$$

a result which implies the form

$$C=tD+t^2H-\tfrac{1}{2}\int d^3x\,\mathfrak{m}\,x^2$$

for the conserved generator of conformal transformations. Upon eliminating $\tilde{T}^{\nu\mu}$ in favor of $T^{\mu\nu}$, (4.5) becomes

$$\frac{\partial}{\partial t}(x_\nu t\,\mathfrak{p}^\nu-t^2\mathfrak{h}-\tfrac{1}{2}\mathfrak{m}x^2)+\nabla_\nu\left[-t^2\mathfrak{h}^\nu-\tfrac{1}{2}x^2m^\nu+x_\mu tT^{\mu\nu}+t\sigma^\nu+\frac{i}{2}\epsilon^{\nu\alpha\beta}x_\beta\phi_S^{m+}\begin{pmatrix} m & 1 & S\\ S & \alpha & m'\end{pmatrix}\phi_S^{m'}\left(1-\frac{1}{S}\right)\right]=0\,,\tag{4.6}$$

which by reference to (3.3), (3.5), and (3.7) is seen to require the conditions (3.11) and (4.3). Thus for a local interaction, conformal invariance clearly gives no restrictions on the theory beyond those already required by Galilean and dilatation invariance. It may be remarked that upon using the commutation (or anticommutation) relations of the independent components

$$[\phi_S^m(\vec{x},t),\phi_S^{m'\dagger}(\vec{x}',t)]_\mp=\delta^m_{m'}\delta(\vec{x}-\vec{x}')\,,$$

one can verify that the commutation relations of the generators in the general-spin case are identical to those found for scalar fields provided that both are written in a common Cartesian or spherical basis.

Inasmuch as it has been seen that conformal invariance in the Galilean case gives no information not already implied by scale invariance, one expects the redefined operators $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}_\nu$ to allow one to cast (4.6) into a somewhat simpler form. In fact one can trivially verify that the equation

$$\frac{\partial}{\partial t}(tx_\nu\mathfrak{p}^\nu-t^2\tilde{\mathfrak{h}}-\tfrac{1}{2}\mathfrak{m}x^2)+\nabla_\nu\left[-t^2\tilde{\mathfrak{h}}^\nu-\tfrac{1}{2}\mathfrak{m}^\nu x^2+x_\mu tT^{\mu\nu}+\frac{i}{2}\left(1-\frac{1}{S}\right)\epsilon^{\nu\alpha\beta}x_\beta(S\|S\|S)\phi_S^{m\dagger}\begin{pmatrix} m & 1 & S\\ S & \alpha & m'\end{pmatrix}\phi_S^{m'}\right]=0$$

represents the desired reformulation of the condition for conformal invariance. This, however, still leaves open the larger question raised in Ref. 1 concerning whether the sucessful imposition of a zero-trace condition necessarily implies finite matrix elements for the redefined operator densities which have been introduced here. This point is considered in the following paper.

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*Research supported in part by the U. S. Atomic Energy Commission.

[1]C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).

[2]The reader who desires a more detailed exposition is referred to J. M. Lévy-Leblond, J. Math. Phys. 4, 776

(1963).

[3]Field theories of spin-zero particles have been considered by J. M. Lévy-Leblond, Commun. Math. Phys. 4, 157 (1967).

[4]We take $\hbar = 1$ and employ the customary notation $E = i\partial/\partial t$, $\vec{p} = (1/i)\vec{\nabla}$.

[5]In the case of several interacting fields of different masses, the possibility of deleting the factor m from the definitions of $\mathfrak{m}$ and $\vec{\mathfrak{p}}$ clearly no longer exists.

[6]J. Schwinger, Phys. Rev. 91, 713 (1953).

[7]It may be useful to remark here that in contrast to a number of recent papers on scale invariance we decline here to recognize any distinction between the "naive" dimensionality implied by the canonical commutation relations and that determined by high-energy behavior of propagators. That so-called anomalous dimensions can be displayed in such analyses is, of course, intimately related to the occurrence of divergences in perturbation theory. On the other hand, the existence of apparent contradictions in formal field-theory manipulations which involve high-energy limits is a well-known phenomenon even in renormalizable theories and one whose history considerably predates current discussions of scale invariance. The view which is insisted upon here is that either such theories do not constitute acceptable field-theoretical models or that their apparent divergences cancel by the Gell-Mann–Low mechanism, thereby rendering perturbation-theory calculations entirely misleading at high energies. The fact that such cancellations cannot occur in Galilean theories (perturbation theory being exact) suggests that the significance of scale-invariant couplings in this case is moot.

[8]In analogy to the case of special relativity, where finite conformal transformations fail to preserve the sign of $x^\mu x_\mu$, one finds that inasmuch as (2.13) implies $1/t' = 1/t - c$ the temporal ordering of events is not in general preserved by conformal transformations.

[9]G. Mack and A. Salam, Ann. Phys. (N.Y.) 53, 174 (1969).

[10]J. M. Lévy-Leblond, Commun. Math. Phys. 6, 286 (1967).

[11]In order to dispel any doubts the reader might have concerning the significance of minimal theories, it may be remarked that for spins $\frac{1}{2}$ and 1, these coincide with the theories one obtains upon taking nonrelativistic limits of the usual Lorentz-covariant equations describing these same spin values.

[12]C. R. Hagen, Commun. Math. Phys. 21, 219 (1971).

[13]No confusion should arise here from the use of the letter m as a dummy index as well as a label for the mass.

[14]E. P. Wigner, *Group Theory* (Academic, New York, 1959).

[15]C. R. Hagen and W. J. Hurley, Phys. Rev. Letters 24, 1381 (1970).

[16]In order that it not be thought that the conventional choice of d eliminates the possibility of constructing scale-invariant interactions, the reader is reminded that coupling terms need not be of the polynomial form. Thus it is trivial to construct local couplings (e.g., $\phi_1^\dagger\phi_2\phi_3\phi_4{}^{1/3}$) which satisfy all the conditions essential for scale (and conformal) invariance. It may also be useful to point out that all of the global conservation laws of the Galilei group, including dilatations, remain valid even for the case of a (spatially) nonlocal interaction provided that the nonlocality is of a prescribed form. However, inasmuch as these couplings are not compatible either with (a) conformal invariance or (b) the existence of local conservation laws for the operator densities associated with the Galilei group, such interaction terms are not considered in this paper.