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# Relativistic Three-Particle Equations

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A functional derivation is presented of relativistic three-particle equations in configuration space. The equations are formulated in terms of two-particle attributes and the inhomogeneous terms are factorized in terms of the two-particle $T$ matrix. The equations are furthermore exhibited in a practicable form in which the kernel is connected and the inhomogeneous terms contain multiple-scattering contributions.

## I. INTRODUCTION

The investigation of dynamical models[1] in separable approximations and the formulation of Faddeev[2] have greatly extended the understanding of the three-particle problem. These and subsequent investigations have revealed a number of features to which a three-particle theory should aspire and a number of equations have been suggested particularly for treating the nonrelativistic three-particle problem.[2,3] Equations which define the three-particle scattering amplitude in terms of the two-particle $T$ matrix and kernel are of particular interest for they allow the introduction of pole approximations and justify the use of separable models. The relativistic three-particle theory[1,3,4] has been approached by relativistic equations in analogy to potential scattering or by graphical techniques. The most useful applications are in the scattering of particles from two-particle bound states.[1] A derivation of relativistic versions of the eikonal approximation for three-particle scattering processes has also been given.[5]

In this paper a functional-derivative formulation is given of relativistic three-particle equations which are defined in terms of two-particle attributes and which allow the description of a number of processes involving composite particles. The equations originate from a study of the radiative decays[6] of composite particles. The paper is organized as follows: In Sec. II the functional-derivative technique[7] is introduced and employed to derive two-particle equations and identities which are required in the development of the three-particle theory of Sec. III. It is found that the technique lends itself to the derivation of three-particle equations of appropriate form in which the clustering of the three-particle amplitude in terms of two-particle attributes is exhibited. The implications of this clustering decomposition[8] for practical applications are apparent for it allows the introduction of pole approximations in the two-particle amplitudes. The appearance of disconnected contributions to the kernel of the integral equation prevents the existence of the Hilbert-Schmidt norm. In Sec. IV this difficulty is circumvented by formulating relativistic three-particle equations with connected kernels. In the concluding section of the paper, V, the relation to previous work is given.

## II. TWO-PARTICLE EQUATIONS

The formulation proceeds from a spinor equation of motion which may serve as a starting point in many-body theory, a quark model,[9] or the unified theory of elementary particles.[10] This equation may be expressed in the following compact notation:

$$D_{ik}\psi_k + V_{im,ln}\bar{\psi}_m\psi_n\psi_l = 0 \tag{2.1}$$

and

$$D_{ik} = -i(\gamma^\mu)_{ik}\delta(x_i - x_k)\frac{\partial}{\partial x_k^\mu}\,. \tag{2.2}$$

To the extent that the three-particle equations are expressible in terms of two-particle attributes, they may be regarded as being independent of the starting equations.

Consider the generating functional

$$U = \exp(iq_{lk}\psi_k\bar{\psi}_l) \tag{2.3}$$

and the Green's function

$$G_{kl} = iW^{-1}\langle 0|T\psi_k\bar{\psi}_l U|0\rangle\,, \tag{2.4}$$

where

$$W = \langle 0|TU|0\rangle \tag{2.5}$$

from which the more-than-two-particle amplitudes may be derived by functional-derivative techniques with respect to the "classical source," $q_{lk}$. In view of the fermion nature of the fields the classical source is antisymmetrical under change of indices and the variation is given by

$$\frac{\delta q_{mn}}{\delta q_{ik}} = \delta_{im}\delta_{nk} - \delta_{mk}\delta_{ni}\,. \tag{2.6}$$

In order to keep the formulation as simple as possible, only the terms generated by the first term of Eq. (2.6) are retained. When the need arises, the equations may be appropriately symmetrized. The Green's function satisfies the equation

$$(D+q+M)_{ik}G_{ks}=\delta_{is}\,, \tag{2.7}$$

where the mass operator, $M$, is given by

$$M_{ik}G_{ks}=iV_{im,kn}G_{kn,sm} \tag{2.8}$$

and the four-particle Green's function is defined by

$$G_{kn,sm}=W^{-1}\langle 0|T\psi_k\psi_n\overline{\psi}_s\overline{\psi}_m U|0\rangle\,. \tag{2.9}$$

Forming the functional derivatives of Eq. (2.7) with respect to the source $q$ and using the fact that the mass operator depends on $q$ only via $G$, it follows that

$$F_{iq,np}+G_{ip}G_{qn}+iG_{ik}K_{ks,lr}F_{rq,sp}G_{ln}=0\,, \tag{2.10}$$

where the four-point function, $F$, is given by

$$F_{rq,sp}=\frac{\partial G_{rs}}{\partial q_{pq}}\,. \tag{2.11}$$

The effective two-particle interaction, $K$, is defined by

$$iK_{ks,lr}=\frac{\partial M_{kl}}{\partial G_{rs}}\,. \tag{2.12}$$

The two-particle $T$ matrix is in turn defined by

$$F_{iq,mp}=-G_{ip}G_{qm}+iG_{ij}G_{qn}T_{jn,fg}G_{fm}G_{gp} \tag{2.13}$$

and satisfies the integral equation

$$T_{da,bc}=K_{da,bc}-iK_{ds,br}G_{rj}T_{ja,fc}G_{fs}\,. \tag{2.14}$$

Relationships which are needed in the sequel are now derived.

(a) Employing the result

$$\frac{\partial M_{kl}}{\partial q_{pq}}=iK_{ks,lr}F_{rq,sp} \tag{2.15}$$

and taking the last two equations into account, it follows that

$$\frac{\partial M_{kl}}{\partial q_{pq}}=-iG_{qn}T_{kn,lm}G_{mp}\,. \tag{2.16}$$

(b) Utilizing Eqs. (2.8), (2.9), (2.11), (2.13), and (2.16) it follows that

$$M_{ij}=iV_{ip,jq}G_{qp}-iV_{ip,rj}G_{rp}-iV_{ip,rq}G_{rk}\frac{\partial M_{kj}}{\partial q_{pq}}\,. \tag{2.17}$$

The last equation may be iterated to exhibit the structure of the mass operator.

## III. THREE-PARTICLE EQUATIONS

Three-particle equations may be derived by functional-derivative techniques starting from Eq. (2.11). However, in order to avoid disconnected contributions it is advantageous to proceed from the following truncated four-point function:

$$\eta_{rq,sp}=F_{rq,sp}+G_{rp}G_{qs} \tag{3.1}$$

$$=iG_{rj}G_{qn}T_{jn,fg}G_{fs}G_{gp} \tag{3.2}$$

$$=-G_{rj}\left(\frac{\partial M_{jf}}{\partial q_{pq}}\right)G_{fs}\,. \tag{3.3}$$

We next define a six-point function by

$$S_{rqa,spb}=\frac{\partial}{\partial q_{ba}}(\eta_{rq,sp})\,. \tag{3.4}$$

Employing the two-particle equations of Sec. II, it follows that

$$\frac{\partial}{\partial q_{ba}}\left(\frac{\partial M_{jf}}{\partial q_{pq}}\right)=L_{jhd,fcg}F_{ga,db}F_{cq,hp}+iK_{jh,fc}\left[S_{cqa,hpb}-\left(\frac{\partial G_{cp}}{\partial q_{ba}}\right)G_{qh}-G_{cp}\left(\frac{\partial G_{qh}}{\partial q_{ba}}\right)\right], \tag{3.5}$$

where

$$L_{jhd,fcg}=\frac{\partial^2 M_{jf}}{\partial G_{gd}\partial G_{ch}} \tag{3.6}$$

denotes a three-particle effective interaction which derives from the two-body effective interaction. It therefore follows that the six-point function satisfies the equation

$$S_{rqa,spb}+iG_{rj}K_{jh,fc}S_{cqa,hpb}G_{fs}=I_{rqa,spb}\,, \tag{3.7}$$

where the inhomogeneous term is given by

$$I_{rqa,spb}=(A+B+C)_{rqa,spb} \tag{3.8}$$

and

$$A_{rqa,spb}=i\left(\frac{\partial G_{rj}}{\partial q_{ba}}\right)G_{qn}T_{jn,fg}G_{fs}G_{gp}\,, \tag{3.9}$$

$$B_{rqa,spb}=iG_{rj}G_{qn}T_{jn,fg}G_{gp}\left(\frac{\partial G_{fs}}{\partial q_{ba}}\right), \tag{3.10}$$

$$C_{rqa,spb}=-G_{rj}[L_{jhp,fc\lambda}F_{\lambda a,pb}F_{cq,hp}-iK_{jh,fc}(F_{ca,pb}G_{qh}+G_{cp}F_{qa,hb})]G_{fs}\,. \tag{3.11}$$

The inhomogeneous terms may be further exhibited. Utilizing equations (2.11) and (2.13), it follows that

$$B_{rqa,spb}=-iG_{as}G_{rj}T_{jn,fg}G_{fb}G_{gp}G_{qn}-G_{rj}G_{qn}T_{jn,fg}G_{gp}G_{fl}G_{ah}T_{lh,cx}G_{cs}G_{xb} \tag{3.12}$$

and the crossed term

$$A_{rqa,spb}=-iG_{rb}G_{aj}G_{qn}T_{jn,fg}G_{fs}G_{gp}-G_{ri}G_{ah}T_{ih,xy}G_{xj}G_{yb}G_{qn}T_{jn,fg}G_{fs}G_{gp}\,. \tag{3.13}$$

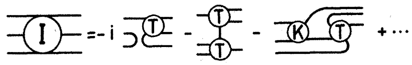


FIG. 1. Graphical representation of some representative terms of the inhomogeneous term Eq. (3.8) exhibiting the clustering decomposition.

Representative terms of the inhomogeneous term Eq. (3.8) are depicted in Fig. 1.

The structure of the three-particle effective interaction may be illustrated by iterating Eq. (2.17) and applying Eq. (3.6) with the effect

$$\begin{aligned} L_{jh\rho,fc\lambda} = &-V_{j\rho,nc}G_{nx}V_{xh,f\lambda} \\ &-V_{jm,nh}G_{nx}V_{x\rho,fu}G_{uz}V_{zy,\lambda c}G_{my} \\ &-V_{j\lambda,nh}G_{nx}V_{xo,fu}G_{uz}G_{yo}V_{z\rho,yc}+\cdots . \end{aligned} \tag{3.14}$$

Defining

$$S_{rqa,spb}=G_{rx}G_{az}X_{xqz,ypb}G_{ys}\,, \tag{3.15}$$

it follows that

$$G_{az}G_{rx}X_{xqz,ypb}G_{ys}+iG_{az}G_{rj}K_{jd,fc}G_{cx}X_{xqz,ypb}G_{yd}G_{fs} = I_{rqa,spb}\,, \tag{3.16}$$

which is the configuration-space representative of the desired relativistic three-particle equation defined in terms of the two-particle kernel. Equations (3.12) to (3.14) exhibit the inhomogeneous term factorized in terms of two-particle attributes.

## IV. RELATIVISTIC THREE-PARTICLE EQUATIONS WITH CONNECTED KERNELS

The relativistic three-particle equations of Sec. III are not suitable for many practicle applications in view of the disconnected nature of the kernel. In order to circumvent this deficiency it is first of all necessary to symmetrize the kernel in order to take into account the role of the spectator particle that accompanies the two-particle kernel.

It follows from Eq. (3.16) that the appropriately symmetrized equation which exhibits the first particle, the second particle, and finally the third particle as a spectator, respectively, is given by

$$\begin{aligned} G_{az}G_{rx}X_{xqz,ypb}G_{ys} = I_{rqa,spb} &- iG_{az}G_{cx}G_{yd}G_{rm}K_{md,gc}G_{gs}X_{xqz,ypb} \\ &- iG_{rm}K_{me,gc}G_{ga}G_{ez}G_{cx}G_{ys}X_{xqz,ypb} \\ &- iG_{sm}K_{me,gd}G_{ga}G_{ez}G_{rx}G_{yd}X_{xqz,ypb}\,. \end{aligned} \tag{4.1}$$

In view of the dependence of the kernel of the equation on the two-particle kernel itself, a partial inversion of the equation may be effected by taking into account the two-particle relativistic equation

$$T_{me,gc}=K_{me,gc}-iT_{ma,gr}G_{rj}K_{je,fc}G_{fa} \tag{4.2}$$

according to Eq. (2.14).

The convolution $T_{ma,gs}I_{rqa,spb}$ as determined by Eq. (4.1) is next to be considered. The contribution which derives from the last term of Eq. (4.1), in particular, reads

$$iT_{ma,gs}G_{sj}K_{je,fd}G_{fa}G_{ez}G_{rx}G_{yd}X_{xqz,ypb} \tag{4.3}$$

which, by virtue of Eq. (4.2), equals

$$(K_{me,gd}-T_{me,gd})G_{ez}G_{rx}G_{yd}X_{xqz,ypb}\,. \tag{4.4}$$

It therefore follows that

$$\begin{aligned} K_{me,gd}G_{ez}G_{rx}G_{yd}X_{xqz,ypb} = T_{mh,gt}I_{rqh,tpb} &- iT_{mh,gt}G_{hz}G_{cx}G_{yd}G_{rj}K_{jd,fc}G_{ft}X_{xqz,ypb} \\ &- iT_{mh,gt}G_{rj}K_{je,fc}G_{fh}G_{ez}G_{cx}G_{yt}X_{xqz,ypb}\,. \end{aligned} \tag{4.5}$$

Application of the above procedure furthermore yields the equations

$$\begin{aligned} G_{az}G_{cx}G_{yd}K_{md,gc}X_{xqz,ypb} = T_{mt,gn}I_{nqa,tpb} &- iT_{mt,gn}G_{nj}K_{je,fc}G_{fa}G_{ez}G_{cx}G_{yt}X_{xqz,ypb} \\ &-iT_{mt,gn}G_{tj}K_{je,fd}G_{fa}G_{ez}G_{nx}G_{yd}X_{xqz,ypb} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} K_{me,gc}G_{ez}G_{cx}G_{ys}X_{xqz,ypb} = T_{mh,gn}I_{nqh,spb} &- iT_{mh,gn}G_{hz}G_{cx}G_{yd}G_{nj}K_{jd,fc}G_{fs}X_{xqz,ypb} \\ &- iT_{mh,gn}G_{sj}K_{je,fd}G_{fa}G_{ez}G_{nx}G_{yd}X_{xqz,ypb}\,. \end{aligned} \tag{4.7}$$

A graphical representation of the last equation is given in Fig. 2. Equations (4.5) to (4.7) may be regarded as the relativistic analog of the Faddeev equations. The importance of the equations derives from the fact that they incorporate the two-particle dynamics of Eq. (2.14) which in turn implies a connected three-particle kernel. This result may be achieved by substitution of Eqs. (4.5)–(4.7) into Eq. (4.1) with the result that

$$\begin{aligned} G_{az}G_{rx}X_{xqz,ypb}G_{ys} = I_{rqa,spb} &- iG_{rm}G_{gs}T_{mt,gn}I_{nqa,tpb} - iG_{rm}G_{ga}T_{mh,gn}I_{nqh,spb} - iG_{sm}G_{ga}T_{mh,gt}I_{rqh,tpb} \\ &- [G_{rm}G_{gs}T_{mt,gn}G_{fa}G_{ez}(G_{nj}K_{je,fc}G_{cx}G_{yt}+G_{tj}K_{je,fd}G_{nx}G_{yd})]X_{xqz,ypb} \end{aligned}$$

$$- [G_{rm} G_{ga} T_{mh,gn} G_{yd} G_{fs} (G_{hz} G_{cx} G_{nj} K_{jd,fc} + K_{fe,jd} G_{jh} G_{ez} G_{nx})] X_{xqz,ypb}$$
$$- [G_{sm} G_{ga} T_{mh,gt} G_{cx} G_{rj} (G_{hz} G_{yd} K_{jd,fc} G_{ft} + G_{rj} K_{je,fc} G_{fh} G_{ez} G_{yt})] X_{xqz,ypb} . \quad (4.8)$$

Equation (4.8) is depicted in Fig. 3. This equation represents a linear relativistic three-particle equation defined in terms of a "known" kernel and inhomogeneous terms which are factorized in terms of two-particle attributes. In contrast to Eq. (4.1), for example, it possesses a connected kernel accompanied by the fact that the Hilbert-Schmidt character of the kernel is not aggravated by the more-than-two-particle nature of the theory. This circumstance is also reflected by the structure of the inhomogeneous terms of the new equation. These terms represent multiparticle scattering contributions. Once the contributions of the cuts associated with these multiple-scattering processes have been separated from the kernel, they do not interfere with the convergence of the Fredholm or quasiparticle methods. The relativistic three-particle scattering problem is reduced to the solution of three-particle equations which may be solved by standard methods provided the two-particle Bethe-Salpeter kernel is sufficiently well behaved.

## V. CONCLUSIONS

A functional-derivative formulation of a relativistic three-particle theory has been developed. Relativistic three-particle equations have been derived and the main difficulties that arise in more-particle theory were avoided by rendering the three-particle kernel connected. The equations provide a basis for the description of many different physical processes such as three-particle bound states and scattering, scattering of composite particles, rearrangement collisions as well as the decay of unstable composite particles.

It is of interest, in conclusion, to discuss the relationship of this work to related investigations in three-particle theory.[1] It is appropriate to employ a symbolic notation for this purpose. Equation (4.1) in a symbolic notation reads

$$S = I - iG(K_i + K_j + K_k)GS , \quad (5.1)$$

where the $K_i$ denotes the two-particle kernel accompanied by particle $i$ as a spectator. In Sec. IV a method has been developed for rendering the kernel connected. In order to make contact with related investigations, it is first of all to be noticed that, according to Eqs. (3.12) and (3.13), the inhomogeneous term may be separated into connected and disconnected parts.

Consider a separation of the form

$$I = I_D + I_R , \quad (5.2)$$

where

$$I_D = \sum_{i=1}^{3} I^i . \quad (5.3)$$

In Eq. (5.3), the $I^i$ denotes the two-particle $T$ matrix connecting the particles $(j, k)$ accompanied by the $i$th particle as a spectator.

It follows from Eqs. (5.1) and (5.2) that the three-particle amplitude may be decomposed in the form

$$S = S_D + S_R , \quad (5.4)$$

where $S_D$ and $S_R$ satisfy the equations

$$S_D = I_D - iG(K_i + K_j + K_k)GS_D \quad (5.5)$$

and

$$S_R = I_R - iG(K_i + K_j + K_k)GS_R . \quad (5.6)$$

Relativistic Faddeev equations may be derived from the equation for the disconnected part Eq. (5.5) and Eq. (5.3).

Define the disconnected three-particle amplitudes as follows:

$$S_D = \sum_{i=1}^{3} S^i . \quad (5.7)$$

Substitution of Eq. (5.7) into Eq. (5.5) yields the equation for the disconnected part:

$$S^i = I^i - iGK_iG(S^i + S^j + S^k) . \quad (5.8)$$

Partial inversion of the equation by means of the two-particle equation

$$T_i = K_i - i T_i G K_i G \quad (5.9)$$

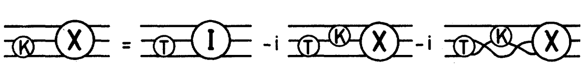

FIG. 2. Graphical representation of the three-particle Eq. (4.7) with final-state interaction.

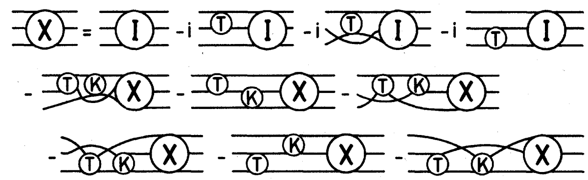

FIG. 3. Graphical representation of the relativistic three-particle Eq. (4.8).

yields a coupled set of relativistic Faddeev equations[2]

$$S^i = (I^i - iGT_iI^iG) - i\sum_{\alpha=i,j,k} GT_iG(1-\delta_{i\alpha})S^\alpha . \tag{5.10}$$

Finally, consider the Eq. (5.6). The kernel of the equation may be cast into a connected form by the method of Sec. IV. The inhomogeneous term contains, among other terms, pole terms connecting the two-particle $T$ matrix. These terms generate relativistic equations for the connected part of the three-particle amplitude in a "ladder" approximation. This equation may be further simplified by considering the two-particle $T$ matrix in the proximity of its bound-state poles. The resulting equation describes the scattering of a particle from a bound-state analogous to the Amado model[1] introduced in connection with nucleon-deuteron scattering. The multiple-scattering terms that occur in the connected equations generate dynamical singularities, the role of which have been investigated in the triton bound-state problem.[1] These singularities also play a role in the treatment of unstable composite particles.[6]

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[9]See, e.g., G. Morpurgo, in *Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 226.

[10]W. Heisenberg, *Introduction to the Unified Field Theory of Elementary Particles* (Wiley, New York, 1966).