Wightman Formulation for the Quantization of the Gravitational Field

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A generalization of Wightman field theory is formulated which makes the theory also applicable to the gravitational field. Strongly geodesically complete manifolds are found to be the most suitable for description of curved space-time in our approach. After the formulation of generalized axioms, the schemes of proofs of the fundamental theorems of the theory (Bargmann-Hall-Wightman theorem, θ theorem, main reconstruction theorem, etc.) are given. The paper ends with hints on the possible ways of constructing the theory of asymptotic states for the quantized gravitational field.

I. INTRODUCTION

Historically the quantization of the gravitational field has been pushed in two main directions: that in which the gravitational field is described in a canonical formalism, and that in which the formalism is covariant. The first one began with Bergmann,[1] who looked for a canonical formulation of the gravitational field in order to quantize it (it was then the only known way to quantize a field), but was stopped by the problem of constraints (namely, that some of the field variables have no conjugate momenta, and those of the other variables are not independent). Dirac, Pirani, and Schild later worked on this problem, and Dirac[2] gave a partial solution to it. The main disadvantage of this kind of formalism is that it fixes the time, and as a consequence reduces the number of field variables from the 10 $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) to the six variables g_{ij} ($i, j = 1, 2, 3$). On the other hand, we have the advantage of getting the canonical commutation relations for the field.

The second direction is generally called the manifestly covariant formalism; as contrary to the canonical formalism, it preserves at every step the formal covariance of general relativity under the changes of local coordinates. The aim of this formalism is to calculate by perturbation methods the scattering matrix of the gravitational field, the gravitational self-energy, etc. In order to derive these quantities, we need the commutation relations of the gravitational field. DeWitt and DeWitt,[3] Feynman,[4] and Lichnérowicz[5] were able to get commutation relations for the first variation of the gravitational field (which is a perturbation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ of the geometrical background field $g_{\mu\nu}$) and of the total field in some particular cases. The main difficulty in deriving these commutation relations results from the nonlinearity of Einstein's equations. The calculations of the physical quantities mentioned above were presented by DeWitt,[6,7] Mandelstam,[8] and Popov and Faddeev.[9]

Today, we are far from having a complete (even from the formal point of view) calculation of these quantities associated with the total gravitational field (and not with the variation $\delta g_{\mu\nu}$).

The aim of this work is to give an axiomatic Wightman-like formulation to the quantized gravitational field and its consequences. The Wightman theory of a quantized field (on Minkowski space) cannot be formulated on a curved space-time in a straightforward manner. We are interested in such a formulation for the following two reasons:

(a) to give a *rigorous* mathematical formulation to the problem of quantization of the gravitational field (a thing which has not been done before);

(b) to examine the stability of the Wightman theory under curvature. It is believed by us that if the Wightman theory is a good theory, then the flat space-time cannot be singular, in the sense that the formulation and results of the theory are extensible also to a curved space-time manifold with "close" structure.

If we want to stay in the spirit of general relativity, it is necessary that the gravitational field will dictate the geometry of space-time. To introduce this feature we assume that the gravitational field is a distribution-metric, namely, an operator valued *covariant symmetric tensor* distribution to which is canonically associated a light-cone field. As we shall see later, this notion of local light cone is necessary in order to define the locality of the field as well as the spectral condition for the 4-momentum operator.

To define the notion of remote past or future we have introduced the notion of geodesically complete metrics. These are metrics the geodesics of which are homeomorphic, via their parametrization, to the real line. This kind of metrics permits us to relate the tangent space with the manifold itself in a satisfactory manner and in particular to connect the locality on the manifold with that in the tangent space.

The covariance group of the theory is a semidirect product of the Poincaré group (operating on the tangent space) by a group of C^∞ functions on the tangent space at the same point (which correspond to the changes of unit of measure). This choice of covariance group can at first sight look strange, the general relativity being covariant under the group of all the diffeomorphisms. We chose this group as the minimal group which is necessary in order to formulate the Wightman theory in a *covariant* manner (the word covariant here means covariant relative to the cone field), and therefore its structure is associated with the gravitational field itself. In this connection note also that the gravitational field has conformal degree[10] 2 relative to any flat metric on V (this results from the fact that it is a tensor).

Now the classical gravitational field is covariant under discrete symmetries (for instance the geodesic symmetry); we shall see in the TCP theorem that the quantized generalized gravitational field also possesses a discrete symmetry θ (usually called TCP).

II. DEFINITIONS AND GENERAL PROPERTIES [11]

Let V be a C^∞ four-dimensional connected paracompact manifold, and denote by V_x the tangent space at x on V. A cone field $\mathcal{C}$ on V is a mapping which associates to every $x\in V$ a cone $C_x \subset V_x$ (we define here a cone in a vector space as the set of isotropic vectors, for a hyperbolic normal quadratic form) such that there exists a C^∞ metric g with signature $(+---)$ such that for every $x\in V$, C_x is the light cone of g at x. Such a metric g is said to be compatible with $\mathcal{C}$.

Given a cone field $\mathcal{C}$ on V, a C^∞ path is said to be timelike (isotropic) relative to $\mathcal{C}$ if the tangent vector at every point x of this path is inside C_x (on C_x). A C^∞ path is said to be physically spacelike if given any couple of points α, β on it there is no timelike or isotropic C^∞ path joining α and β.

Two C^∞ vector fields T and T' over V, the vectors at every point x of which are timelike, are called equivalent if at every point $x \in V$, $T(x)$ and $T'(x)$ belong to the same connected component of $V_x - C_x$. An equivalence class is called a time orientation of $\mathcal{C}$. V being connected, there are two time orientations of $\mathcal{C}$. Given an orientation of $\mathcal{C}$ we call future C_x^+ in V_x the closure of the connected component of $V_x - C_x$ which contains a vector $T(x)$, where T is an element of the orientation.

Two points x and y are said to be timelike (isotropically, physically spacelike) separated if there exists a timelike (isotropic, physically spacelike) C^∞ path joining x and y. Two points x and y are said to be spacelike separated if there exists no timelike or isotropic C^∞ path joining x and y.

A C^∞ metric g on V, with signature $(+---)$, is called *geodesically complete* if for every $x\in V$, its exponential mapping $\exp_x$ is a C^∞ diffeomorphism between V_x and an open set $O_x \subset V$. g is said to be *strongly geodesically complete* if in addition any two timelike or physically spacelike separated points are joined by a geodesic.

A cone field $\mathcal{C}$ is called geodesically complete (strongly geodesically complete) if there exists a metric g on V, compatible with $\mathcal{C}$ and geodesically complete (strongly geodesically complete). Such a metric g is called geodesically compatible (strongly geodesically compatible) with $\mathcal{C}$.

We shall now examine a condition under which a geodesically complete metric is strongly geodesically complete.

Definition 1. A C^∞ metric g on V with signature $(+---)$ is called *stationary* if there exists a connected global one-parameter isometry group, with timelike trajectories (called time lines), without invariant point on V, such that:

(a) The time lines are homeomorphic to the real line $\mathbb{R}$.

(b) There exists a three-dimensional manifold V' with the same topological properties as V, such that there exists a C^∞ diffeomorphism between V and $V'\times\mathbb{R}$ in which the image of the time lines are the real lines $\{x'\}\times\mathbb{R}$, where $x'\in V'$.

In that case (Lichnérowicz,[12] p. 110) there exists around every point a local coordinate system x^μ (which is said to be *adapted* to g) satisfying the following conditions:

(i) x^i $(i=1,2,3)$ is an arbitrary coordinate system on V'.

(ii) The manifolds $x^0=\text{const}$ are submanifolds in V, diffeomorphic to V'.

(iii) g does not depend on x^0.

Definition 2. A stationary metric g is called *static* if there exists a local coordinate system x^μ adapted to g (in a neighborhood of every point), such that

$$ds^2 = g_{00}(dx^0)^2 + g_{ij}dx^i dx^j \quad (i,j=1,2,3).$$

Theorem: Let V be simply connected. A geodesically complete static metric g on V is strongly geodesically complete.

Proof. The equations of geodesics in a coordinate system adapted to g in which $g_{0i}=0$ $(i=1,2,3)$ (note that x^0 can be chosen globally on V) are

$$\frac{d^2x^0}{dt^2} + g^{00}\partial_i(g_{00})\frac{dx^0}{dt}\frac{dx^i}{dt}=0,$$
$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} - \tfrac{1}{2}g^{ij}\partial_j(g_{00})\left(\frac{dx^0}{dt}\right)^2=0, \qquad (2.1)$$

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}),$$

which becomes

$$\left|\frac{dx^0}{dt}\right| = c g^{00} \text{ where } c \text{ is a positive constant},$$
$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + \tfrac{1}{2}c^2 g^{ij}\partial_j g^{00} = 0\,. \tag{2.2}$$

(a) Timelike-separated points: Suppose that x and y are timelike separated; denote by τ a C^∞ timelike segment between x and y, and let $\alpha = \int_\tau dx^0$. Suppose $y \notin \exp_x(V_x)$. If $z \in \exp_x(V_x)$ and $z \in \tau$, denote by $\tau(z)$ the subsegment of τ between x and z, and by $\gamma(z)$ the geodesic segment between x and z. $\tau(z) - \gamma(z)$ is a boundary ∂S because V is simply connected; thus the Stokes formula gives

$$\int_{\tau(z)} dx^0 - \int_{\gamma(z)} dx^0 = \int_{\partial S} d^2x^0 = 0\,.$$

Therefore, $\alpha \geq \int_{\gamma(z)} dx^0$ (if we suppose without loss of generality that $\alpha \geq 0$), and

$$\alpha \geq \int_{\gamma(z)} \frac{dx^0}{dt} dt = \int_{\gamma(z)} c g^{00} dt \geq \int_{\gamma(z)} c^{-1} dt,$$

because on the geodesic we have

$$c^2 g^{00} = 1 - g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} \geq 1\,.$$

But if z_0 is the limit point of $\exp_x(V_x) \cap \tau$, then, when $z \to z_0$, we have $\int_{\gamma(z)} c^{-1} dt \to \infty$ and thus $\alpha = +\infty$, which is absurd. Therefore there exists a geodesic joining x and y.

(b) Remark: Call g' the elliptic metric (g_{ij}) defined on V', and define $h_\lambda = \lambda g'$, where λ is a strictly positive C^∞ function on V'. The equations of geodesics on V' relative to this metric h_λ are (if the canonical parameter t' of the geodesics is given by $\lambda = dt'/dt$)

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} - \tfrac{1}{2}\lambda^{-1} g_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} g^{il}\partial_l \lambda = 0\,. \tag{2.3}$$

On the other hand the diffeomorphism between V and $V' \times \mathsf{R}$ defines canonically a projection π from V onto V' along the time lines. Comparing (2.2) and (2.3) we see that a necessary and sufficient condition for the projection on V' of the isotropic or spacelike geodesics of V to be geodesics of h_λ is $\lambda = K(c^2 g^{00} - \delta)$, where K is a constant and $\delta = 0$ or -1, respectively, for the isotropic or spacelike geodesics of V.

(c) Now let $x, y \in V$ be physically spacelike separated. Denote by Z the time line passing through y. Without loss of generality we can suppose that $x \in \{x^0 = 0\} \simeq V'$ and that $y \in V^+ \equiv \{x^0 \geq 0\}$. We shall define $Z^+ = Z \cap V^+$. If $\delta = -1$ choose $K = 1$, i.e., $\lambda(c) = c^2 g^{00} + 1$. Let γ be a geodesic in V; thus, g being geodesically complete, t runs over all the real line and so does

$$t' = \int_{\mathsf{R}} [c^2 g^{00}(\gamma(t)) + 1] dt\,.$$

Therefore, $h_{\lambda(c)}$ is complete in the usual sense. Thus, as there exists a path joining x and $\pi(y)$, there exists a geodesic γ'_c in V' relative to $h_{\lambda(c)}$ joining x and $\pi(y)$ [and such that its length is equal to the distance $d_c(x, \pi(y))$ between x and $\pi(y)$]. Thus the intersection of Z^+ with the set $A_x(c)$ formed by the geodesics such that $(d\gamma/dt)(0) = X$, $X^0 = c g^{00}(x)$, and $\gamma(0) = x$ [when the parameter t is chosen so that $g(X, X) = -1$] is not empty. Denote by $y(c)$ one point of this intersection; Z^+ is contained in the interior of $\exp_{y(c)}(C_{y(c)})$. As $A_x(c)$ is spacelike, $y(c)$ is the only intersection of $A_x(c)$ with $\exp_{y(c)}(C_{y(c)})$, and *a fortiori* with Z^+.

We show now that Z^+ also intersects $\exp_x(C_x)$ at one and only one point y_∞. The projection of the isotropic geodesics on V' are the geodesics of the metric $g'' = g^{00} g'$; consider

$$\alpha(\gamma) = \int_0^\infty g^{00}(\gamma(t)) dt,$$

where γ is an isotropic geodesic. As

$$c^2 g^{00} = -g_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt},$$

if $|\alpha(\gamma)| < +\infty$, then

$$\left|\int_{\pi(\gamma)} g_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} dt\right| < +\infty\,.$$

But g' is geodesically complete on V'; thus using normal coordinates on V' we see that $\pi(\gamma)$ would have a limit point in V' when $t \to +\infty$; therefore, γ would have a limit point inside V when $t \to +\infty$, and this is absurd because a geodesic cannot have a limit point (it can always be extended from any point). Therefore g'' is complete and there is a geodesic (for g'') joining x and $\pi(y)$, and thus there exists an isotropic geodesic joining x and Z^+. In other words $\exp_x(C_x)$ intersects Z^+ and, for the same reason as above, this intersection is reduced to one point y_∞.

But $\exp_x^{-1}[A_x(c)]$ varies continuously between the spacelike hyperplane $H_0 = \exp_x^{-1}(\{x^0 = 0\})$ for $c = 0$ and C_x (when $c \to +\infty$). Thus $y(c)$ varies continuously from $\pi(y)$ when $c = 0$ to y_∞ when $c \to +\infty$. This implies that every point between $\pi(y)$ and y_∞ on Z^+ is joined to x by a geodesic. But y is necessarily between $\pi(y)$ and y_∞ because otherwise $y^0 > y^0_\infty$ and if γ_∞ is the isotropic geodesic joining x and y_∞, then the path $t \to ((y^0/y^0_\infty)\gamma^0_\infty(t), \gamma^i_\infty(t))$ would be a timelike path joining x and y, which is absurd.

We conjecture that every geodesically complete (hyperbolic) metric is strongly geodesically complete. We hope in the future to be able to give an answer to this problem.

Theorem: Let g and g′ be two metrics, strongly geodesically compatible with a strongly geodesically complete cone field $\mathcal{C}$. *Then* $g' = \lambda g$ *where* λ *is a* C^∞ *strictly positive function on V, and the mapping*

$$(g, g')_x \equiv \exp_x^{-1} \circ \exp'_x$$

(where $x \in V$ *and* $\exp_x$ *and* $\exp'_x$ *are, respectively, the exponential mappings at x of g and g′) is a* C^∞ *diffeomorphism of* V_x. *Its differential* $d((g,g')_x)_0$ *at the origin O of* V_x *is the product of a dilatation in* V_x *by an orthochronous Lorentz transformation on* V_x *(relative to the light cone* C_x*).*

The first statement is clear because g and g' have the same light cone at every point. Since g and g' are strongly geodesically complete, $\exp_x(V_x) = \exp'_x(V_x)$ because they consist of points which are either timelike, or physically spacelike, or isotropically separated with x.

If $X \in C_x$ and $t \to \gamma(t)$ is the geodesic for g such that $\gamma(0) = x$ and $(d\gamma/dt)(0) = X$, we see that if we define $t' = \int_0^t \lambda(\gamma(t))dt$, and a_x by $t'(a_x) = 1$, then the mapping

$$t' \to \overline{\gamma}(t') \equiv \gamma(a_x^{-1} t(t'))$$

is a geodesic for g' such that

$$\overline{\gamma}(0) = x\,,$$

$$(d\overline{\gamma}/dt')(0) = [a_x \lambda(x)]^{-1} X\,,$$

and

$$\gamma(1) = \overline{\gamma}(1)\,.$$

Thus

$$(g, g')_x X = [a_x \lambda(x)]^{-1} X\,.$$

In particular, $(g, g')_x$ preserves the generatrices of C_x and their orientation. $d((g, g')_x)_0$ will be an element of $\mathrm{GL}(V_x)$ which preserves the generatrices of C_x and their orientation. It is therefore the product of a dilatation by an element of the orthochronous Lorentz group associated to C_x.

If $\mathcal{C}$ is a strongly geodesically complete cone field and $x \in V$, denote by P_x the Poincaré group (connected component of the inhomogeneous Lorentz group) on V_x associated to the cone C_x.

Definition. We shall call chronological group at x of $\mathcal{C}$ *the group* G_x *generated by finite products of elements of* P_x *and of diffeomorphisms of the kind* $(g,g')_x$ *when g and g′ are two metrics strongly geodesically compatible with* $\mathcal{C}$.

We put on G_x the topology associated with the uniform convergence on every compact set of V_x of the diffeomorphisms and their partial derivatives. Denote by $\mathcal{S}^2\mathfrak{D}(V)$ the set of the C^∞ twisted 4-forms-2-contravariant symmetric tensor field over V (Ref. 13) with compact support, with the $\mathfrak{D}$-space topology. If F is a topological vector space over $\mathbb{C}$ (complex field), let $\mathcal{S}_2\mathfrak{D}'(V, F)$ be the set of the linear continuous applications over $\mathcal{S}^2\mathfrak{D}(V)$ with values in F. Denote by $\mathfrak{D}(V)$ the set of C^∞ twisted 4-forms over V with compact support. If $\mathcal{G} \in \mathcal{S}_2\mathfrak{D}'(V, F)$, we shall say that a C^∞ vector field X over V is isotropic for $\mathcal{G}$ if for every $\varphi \in \mathfrak{D}(V)$ we have $\mathcal{G}(X \otimes X \otimes \varphi) = 0$. Denote by $\mathcal{I}(\mathcal{G})$ the set of C^∞ vector fields isotropic for $\mathcal{G}$. If $x \in V$, we define $C_x = \{X(x) \mid X \in \mathcal{I}(\mathcal{G})\}$.

Definition. We shall say that $\mathcal{G} \in \mathcal{S}_2\mathfrak{D}'(V, F)$ *is an F-valued distribution metric if the map* $\mathcal{C}: x \to C_x$ *is a cone field.* $\mathcal{C}$ *will be called the cone field associated with* $\mathcal{G}$.

We shall say that $\mathcal{G}$ is geodesically complete (strongly geodesically complete) if $\mathcal{C}$ is geodesically complete (strongly geodesically complete).

Theorem: A necessary and sufficient condition for $\mathcal{G} \in \mathcal{S}_2\mathfrak{D}'(V, F)$ *to be a distribution metric is that* $\mathcal{G} = \lambda g$ *where g is a* C^∞ *metric on V [with signature* $(+---)$*], and* $\lambda \in \mathfrak{D}'(V, F)$ *with support equal to V.*

This theorem is easily checked in local coordinate systems, and then proved in general by gluing together the different distributions obtained in these different local coordinates.

III. THE GRAVITATIONAL FIELD

We give here a definition for the gravitational field with a system of Wightman axioms,[14] which put the accent on the covariance properties of the field and the spectral properties of the energy-momentum operator of the field.

We suppose that the manifold V has the same properties as in Sec. II.

Axiom (0): We suppose we are given a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, called the space of states.

Axiom (1): Call $\mathrm{Op}(\mathcal{H})$ the set of linear operators in $\mathcal{H}$, endowed with the weak topology. The gravitational field is a strongly geodesically complete distribution metric

$$\mathcal{G} \in \mathcal{S}_2\mathfrak{D}'(V, \mathrm{Op}(\mathcal{H}))\,,$$

the associated cone field $\mathcal{C}$ of which is time oriented. We suppose that $\mathcal{G}(\varphi)$ is defined, for every $\varphi \in \mathcal{S}^2\mathfrak{D}(V)$, on a dense domain $D \subset \mathcal{H}$, $\mathcal{G}(\varphi) D \subset D$, and if $\Phi, \Psi \in D$ then $(\Phi, \mathcal{G}(\varphi)\Psi) = (\mathcal{G}(\overline{\varphi})\Phi, \Psi)$, where $\overline{\varphi}$ is the complex conjugate of φ.

Let $x \in V$ and g be a metric strongly geodesically compatible with $\mathcal{C}$. When $\varphi \in \mathfrak{D}(O_x)$, where $O_x = \exp_x(V_x)$, define $\varphi^x \in \mathfrak{D}(V_x)$ by $\varphi^x = (\exp_x)^* \varphi = \varphi \circ \exp_x$. If A, B are C^∞ vector fields on V, define $(A, B)_g^x \in \mathfrak{D}'(V_x, \mathrm{Op}(\mathcal{H}))$ by $(A, B)_g^x(\varphi^x) = (A, B)(\varphi)$, where $(A, B)(\varphi) = \frac{1}{2}\mathcal{G}((A\otimes B + B\otimes A)\otimes\varphi)$.

Axiom (2): For every $x\in V$ there exists a sub-Hilbert space $\mathcal{H}_x$ of $\mathcal{H}$ and a unitary representation U_g^x of G_x on $\mathcal{H}_x$ such that, if $h\in G_x$, A and B are C^∞ vector fields on V, and $\varphi\in\mathfrak{D}(V_x)$, then we have

$$U_g^x(h)(A, B)_g^x(\varphi)U_g^x(h^{-1}) = (A, B)_g^x(\varphi_h), \tag{3.1}$$

where

$$\varphi_h = (h^{-1})^*\varphi = \varphi\circ h^{-1},$$

and

$$g_0(A, B) = \text{const}$$

for a given metric g_0 strongly geodesically compatible with $\mathfrak{C}$. In addition, we suppose, for the algebraic stability of the operations, that if we denote $D_x = \mathcal{H}_x\cap D$, then

$$\mathcal{G}(\varphi)D_x\subset D_x \quad\text{and}\quad U_g^x(h)D_x\subset D_x\,.$$

We must check that this axiom is independent of the choice of g. Indeed, let g' be another metric, strongly geodesically compatible with $\mathfrak{C}$. Define

$$U_{g'}^x(h) = U_g^x((g', g)_x)U_g^x(h)U_g^x((g, g')_x)\,. \tag{3.2}$$

The equation $(A, B)_{g'}^x(\varphi) = (A, B)_g^x((g', g)_x^*\varphi)$ implies then $U_{g'}^x(h)(A, B)_{g'}^x(\varphi)U_{g'}^x(h^{-1}) = (A, B)_{g'}^x(\varphi_h)$.

Remarks: (1) This axiom shows that G_x is the minimal group necessary in order to have an intrinsic theory on V, and we could have taken instead of G_x the group of all the diffeomorphisms of V_x.

(2) If $g_0' = \alpha g_0$ is a metric strongly geodesically compatible with $\mathfrak{C}$, the field $\alpha^{-1}\mathcal{G}$ is a gravitational field if in Axiom (2) we take g_0' instead of g_0.

(3) It is sufficient in (3.1) to take $g = g_0$; the general case follows immediately.

For $x\in V$, denote by T_g^x the restriction of V_g^x to the translation group on V_x. The spectral decomposition of T_g^x on $\mathcal{H}_x$, if t_a is the translation of vector a, is

$$T_g^x(t_a) = \int \exp[i(p, a)_{g(x)}]dE_g^x(p), \tag{3.3}$$

where $(p, a)_{g(x)}$ is the scalar product relative to $g(x)$ and the integral is taken over V_x. The energy-momentum operator is defined by (integration over V_x)

$$P_g(x) = \int p\, dE_g^x(p)\,. \tag{3.4}$$

On the other hand, if g' is defined as above, and

$$T_{g'}^x(t_a) = \int \exp[i(p, a)_{g'(x)}]dE_{g'}^x(p)$$

and

$$P_{g'}(x) = \int p\, dE_{g'}^x(p)$$

(integrations over V_x), then we see that

$$P_{g'}(x) = \lambda^{-1}(x)U_g^x((g', g)_x)P_g(x)U_g^x((g, g')_x) \tag{3.5}$$

if $g' = \lambda g$.

Axiom (3): For every $x\in V$, $p = 0$ is an eigenvalue of $P_g(x)$ with multiplicity unity and if we denote by Ω a corresponding eigenvalue, then $\Omega\in D_x$ and is independent of x. The total spectrum of $P_g(x)$ is equal to C_x^+. The relation (3.5) shows that this axiom is independent of the choice of g.

If M, N are two subsets of V, we shall say that they are spacelike separated if every couple of points $(x, y)\in M\times N$ are spacelike separated relative to $\mathfrak{C}$.

Axiom (4): If $\varphi, \varphi'\in s^2\mathfrak{D}(V)$ have their support spacelike separated, then $[\mathcal{G}(\varphi), \mathcal{G}(\varphi')] = 0$.

Axiom (5): The set of elements of the form $\mathcal{G}(\varphi_1)\cdots\mathcal{G}(\varphi_n)\Omega$ for $n\in\mathbb{N}$ (natural numbers) and $\varphi_1, \ldots, \varphi_n\in s^2\mathfrak{D}(V)$ is dense in $\mathcal{H}$.

Remark: We utilize translations in the *tangent space* to define the energy-momentum operators, since in the case of flat space-time it coincides with the usual definition and since in any case our construction relates very closely translations in the manifold with translations in the tangent space.

IV. THE WIGHTMAN DISTRIBUTIONS

For $\varphi_1, \ldots, \varphi_n\in s^2\mathfrak{D}(V)$, we define

$$W_n(\varphi_1\otimes\cdots\otimes\varphi_n) = (\Omega, \mathcal{G}(\varphi_1)\cdots\mathcal{G}(\varphi_n)\Omega)\,.$$

W_n can be extended by nuclearity to a continuous linear form (Wightman distribution) on the space $s^2\mathfrak{D}(V)_n$ of the C^∞ $n$tuple-twisted 4-forms twice contravariant symmetric tensor field (for every variable $x_1, \ldots, x_n$) (see Ref. 13) with compact support and with the $\mathfrak{D}$-space topology.

One easily checks that these distributions have the following properties:

(1) A necessary and sufficient condition for a C^∞ vector field A to be isotropic relative to $\mathfrak{C}$ is that

$$W_n((A\otimes A\otimes\varphi)(x_1)\otimes\varphi_2(x_2)\cdots\otimes\varphi_n(x_n)) = 0$$

for every $\varphi\in\mathfrak{D}(V)$ and $\varphi_2, \ldots, \varphi_n\in s^2\mathfrak{D}(V)$.

(2) If $\varphi\in s^2\mathfrak{D}(V)_n$, $W_n(\varphi^*) = \overline{W_n(\varphi)}$, where $\varphi^*(x_1, \ldots, x_n) = \overline{\varphi}(x_n, \ldots, x_1)$.

(3) If $\varphi\in\mathfrak{D}(V)_n$ (space of the C^∞ $n$tuple-twisted 4-forms with compact support) and $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ are C^∞ vector fields on V [from now on we suppose that $g_0(A_i, B_i) = \text{const}$], and if we define

$$(A_i, B_i)_n(\varphi) = W_n(((A_1\otimes B_1)(x_1)\otimes\cdots\otimes(A_n\otimes B_n)(x_n))\otimes\varphi(x_1, \ldots, x_n)),$$

then $(A_i, B_i)_n \in \mathfrak{D}'(V)_n$. If g is a metric strongly geodesically compatible with $\mathfrak{C}$, we define for $x \in V$

$$(A_i, B_i)^x_{n,g}(\varphi) = (A_i, B_i)_n(\varphi^x),$$

where

$$\varphi \in \mathfrak{D}(V_x)_n,$$

$$\varphi^x(x_1, \ldots, x_n) = \varphi(\exp_x^{-1}(x_1), \ldots, \exp_x^{-1}(x_n))$$

Then $(A_i, B_i)^x_{n,g} \in \mathfrak{D}'(V_x)_n$ and we have, if $h \in G_x$ and

$$\varphi_h(x_1, \ldots, x_n) = \varphi(h^{-1}(x_1), \ldots, h^{-1}(x_n)),$$

the following:

$$(A_i, B_i)^x_{n,g}(\varphi) = (A_i, B_i)^x_{n,g}(\varphi_h).$$

This allows us to define

$$\{A_i, B_i\}^x_{n-1,g}(\xi_2, \ldots, \xi_n) \in \mathfrak{D}'(V_x)_{n-1}$$

with the change of variables $\xi_k = x_k - x_{k-1}$ in $(A_i, B_i)^x_{n,g}(x_1, \ldots, x_n)$.

(4) If $\varphi \in \mathfrak{D}(V_x)_n$, we define its Fourier transform by the following integral (over V_x^n):

$$\mathfrak{F}\varphi(p_1, \ldots, p_n) = \int \exp\left(i \sum_{k=1}^{n} (p_k, x_k)_{g(x)}\right) \varphi(x_1, \ldots, x_n),$$

and if $T \in \mathfrak{D}'(V_x)_n$ by $T(\varphi) = {}^{\mathfrak{F}}T({}^{\mathfrak{F}}\varphi)$, we have ${}^{\mathfrak{F}}\varphi \in Z(V_x^n)$ and ${}^{\mathfrak{F}}T \in Z'(V_x^n)$, in the notation of Gelfand and Shilov.[15] Then the support of ${}^{\mathfrak{F}}\{A_i, B_i\}^x_{n-1,g}(q_2, \ldots, q_n)$ is contained in the set $\{(q_2, \ldots, q_n) \mid q_k \in C_x^+\}$.

(5) If $\varphi_1, \ldots, \varphi_n \in \mathfrak{s}^2\mathfrak{D}(V)$ and φ_k and φ_{k+1} have spacelike separated supports, then

$$W_n(\varphi_1 \otimes \cdots \otimes \varphi_k \otimes \varphi_{k+1} \otimes \cdots \otimes \varphi_n) = W_n(\varphi_1 \otimes \cdots \otimes \varphi_{k+1} \otimes \varphi_k \otimes \cdots \otimes \varphi_n).$$

(6) If $\varphi_n \in \mathfrak{s}^2\mathfrak{D}(V)_n$, $n = 0, \ldots, N$,

$$\sum_{m,n \geq 0} W_{m+n}(\varphi_n^* \otimes \varphi_m) \geq 0.$$

(7) If the support of ${}^{\mathfrak{F}}\varphi(p_1, \ldots, p_n)$ $[\varphi \in \mathfrak{D}(V_x)_n]$ relative to the variable $p_1 + \cdots + p_n$ intersects C_x^+ at most at the point $p_1 + \cdots + p_n = 0$, then $(A'_k, B'_k)^x_{2n,g}(\varphi^* \otimes \varphi) = |(A_i, B_i)^x_{n,g}(\varphi)|^2$, where

$$(A'_k) = (A_1, \ldots, A_n, A_1, \ldots, A_n)$$

and

$$(B'_k) = (B_1, \ldots, B_n, B_1, \ldots, B_n).$$

Reconstruction Theorem: Suppose we are given a system of distributions $W_n \in \mathfrak{D}'(V)_n$, $n \in \mathbf{N}$, satisfying (1′) the C^∞ vector fields, such that

$$W_n((A \otimes A \otimes \varphi)(x_1) \otimes \varphi_2(x_2) \otimes \cdots \otimes \varphi_n(x_n)) = 0$$

for every $\varphi \in \mathfrak{D}(V)$, $\varphi_2, \ldots, \varphi_n \in \mathfrak{s}^2\mathfrak{D}(V)$ and $n \in \mathbf{N}$ are the isotropic vector fields of a strongly geodesically complete cone field $\mathfrak{C}$ on which we choose a time orientation, and the properties (2) to (7) above, then there exists a unique quantized gravitational field $\mathcal{G}$, the Wightman distribution of which are these W_n.

We take as Hilbert space $\mathcal{H}$ the Hausdorff completed of the algebraic sum $\oplus_n(\mathfrak{s}^2\mathfrak{D}(V)_n)$ relative to the scalar product defined for $\varphi = (\varphi_n)$ and $\psi = (\psi_n)$ by

$$(\varphi, \psi) = \sum_{m,n \geq 0} W_{m+n}(\varphi_n^* \otimes \psi_m),$$

which is well defined as this sum consists of only a finite number of nonzero terms. Denote by π the canonical mapping $\oplus_n(\mathfrak{s}^2\mathfrak{D}(V)_n) \to \mathcal{H}$ and by D its range. The gravitational field is defined, for $\varphi \in \mathfrak{s}^2\mathfrak{D}(V)$ and $\psi \in \mathfrak{s}^2\mathfrak{D}(V)_n$, by $\mathcal{G}(\varphi)\pi(\psi) = \pi(\varphi \otimes \psi)$. We thus obtain an operator-valued distribution.

The representation U_g^x of G_x is defined, for $\varphi \in \mathfrak{D}(V_x)_n$ by

$$U_g^x(h)\pi([(A_1 \otimes B_1)(x_1) \otimes \cdots \otimes (A_n \otimes B_n)(x_n)] \otimes [(\exp_x^{-1})^*\varphi](x_1, \ldots, x_n)) = \pi([(A_1 \otimes B_1)(x_1) \otimes \cdots \otimes (A_n \otimes B_n)(x_n)] \otimes [(\exp_x^{-1})^*\varphi_h](x_1, \ldots, x_n)).$$

This defines by extension a unitary operator on $\mathcal{H}_x = \overline{D}_x$, where D_x is the set of the elements of the form $\sum_n \pi((\exp_x^{-1})^*\varphi_n)$, where $(\varphi_n) \in \oplus_n(\mathfrak{s}^2\mathfrak{D}(V)_n)$. The continuity in h results from the choice we made for the topology of G_x in Sec. II. The other properties are straightforward.

We remark that, V being paracompact, $\mathcal{H}$ is a separable Hilbert space because $\oplus_n(\mathfrak{s}^2\mathfrak{D}(V)_n)$ is countable.

The passage from the Wightman distributions to the Wightman functions can now be realized as in the flat-space case.

V. THE *TCP* THEOREM FOR THE GRAVITATIONAL FIELD

A. The Bargmann-Hall-Wightman (BHW) Theorem

Let $z_k \in V_x^c$ (complexification of V_x), $z_k = x_k + iy_k$, $\eta_k = y_k - y_{k-1}$ if $k \geq 2$ and $\eta_1 = y_1$, and let $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ be C^∞ vector fields on V; we define

$$(A_i, B_i)^x_{n,g}(z_1, \ldots, z_n) \equiv (A_i, B_i)^x_{n,g}(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n)$$
$$\equiv \int \exp\Big(i\sum_k (p_k, z_k)_{g(x)}\Big)^{\mathfrak{F}}(A_i, B_i)^x_{n,g}(p_1, \ldots, p_n)dp_1 \cdots dp_n$$

(integration over V_x^n), which is an holomorphic function for $\eta_k \in C_x^+$.

$(A_i, B_i)^x_{n,g}(x_1, \ldots, x_n)$ (defined in Sec. IV) is the limit in $\mathfrak{D}'(V)_n$ of

$$(A_i, B_i)^x_{n,g}(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n)$$

when $\eta_1, \ldots, \eta_n$ go to zero. If we define

$$\{A_i, B_i\}^x_{n-1,g}(\zeta_2, \ldots, \zeta_n) = (A_i, B_i)^x_{n,g}(z_1, \ldots, z_n),$$

where $\zeta_k = z_k - z_{k-1}$, these functions are analytic for

$$\zeta_k \in \mathfrak{I} \equiv \{\zeta \,|\, \mathrm{Im}(\zeta) \in C_x^+\}.$$

The distributions $\{A_i, B_i\}^x_{n-1,g}(\xi_2, \ldots, \xi_n)$ (defined in Sec. IV) are limits of the $\{A_i, B_i\}^x_{n-1,g}(\zeta_2, \ldots, \zeta_n)$ when $\mathrm{Im}(\zeta_k)$ go to zero.

We shall call $L_+(\mathbf{C})$ the connected component of identity of the complex Lorentz group. Then the BHW theorem takes the following form:

Theorem: If $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ are C^∞ vector fields on V, the functions $\{A_i, B_i\}^x_{n-1,g}(\zeta_2, \ldots, \zeta_n)$, which are invariant under the connected component of identity of the real Lorentz group and analytic in $\mathfrak{I}^{n-1}$, have an analytic extension which is invariant under $L_+(\mathbf{C})$ to the domain

$$\mathfrak{I}'_{n-1} = \bigcup_{\Lambda \in L_+(\mathbf{C})} \Lambda \mathfrak{I}^{n-1}.$$

The functions $\{A_i, B_i\}^x_{n-1,g}(\zeta_2, \ldots, \zeta_n)$ have the same properties as the usual Wightman functions for the scalar field. Thus the demonstration of the BHW theorem in our case is the same as that given (e.g.) in Jost.[14]

B. The *TCP* Theorem

To bring us back to a situation similar to the flat case, we begin with the following:

Lemma: Given two points x and y in V which are physically spacelike separated, there exist two neighborhoods M_x and M_y of these points which are spacelike separated.

Define $x' = \exp_y^{-1}(x)$; there exists an open neighborhood O' of x' in V_y which is spacelike separated (in the Minkowskian sense relative to C_y) with the origin of V_y. Then $\exp_y(O')$ and y are spacelike separated. Choose x_1 and x_2 into two different connected components of the intersection of the image under $\exp_x$ of the interior of C_x with $\exp_y(O')$. Denote by A_i and B_i ($i=1, 2$) the images under the exponential mapping at x_i of the interior and the exterior (respectively) of the light cone at x_i;

$$\exp_y(O') \cap A_1 \cap A_2 = M_x$$

and

$$\exp_y(V_x) \cap B_1 \cap B_2 = M_y$$

are two open sets of V containing x and y (respectively) and which are spacelike separated.

Corollary: For any $x \in V$, choose $a \in V_x$, spacelike separated (in the Minkowskian sense) in V_x with the origin 0 of V_x. There exist two open sets O_1 and O_2 which contain, respectively, 0 and a, are spacelike separated in the Minkowskian sense in V_x, and are such that $\exp_x(O_1)$ and $\exp_x(O_2)$ are spacelike separated in V.

Let S_1 and S_2 be two Minkowskian spacelike-separated open sets in V_x which contain, respectively, 0 and a; put $y = \exp_x(a)$ and choose M_x and M_y, two open sets containing x and y and which are spacelike separated in V. The open sets

$$O_1 = S_1 \cap \exp_x^{-1}(M_x) \quad \text{and} \quad O_2 = S_2 \cap \exp_x^{-1}(M_y)$$

answer the question.

When $x \in V$, $\varphi \in \mathscr{E}^2\mathfrak{D}(V_x)$, and g is a metric strongly geodesically compatible with $\mathfrak{C}$, we shall define

$$\mathfrak{G}_g^x \in \mathscr{E}_2\mathfrak{D}'(V_x, \mathrm{Op}(\mathcal{H}))$$

by

$$\mathfrak{G}_g^x(\varphi) = \mathfrak{G}((\exp_x^{-1})^*\varphi).$$

Let $a \in V_x$ be spacelike separated with the origin 0 of V_x and let O_1 and O_2 be two open sets satisfying the conditions of the above corollary. If $\varphi, \varphi' \in \mathscr{E}^2\mathfrak{D}(V_x)$ with $\mathrm{supp}(\varphi) \subset O_1$ and $\mathrm{supp}(\varphi') \subset O_2$, we have

$$[\mathfrak{G}_g^x(\varphi), \mathfrak{G}_g^x(\varphi')] = 0.$$

This last property together with the BHW theorem implies (cf. Jost,[14] theorem, p. 85) the locality of the field $\mathfrak{G}_g^x(\varphi)$ in the tangent space V_x.

Before formulating the *TCP* theorem, and for the sake of completeness, we give the following definition:

Suppose $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ are C^∞ vector fields on V, and $\varphi_1, \ldots, \varphi_n \in \mathfrak{D}(V)$. Then we define

$$[A_i, B_i]_n(\varphi_1 \otimes \cdots \otimes \varphi_n) = \mathfrak{G}(A_1 \otimes B_1 \otimes \varphi_1) \cdots \mathfrak{G}(A_n \otimes B_n \otimes \varphi_n)\Omega.$$

These $[A_i, B_i]_n$ can be extended by nuclearity to distributions over $\mathfrak{D}(V)_n$. We then define, if $\varphi \in \mathfrak{D}(V_x)_n$,

$$[A_i, B_i]^x_{n,g}(\varphi) = [A_i, B_i]_n(\varphi^x),$$

where $\varphi^x = (\exp_x{}^{-1})^*\varphi$.

TCP Theorem: To every $x \in V$ is associated an antilinear operator θ_g^x on $\mathcal{H}_x$ obtained by antilinear extensions of

$$\theta_g^x \Omega = \Omega,$$

$$\theta_g^x [A_i, B_i]_{n,g}^x(\varphi) = [A_i, B_i]_{n,g}^x(\varphi^-),$$

where

$$\varphi \in \mathfrak{D}(V_x)_n,$$

$$\varphi^-(x_1, \ldots, x_n) = \bar{\varphi}(-x_1, \ldots, -x_n).$$

This operator has the following action on the field $\mathcal{G}$: Let $x \in V$ and $O_x = \exp_x(V_x)$(recall that O_x is independent of the choice of g strongly geodesically compatible with $\mathfrak{C}$), and denote by S_g^x the geodesic symmetry (relative to the metric g) at the point x; then if A, B are C^∞ vector fields on V and $\varphi \in \mathfrak{D}(O_x)$, we have

$$\theta_g^x\, \mathcal{G}(A \otimes B \otimes \varphi)\theta_g^x = \mathcal{G}(A \otimes B \otimes (\varphi \circ S_g^x)).$$

Remark. Though for the sake of simplicity we utilize strong locality here, the generalization in our case to weak locality is straightforward.

VI. POSSIBLE DEVELOPMENT OF A THEORY OF ASYMPTOTIC STATES

(1) The equations satisfied by the asymptotic fields are Einstein equations $R(\mathcal{G}) = 0$, where R is the Ricci tensor of $\mathcal{G}$. However, this expression does not make sense as a distribution equation. We shall therefore give to these equations the following sense:

We say that $\mathcal{G}$ is a weak solution of the equation $R(\mathcal{G}) = 0$ if there exists a sequence $\mathcal{G}_n$ of C^∞ operator-valued metrics with signature $(+ - - -)$, which converge in $\mathcal{S}_2\mathfrak{D}'(V, \mathrm{Op}(\mathcal{H}))$ to $\mathcal{G}$ (for the strong topology in $\mathfrak{D}'$), and such that $R(\mathcal{G}_n)$ tends to zero in $\mathcal{S}^2\mathfrak{D}'(V, \mathrm{Op}(\mathcal{H}))$.

However, the asymptotic properties of the solution of Einstein's equations are not known. It is therefore impossible at present to give a general theory of asymptotic states. Nevertheless, the following remarks seem of relevance to a possible future.

(2) Let h be a metric on V and let f be a C^∞ $\mathrm{Op}(\mathcal{H})$-valued function on V such that $[f(x)]^{-1}$ exists at every point of V; the Christoffel symbols of the metric fh, in a coordinate system, are given by

$$\left\{ {\alpha \atop \beta\gamma} \right\}(fh) = \left\{ {\alpha \atop \beta\gamma} \right\}(h) + \frac{1}{2f}\,[\delta_\gamma^\alpha \partial_\beta(f) + \delta_\beta^\alpha \partial_\gamma(f) - h_{\beta\gamma} h^{\alpha\rho}\partial_\rho(f)],$$

where $(1/f)$ is the function $x \to [f(x)]^{-1}$ and

$$\left\{ {\alpha \atop \beta\gamma} \right\}(h) = \tfrac{1}{2} h^{\alpha\rho}[\partial_\beta(h_{\gamma\rho}) + \partial_\gamma(h_{\beta\rho}) - \partial_\rho(h_{\gamma\beta})],$$

and where the Ricci tensor $R_{\alpha\beta}(fh)$ of fh is given by

$$R_{\alpha\beta}(fh) = R_{\alpha\beta}(h) - \frac{1}{2f}\partial_\rho(h_{\alpha\beta}\partial^\rho f) - \frac{1}{f}\partial_\alpha\partial_\beta f - \frac{3}{f^2}(\partial_\alpha f)(\partial_\beta f) + \frac{1}{2f}\partial^\rho f\left(\partial_\alpha(h_{\beta\rho}) + \partial_\beta(h_{\alpha\rho}) - h_{\alpha\beta}\left\{ {\gamma \atop \gamma\rho} \right\}(h)\right).$$

Put

$$A_{\alpha\beta}(f, h) = -\partial_\rho(h^{\rho\gamma}\partial_\gamma f)h_{\alpha\beta} - 2\partial_\alpha\partial_\beta f + \partial_\rho f\left(2\left\{ {\rho \atop \alpha\beta} \right\}(h) - 2h_{\alpha\beta}h^{\rho\gamma}\left\{ {\pi \atop \pi\gamma} \right\}(h)\right) + \frac{6}{f}(\partial_\alpha f)(\partial_\beta f). \tag{6.1}$$

Suppose that we are given a strongly geodesically complete metric g satisfying $R(g) = 0$ and a weak solution λ of $A(\lambda, g) = 0$ {e.g., a sequence λ_n of C^∞ $\mathrm{Op}(\mathcal{H})$-valued functions on V such that $[\lambda_n(x)]^{-1}$ exists for every $x \in V$, which converges to λ in $\mathfrak{D}'(V, \mathrm{Op}(\mathcal{H}))$, and with $\mathrm{supp}\lambda = V$}. Then $\mathcal{G} = \lambda g$ is a distribution metric which is a weak solution of $R(\mathcal{G}) = 0$.

We note that the equation $A(\lambda, g) = 0$ contains only one nonlinear term in λ, which is far less complicated than the nonlinear terms of the general equation $R(\mathcal{G}) = 0$. So it seems possible, in that particular case, to make a treatment of asymptotic states (at least formally).

(3) Another possible direction is to suppose that the asymptotic field is asymptotically flat. We can, for instance, suppose, if g is a metric strongly geodesically compatible with $\mathfrak{C}$, and (e_α) a Cartesian basis in V_x for a chosen $x \in V$, that

$$\Box_y(\mathcal{G}_g^x)_{\alpha\beta} - \frac{1}{2}\frac{\partial^2}{\partial y^\alpha \partial y_\rho}(\mathcal{G}_g^x)_{\rho\beta} - \frac{1}{2}\frac{\partial^2}{\partial y^\beta \partial y_\rho}(\mathcal{G}_g^x)_{\rho\alpha} = T, \tag{6.2}$$

where $(\mathcal{G}_g^x)_{\alpha\beta}$ is defined by

$$(\mathcal{G}_g^x)_{\alpha\beta}(\varphi) = \tfrac{1}{2}(\mathcal{G}_g^x)[(e_\alpha \otimes e_\beta + e_\beta \otimes e_\alpha) \otimes \varphi]$$

for $\varphi \in \mathfrak{D}(V_x)$, and T is a distribution which satisfies the following condition: For every $\epsilon > 0$ there exists a compact $K_\epsilon \subset V_x$ which contains the origin such that if $\varphi \in \mathfrak{D}(V_x)$ with $\mathrm{supp}(\varphi) \subset \tilde{C}_x \cap (V_x - K_\epsilon)$, where $\tilde{C}_x$ is the set of timelike and isotropic vectors at x, we have for all $\Psi \in D$

$$\| T(\varphi)\Psi \| \leq \epsilon \left[\sum_{\rho\gamma\delta} \| \partial_\rho (\mathcal{G}_g^x)_{\gamma\delta}(\varphi)\Psi \| + \sum_{\rho\gamma\delta\pi} \| \partial_{\rho\gamma}{}^2 (\mathcal{G}_g^x)_{\delta\pi}(\varphi)\Psi \| \right]. \qquad (6.3)$$

One then finds the following commutation relations:

$$[(\mathcal{G}_g^x)_{\alpha\beta}(y), (\mathcal{G}_g^x)_{\gamma\delta}(y')] = -i\hbar[K_{\alpha\beta\gamma\delta}(y,y') - g_{\alpha\beta}(y) g_{\gamma\delta}(y') G^0(y,y')] + T', \qquad (6.4)$$

where G^0 and K are, respectively, the scalar and the symmetric tensorial propagators[5] of the operators $\Box_y = g_{\alpha\beta}(y)\partial_\alpha\partial_\beta$, and T' is a distribution with the same properties as T arising from higher-order terms. Here we have asymptotically (when $t \to \pm\infty$) a *linear* equation which after contraction of the nonoperatorial spin-2 degrees of freedom gives rise to scalar "gravitons." In spite of the difficulty of getting an asymptotic-state theory for zero-mass particles, the asymptotic linearity gives an important simplification and therefore there is a good hope in this direction.

VII. CONCLUSIONS

We saw that it was possible to give a Wightman formulation to the gravitational field in which the geometry of space-time is directly created by the quantized field (by geometry we mean here the cone field). This implies some particularities in comparison with the usual theory of a flat space-time; for instance, the fact that the group G_x cannot (in general) operate everywhere on $\mathcal{H}$, namely, that to every $x \in V$ is associated a Hilbert space $\mathcal{H}_x \subset \mathcal{H}$. In general, $\mathcal{H}_x \neq \mathcal{H}$ if $\exp_x(V_x) \neq V$, a thing associated with the fact that $\mathcal{H}_x$ is connected with a space of forms the support of which is contained in $O_x = \exp_x(V_x)$. This set containing only the points which are connected to x by geodesics compatible with the cone field, the space $\mathcal{H}_x$ thus can be interpreted as the space of states that *a priori* can be physically influenced by the event x.

Note also that the decomposition $\mathcal{G} = \lambda g$ implies that $\mathcal{G}$ contains only one quantized operational degree of freedom, which is in the λ part. The g part itself, consisting of C^∞ functions, contains the spin degrees of freedom. In other words, though in our quantization procedure the classical field has spin 2, the quantized field has spin 0.

In this connection one has to remark that for a *given* generalized metric, $\mathcal{G}$-null directions at each point of V are fixed. However, $\mathcal{G}$ itself can be uniquely determined only if the Cauchy data of Einstein's equations are known, and this is not supposed here.

Concerning the results, we get (except for the asymptotic-states theory) the usual results of the Wightman theory on Minkowski space. The theory of asymptotic states contains some difficulties, the first resulting from the nonlinearity of Einstein's equations (this problem is not a new one and one can try to solve it by perturbation-theory methods), and the second being the fact that the gauge group in non-Abelian. In that direction Popov and Faddeev[9] gave an interesting formal technique of renormalization.

Our Wightman theory of gravitation contains the usual difficulties: Does there exist a nontrivial quantized field having essentially self-adjoint operators? Can we (for instance in the asymptotically flat case) have the Haag-Ruelle kind of results also for zero-mass particles? etc.

To sum up we see that the problems which arose in our Wightman theory of the gravitational field are problems which existed already for a field on the Minkowski space-time which satisfies nonlinear equations of motion (as the Yang-Mills field, for example).

Therefore, any progress in the Wightman theory or in the knowledge of properties of solutions of the classical Einstein equations will contribute to a complete solution to the problem of axiomatic quantization of the gravitational field.

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