[10] One easily verifies that the restriction on the range of the y integration implied by Eq. (8) can be neglected provided $\epsilon \ll 1$.

[11] This justifies our statement in Sec. II that the two-body S-matrix factor strongly damps chains from which pions are produced with large relative rapidities.

[12] Notice that the contribution from the "grey ring" can be made arbitrarily small by taking g to be large.

[13] Notice that the bootstrap solution given by Eqs. (41) and (42) holds for all values of R_0.

[14] T. L. Neff, R. Savit, and R. Blankenbecler, SLAC Report No. 988 (unpublished).

PHYSICAL REVIEW D VOLUME 5, NUMBER 12 15 JUNE 1972

Where Are the Corrections to the Goldberger-Treiman Relation?*

Heinz Pagels and A. Zepeda†

The Rockefeller University, New York, New York 10021

(Received 9 February 1972)

On the basis of unsubtracted dispersion relations we point out that theoretical estimates fail to account for the corrections to the Goldberger-Treiman relation $\Delta = 1 - Mg_A/gf_\pi \simeq 0.08 \pm 0.02$ by one to two orders of magnitude. As a new result we prove an exact threshold theorem for the absorptive part of $K(t)$, the pion-nucleon form factor, by doing the full three-body angular integrations and using current-algebra low-energy theorems for the 3π processes. In the chiral-SU(2)⊗SU(2)-symmetry world in which the pion mass vanishes the result is $\mathrm{Im}K(t) = [gt^2/3(8\pi)^3 f_\pi^{\,4}][\frac{1}{3}g_A^{\,2}(\frac{5}{2} - \frac{17}{35}\pi^2) - \frac{7}{8}]$ as $t \to 0$, the threshold point. This leads to an estimate of Δ which fails by two orders of magnitude. We also show the pion spectral function in the chiral-symmetric world behaves like $\rho_\pi(t) = 8t/3(8\pi)^4 f_\pi^{\,4}$ as $t \to 0$. As a way out of this negative result we consider (i) subtractions in the dispersion relation, (ii) new values for g and g_A, and (iii) a heavy pion as possibilities.

I. CORRECTIONS TO THE GOLDBERGER-TREIMAN RELATION

This paper will be devoted to an examination of the corrections to the Goldberger-Treiman relation,[1] which are about 8%[2]:

$$\Delta = 1 - \frac{Mg_A}{f_\pi g} = +0.08 \pm 0.02\,.$$

The approximate validity of the relation $Mg_A \simeq f_\pi g$ today is understood as a consequence of a slightly broken chiral SU(2)⊗SU(2) symmetry with the pion in the role of the Nambu-Goldstone boson.[3]

The main point of this article is to point out that on the basis of unsubtracted dispersion relations we have no theoretical understanding of these corrections. Theoretical estimates are too small to account for the observed value by one to two orders of magnitude.[4] If the experimental numbers are indeed correct then this discrepancy poses a theoretical problem to which we presently have no answer.

As a major result in this study we have established an exact threshold theorem for the absorptive part of $K(t)$, the pion-nucleon form factor, in the SU(2)⊗SU(2)-symmetry limit for which the pion mass vanishes. This result was obtained from the 3-pion intermediate state utilizing the exact chiral-symmetry low-energy theorems for the 3-pion processes and performing the exact three-body angular integrations over the appropriate matrix elements. The result is the threshold theorem

$$\mathrm{Im}K(t) \underset{t\to 0}{\sim} \frac{gt^2}{3(8\pi)^3 f_\pi^{\,4}}\left[\tfrac{1}{3}g_A^{\,2}(\tfrac{5}{2} - \tfrac{17}{35}\pi^2) - \tfrac{7}{8}\right], \quad (1.1)$$

valid in the exact SU(2)⊗SU(2)-symmetry limit in which the 3π threshold is at $t=0$. Using this exact result we estimate the 3-pion contribution from threshold states to be too small by two orders of magnitude. Further, this result shows that the corrections are analytic to leading order in chiral-symmetry breaking.

We begin our discussion by defining the matrix element of the divergence of the axial-vector current between nucleon states:

$$\langle N(p')\,|\,i\partial^\mu A_\mu^a(0)\,|\,N(p)\rangle = \bar{u}(p')D(t)(\tfrac{1}{2}\tau^a)\gamma_5 u(p),$$
$$t=(p'-p)^2\,. \quad (1.2)$$

Here $D(t)$ satisfies $D(0)=2Mg_A$, where M is the nucleon mass and g_A the axial-vector coupling constant related to the rate of Gamow-Teller transitions in neutron β decay. Defining the pion field according to partial conservation of axial-vector current (PCAC)

$$\partial^\mu A_\mu^a(x) = \mu^2 f_\pi \pi^a(x)\,, \quad (1.3)$$

where μ is the pion mass and f_π the pion decay constant,[5] we may define the pion nucleon form factor

$$K(t)\bar{u}(p')\gamma_5\tau^a u(p) = i\langle N(p')|(\Box+\mu^2)\pi^a(0)|N(p)\rangle$$

so that

$$D(t) = \frac{2\mu^2 f_\pi K(t)}{-t+\mu^2} \tag{1.4}$$

and $K(\mu^2) = g$, the pion-nucleon coupling constant.

As a fundamental assumption we presume that $\partial^\mu A_\mu^a(x)$ is a gentle operator and that $D(t)\to 0$ as $t\to\infty$. This implies an unsubtracted dispersion relation,

$$D(t) = \frac{1}{\pi}\int_0^\infty \frac{dt'\,\mathrm{Im}D(t')}{t'-t}, \tag{1.5}$$

and extracting the π pole term and setting $t=0$, we find

$$Mg_A - f_\pi g = \frac{1}{2\pi}\int_{(3\mu)^2}^\infty \frac{dt}{t}\,\mathrm{Im}D(t). \tag{1.6}$$

We will find it more convenient to discuss $K(t)$ defined by (1.4) so that the corrections to the Goldberger-Treiman relation,

$$\Delta = 1 - \frac{Mg_A}{f_\pi g}, \tag{1.7}$$

may be expressed by

$$\Delta = \frac{\mu^2}{g\pi}\int_{(3\mu)^2}^\infty \frac{dt\,\mathrm{Im}K(t)}{t(t-\mu^2)}. \tag{1.8}$$

The only assumption going into this expression for Δ is the absence of subtractions for $D(t)$. It should be remarked here that electromagnetic corrections have been estimated.[6,7] They are finite[8] and are typically of order $\alpha/4\pi$, too small by at least an order of magnitude. We may continue to ignore them.

One might suppose, in view of our total ignorance of the high-frequency part of the dispersion integral (1.8), that one could say nothing about this part of Δ. However, one can establish a rigorous bound on the high-frequency part

$$\Delta_H = \frac{\mu^2}{g\pi}\int_{(2M)^2}^\infty \frac{dt\,\mathrm{Im}K(t)}{t(t-\mu^2)}. \tag{1.9}$$

This bound is[4]

$$|g\Delta_H|^2 < \frac{2\mu^4}{M^2}\int_{(2M)^2}^\infty \frac{dt}{t}\rho_\pi(t). \tag{1.10}$$

Here $\rho_\pi(t) > 0$ is the pion spectral function where the interpolating field for the pion is defined by PCAC [Eq. (1.3)]. The pion propagator is

$$\Delta_\pi(t) = \frac{1}{t-\mu^2} + \int_{(3\mu)^2}^\infty \frac{dt'\rho_\pi(t')}{t-t'}. \tag{1.11}$$

If approximate SU(2) ⊗ SU(2) is valid we may assume that $\Delta_\pi(0)$ is dominated by the pion pole. This requires

$$1 \geq \mu^2\int_{(3\mu)^2}^\infty \frac{dt}{t}\rho_\pi(t), \tag{1.12}$$

where equality holds only if the continuum equals the pion-pole contribution to $\Delta_\pi(0)$. From (1.12) and (1.10) we obtain

$$|\Delta_H| \leq \frac{\sqrt{2}}{g}\left(\frac{\mu}{M}\right) \simeq 0.014. \tag{1.13}$$

Had we assumed that the right-hand side of (1.12) was $\lesssim 0.1$, a more reasonable estimate in view of the success of current algebra, then $|\Delta_H| < 0.004$. In either case the high-frequency part can safely be ignored so that

$$\Delta \simeq \frac{\mu^2}{g\pi}\int_{(3\mu)^2}^{(2M)^2} \frac{dt\,\mathrm{Im}K(t)}{t(t-\mu^2)}. \tag{1.14}$$

The contributions of intermediate $\rho\pi$ and $\sigma\pi$ states have been estimated and found to be small.[9] The $\sigma\pi$ state will contribute to Δ with the wrong sign, as can be argued as follows. The $\sigma\pi$ intermediate-state contribution to Δ essentially is the influence of the σ term in π-N scattering on Δ.[4] This σ term as estimated by Cheng and Dashen[10] is +110 MeV, although it may be smaller by a factor of 2 or 3.[11] The σ term is the nucleonic mass shift δM when one turns on SU(2) ⊗ SU(2)-violating forces. From our definition of Δ (1.7) such a shift in the nucleonic mass contributes to $\Delta_\sigma \simeq -\delta M/M$, which is negative or in the wrong direction if the *sign* of the Dashen-Cheng calculation is correct.

Finally we turn to the 3π intermediate state. In Sec. II we will prove the theorem on the absorptive part $\mathrm{Im}K(t)$ in the chiral limit which is given by (1.1). From this result and from the dispersion integral (1.8) we see that if we develop Δ in an expansion in μ^2 then

$$\Delta \underset{\mu^2\to 0}{\sim} C_1\mu^2 + C_2\mu^4\ln\mu^2 + \cdots, \tag{1.15}$$

where $C_{1,2}$ are finite in the chiral SU(2) ⊗ SU(2) limit. Hence Δ is analytic to leading order in chiral-symmetry breaking[12] and we have from a cutoff dispersion integral

$$C_1 \simeq \frac{1}{g\pi}\int_0^{(2\Lambda)^2} \frac{dt}{t^2}\,\mathrm{Im}K(t). \tag{1.16}$$

If we estimate Δ using this result we obtain from a cutoff dispersion relation (1.16)

$$\Delta \simeq \frac{\mu^2 4\Lambda^2}{3\pi(8\pi)^3 f_\pi^4}\left[\tfrac{1}{3}g_A^2\left(\tfrac{5}{2} - \tfrac{17}{35}\pi^2\right) - \tfrac{7}{8}\right] = -0.001, \quad \Lambda \sim M \tag{1.17}$$

which fails to account for the observed corrections, in sign and by two orders of magnitude. The main reason for this small magnitude is that the three-body phase space is very small.

II. THRESHOLD THEOREM

In this section we will describe the calculation of our threshold theorem (1.1). Our starting point is the unitarity condition for $\operatorname{Im}K(t)$, which is

$$\operatorname{Im}K(t)=\tfrac{1}{4}\sum_{\text{spins},d,n}\hat{P}^d(2\pi)^4\delta^4(p'-p-p_n)\times\langle N(p')|\bar{\eta}(0)|n\rangle\langle n|J^d(0)|0\rangle u(p). \tag{2.1}$$

Here we have introduced a projection operator

$$\hat{P}^d=-\tfrac{2}{3}(M^2/t)\bar{u}(p)\tau^d i\gamma_5 u(p'), \tag{2.2}$$

where τ^d are the usual isospin matrices $[\tau_a,\tau_b]_+=2\delta_{ab}$ $(a,b=1,2,3)$ and the nucleon wave functions are normalized so that $\sum_{\text{spin}}u(p)\bar{u}(p)=(\not{p}+M)/2M$. Also, $\bar{\eta}(x)=\bar{\psi}(x)(-i\overleftarrow{\not\partial}-M)$ and $J^d(x)=(\Box+\mu^2)\pi^a(x)$, with the pion field defined by $\mu^2 f_\pi\pi^a(x)=\partial^\mu A_\mu^a(x)$. We here will compute the behavior of $\operatorname{Im}K(t)$ in the chiral limit, $\mu^2=0$, as $t\to 0$, which is the threshold in the chiral limit. The leading behavior of $\operatorname{Im}K(t)$ is obtained by retaining just the 3π intermediate state which contributes proportional to t^2. The contribution from the 5π and 7π states begins to contribute only in orders t^4 and t^6 so the leading behavior involves only the 3π state. From the 3π state one has (see Fig. 1)

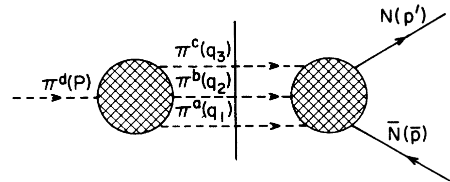

FIG. 1. Unitarity condition.

$$\operatorname{Im}K(t)=\frac{1}{4}\frac{1}{3!}\sum_{\substack{a,b,c,d,\\ \text{spins}}}\frac{\hat{P}^d}{(2\pi)^5}\int\frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}\frac{d^3q_3}{2q_3^0}\delta^4(p'-p-(q_1+q_2+q_3))\times\langle N(p')|\bar{\eta}(0)|\pi^a(q_1)\pi^b(q_2)\pi^c(q_3)\rangle\langle 0|J^d(0)|\pi^a(q_1)\pi^b(q_2)\pi^c(q_3)\rangle^* u(p). \tag{2.3}$$

In the chiral limit the matrix elements in this unitarity condition for $\pi^d\to\pi^a+\pi^b+\pi^c$ and (by crossing) for $\pi^a+\pi^b+\pi^c\to N+\bar{N}$ can be established at threshold by the standard techniques for proving low-energy theorems. It is useful to cross one of the nucleon legs and define $\bar{p}=-p$ and $P=\bar{p}+p'$ so $P^2=t$. Using Weinberg's method[13] for low-energy pion scattering, one finds

$$\langle 0|J^d(0)|\pi^a(q_1)\pi^b(q_2)\pi^c(q_3)\rangle=-\frac{1}{f_\pi^{\,2}}\{\delta_{ad}\delta_{bc}[\mu^2-(P-q_1)^2]+\delta_{bd}\delta_{ac}[\mu^2-(P-q_2)^2]+\delta_{cd}\delta_{ab}[\mu^2-(P-q_3)^2]\}. \tag{2.4}$$

To establish the low-energy theorem for the process $3\pi\to\bar{N}N$ we have found it expedient to separate this process into proper and improper parts. The improper part is defined as that part of the amplitude that one can cut across a single pion line in the t channel, i.e., those processes for which $3\pi\to\pi\to\bar{N}N$. The proper part is everything else. For the proper part one establishes by standard current algebra a low-energy theorem as $q_{1,2,3}\to 0$, which is the threshold point. With the understanding that the improper piece has been separated out, the result for the proper piece is

$$\begin{aligned}
f_\pi^{\,3}\langle\alpha|\pi^a(q_1)\pi^b(q_2)\pi^c(q_3)\beta\rangle={}&-q_1^\mu q_2^\nu q_3^\lambda\langle\alpha|T(\hat{A}_\mu^a(q_1)\hat{A}_\nu^b(q_2)\hat{A}_\lambda^c(q_3))|\beta\rangle\\
&+\tfrac{1}{2}[q_1^\mu q_3^\lambda\epsilon^{bad}\langle\alpha|T(V_\mu^d(q_1+q_2)\hat{A}_\lambda^c(q_3))|\beta\rangle+q_2^\nu q_3^\lambda\epsilon^{abd}\langle\alpha|T(V_\nu^d(q_1+q_2)\hat{A}_\lambda^c(q_3))|\beta\rangle\\
&+q_2^\nu q_1^\mu\epsilon^{cbd}\langle\alpha|T(V_\nu^d(q_2+q_3)\hat{A}_\mu^a(q_1))|\beta\rangle+q_3^\lambda q_1^\mu\epsilon^{bcd}\langle\alpha|T(V_\lambda^d(q_2+q_3)\hat{A}_\mu^a(q_1))|\beta\rangle\\
&+q_3^\lambda q_2^\nu\epsilon^{acd}\langle\alpha|T(V_\lambda^d(q_1+q_3)\hat{A}_\nu^b(q_2))|\beta\rangle+q_1^\mu q_2^\nu\epsilon^{cad}\langle\alpha|T(V_\mu^d(q_1+q_3)\hat{A}_\nu^b(q_2))|\beta\rangle]\\
&+\tfrac{1}{3}[(P-3q_3)^\mu\delta_{ab}\delta_{cf}+(P-3q_2)^\mu\delta_{ac}\delta_{bf}+(P-3q_1)^\mu\delta_{bc}\delta_{af}]\langle\alpha|\hat{A}_\mu^f(P)|\beta\rangle.
\end{aligned} \tag{2.5}$$

Here $\hat{A}_\mu$ denotes that the axial-vector current has the π pole removed, and $A_\mu^a(q)=\int d^4x\,e^{-iq\cdot x}A_\mu^a(x)$.

There are three distinct terms in (2.5) which will contribute to $\operatorname{Im}K(t)$; we will denote them by $\operatorname{Im}K_{\hat{A}\hat{A}\hat{A}}$, $\operatorname{Im}K_{V\hat{A}}$, and $\operatorname{Im}K_{\hat{A}}$. These three terms taken together with the contribution of the improper part, $\operatorname{Im}K_\pi$, give us the total contribution of the $3\pi\to\bar{N}N$ matrix element.[3] (See Fig. 2.)

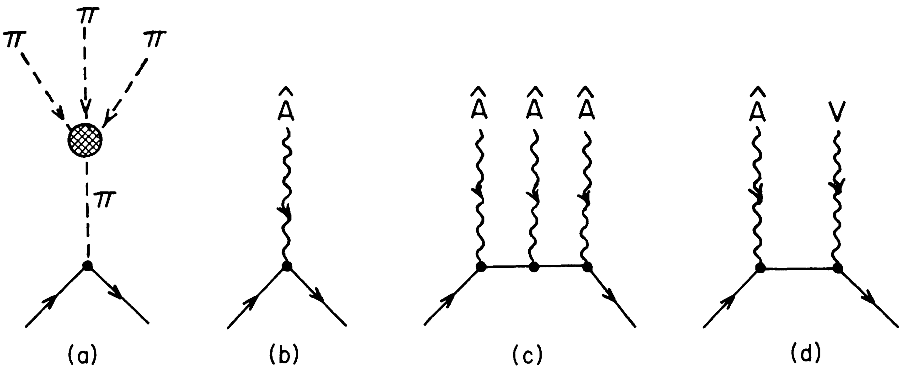

FIG. 2. Low-energy contributions to $T(3\pi \to \bar{N}N)$. (a) Improper graph. (b), (c), (d) Proper graphs corresponding to current products and commutators.

A. Improper Graph

Using the $\pi \to 3\pi$ amplitude (2.4) with $\mu^2=0$, the contribution of the improper graph is

$$\mathrm{Im}\,K_\pi(t)=\frac{1}{4\times 3!\,(2\pi)^5}\sum_{\substack{a,b,c,d,\\ \text{spins}}}\frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}\frac{d^3q_3}{2q_3^0}\delta^4(P-q_1-q_2-q_3)$$

$$\times\frac{2}{3}\frac{M^2}{t}\bar{v}(\bar{p})\tau^d i\gamma_5 u(p')\frac{1}{-t+\mu^2}g\bar{u}(p')i\gamma_5\tau^d v(\bar{p})$$

$$\times\frac{1}{f_\pi^{\,4}}[\delta_{ad}\delta_{bc}(P-q_1)^2+\delta_{bd}\delta_{ac}(P-q_2)^2+\delta_{cd}\delta_{ab}(P-q_3)^2]^2\,.$$

The angular integrations present no difficulties since all angles are in the numerator, and one finds

$$\mathrm{Im}\,K_\pi(t)\underset{t\to 0}{\sim}-\frac{gt^2}{3(8\pi)^3 f_\pi^{\,4}}\,. \tag{2.6}$$

The sign is negative, as is required for all improper graphs.[4] A by-product of this discussion is an exact theorem on the behavior of the pion spectral function defined in (1.11), which is

$$\rho_\pi(t)\underset{t\to 0}{\sim}\frac{8t}{3(8\pi)^4 f_\pi^{\,4}} \tag{2.7}$$

in the chiral limit. This result is obtained from (2.6) by dividing out the pion pole term.

B. $\mathrm{Im}K_{\hat{A}}$ Term

The contribution of this term [Fig. 1(b)] is given by

$$\mathrm{Im}\,K_{\hat{A}}(t)=\frac{1}{4\times 3!\,(2\pi)^5}\sum_{\substack{a,b,c,d,\\ \text{spins}}}\int\frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}\frac{d^3q_3}{2q_3^0}\delta^4(P-q_1-q_2-q_3)\frac{2}{3}\frac{M^2}{t}\bar{v}(\bar{p})\tau^d i\gamma_5 u(p')\frac{1}{f_\pi^{\,3}}\bar{u}(p')g_A\tfrac{1}{2}\tau^f i\gamma_5\gamma_\mu v(\bar{p})$$

$$\times\tfrac{1}{3}[(P-3q_3)^\mu\delta_{ab}\delta_{cf}+(P-3q_2)^\mu\delta_{ac}\delta_{bf}+(P-3q_1)^\mu\delta_{bc}\delta_{af}]$$

$$\times\frac{1}{f_\pi^{\,2}}[(P-q_1)^2\delta_{ad}\delta_{bc}+(P-q_2)^2\delta_{bd}\delta_{ac}+(P-q_3)^2\delta_{cd}\delta_{ab}]\,. \tag{2.8}$$

The angular integrations present no great difficulty, and one obtains

$$\mathrm{Im}\,K_{\hat{A}}(t)\underset{t\to 0}{\sim}\frac{-gt^2}{36(8\pi)^3 f_\pi^{\,4}}\,, \tag{2.9}$$

where we have used $Mg_A=gf_\pi$.

C. $\mathrm{Im}K_{\hat{A}\hat{A}\hat{A}}$ Term

This part confronts us with doing the angular integrations over the nucleon pole terms and is the most difficult to calculate. The unitarity condition reads

$$\mathrm{Im}K_{\hat{A}\hat{A}\hat{A}}(t) = -\frac{1}{4\cdot 3!\,(2\pi)^5}\sum_{a,b,c,d,\ \mathrm{spins}}\int \frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}\frac{d^3q_3}{2q_3^0}\delta^4(P-q_1-q_2-q_3)$$
$$\times\frac{1}{f_\pi^{\,2}}[\delta_{ad}\delta_{bc}(P-q_1)^2+\delta_{bd}\delta_{ac}(P-q_2)^2+\delta_{cd}\delta_{ab}(P-q_3)^2]$$
$$\times\hat{P}^d\left(\frac{g_A}{f_\pi}\right)^3\left[\left(\bar{u}(p')\not{q}_a i\gamma_5\tfrac{1}{2}\tau_a\frac{\not{p}'-\not{q}_a+M}{(p'-q_a)^2-M^2}\not{q}_b i\gamma_5\tfrac{1}{2}\tau_b\frac{(-\not{\bar{p}}+\not{q}_c+M)}{(\bar{p}-q_c)^2-M^2}\right)\right.$$
$$\left.\times\not{q}_c i\gamma_5\tfrac{1}{2}\tau_c v(\bar{p})+\text{permutations of } abc\right].$$

Carrying out the traces, this expression reduces to

$$\mathrm{Im}K_{\hat{A}\hat{A}\hat{A}}(t)=\frac{gg_A^{\;2}}{16(2\pi)^5f_\pi^{\;4}t}[T_1(t)+T_2(t)],$$

with $T(t)=-2\,\mathrm{Re}F_1(t)$, $T_2=2M^2\mathrm{Re}F_2(t)$,

$$F_1(t)=\int\frac{d^4q_1\theta(q_1^0)\delta(q_1^{\,2})}{(\bar{p}\cdot q_1)(\bar{p}^*\cdot q_1)}\int d^4q_2\theta(q_2^0)\delta(q_2^{\,2})\theta(P_0-q_1^0-q_2^0)$$
$$\times\delta((P-q_1)^2-2q_2\cdot(P-q_1))(-P^2+8P\cdot q_2)[P^2(q_1\cdot q_2)(P\cdot q_1)-4\bar{p}\cdot q_2(\bar{p}^*\cdot q_1)^2],$$

$$F_2(t)=\int d^4q_1\theta(q_1^0)\delta(q_1^2)[-P^2+8(P-q_1)^2]\frac{(P-q_1)^2}{\bar{p}^*\cdot q_1}\int d^4q_2\theta(q_2^0)\delta(q_2^{\,2})\theta(P_0-q_1^0-q_2^0)\delta((P-q_1-q_2)^2)\frac{(P-q_2)^2}{\bar{p}\cdot q_2}.$$

In order to carry out the remaining integration we note that the expressions under $\int d^4q_2$ are Lorentz-invariant. We carry out this integration in the Lorentz frame Γ' in which $\vec{P}'=\vec{q}_1'$, for which the δ function is simple. By explicitly carrying out the Lorentz transformation after integration removing the δ function, we evaluate the result in the barycentric frame in which $\vec{P}=0$ and perform the $\int d^4q_1$ integration. After some manipulation one finds

$$F_1(t)=\tfrac{1}{64}P^5\int_0^1dx(1-x)\int_0^\pi\frac{\sin\theta_1d\theta_1}{p_0^{\,2}+|\bar{p}|^2\cos^2\theta_1}\int_0^{2\pi}d\phi_1\int_0^\pi\sin\theta_2d\theta_2\int_0^{2\pi}d\phi_2[1+2x-2(1-x)\cos\theta_2]$$
$$\times[\tfrac{1}{2}P^3(1-\cos\theta_2)-4(p_0+i|\vec{p}|\cos\theta_1)^2\{P_0\tfrac{1}{8}(1+x)-P_0\tfrac{1}{8}(1-x)\cos\theta_2+i|\vec{p}|\tfrac{1}{4}(1-x)\cos\theta_1$$
$$-i\tfrac{1}{2}|\vec{p}|[\sqrt{x}\sin\theta_1\sin\theta_2\cos\phi_2+\tfrac{1}{2}(1+x)\cos\theta_2\cos\theta_1]\}],$$

$$F_2(t)=\tfrac{1}{8}P^6\int_0^1dx(1-x)(-1+8x)x\int_0^\pi\frac{\sin\theta_1d\theta_1}{P_0+i|\vec{p}|\cos\theta_1}\int_0^\pi\sin\theta_2d\theta_2\int_0^{2\pi}d\phi_1\int_0^{2\pi}d\phi_2$$
$$\times\frac{1+\cos\theta_2}{P_0[1+x-(1-x)\cos\theta_2]+2i|\vec{p}|\{[1-x-(1+x)\cos\theta_2]\cos\theta_1-2\sqrt{x}\,\sin\theta_1\sin\theta_2\cos\phi_2\}},$$

where $P_0=2p_0=\sqrt{P^2}$ and $|\vec{p}|=(M^2-\frac{1}{4}t)^{1/2}$. In the limit $t=P^2\to 0$ these integrals can be done exactly, with the result

$$\mathrm{Re}F_1(t)\underset{t\to 0}{\sim}\tfrac{13}{36}\pi^2t^3,$$

$$2M^2\mathrm{Re}F_2(t)\underset{t\to 0}{\sim}\pi^2t^3(1-\tfrac{17}{315}\pi^2).$$

The result is

$$\operatorname{Im} K_{\hat{A}\hat{A}\hat{A}} \underset{t\to 0}{\sim} \frac{g g_A{}^2 t^2}{9(8\pi)^3 f_\pi{}^4}\left(\tfrac{5}{2} - \tfrac{17}{35}\pi^2\right). \tag{2.10}$$

D. Im$K_{\hat{A}V}$ Term

The unitarity condition for this term reads

$$\operatorname{Im} K_{\hat{A}V} = \frac{g_A}{16(2\pi)^5} \sum_{a,b,c,d,\text{ spins}} \int \frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}\frac{d^3q_3}{2q_3^0}\,\delta^4(P-q_1-q_2-q_3)\hat{P}^d \frac{\bar{u}(p')}{f_\pi{}^3}$$

$$\times\Big[\epsilon^{cbe}(\tfrac{1}{2}\tau^a)i\gamma_5\not{q}_1\frac{(\not{p}'-\not{q}_1+M)}{(p'-q_1)^2-M^2}(\tfrac{1}{2}\tau^e)i\not{q}_2+\epsilon^{cbe}(\tfrac{1}{2}\tau^e)i\not{q}_2\frac{\not{q}_1-\bar{\not{p}}+M}{(q_1-\bar{p})^2-M^2}(\tfrac{1}{2}\tau^a)i\gamma_5\not{q}_1$$

$$+\epsilon^{cae}(\tfrac{1}{2}\tau^b)i\gamma_5\not{q}_2\frac{\not{p}'-\not{q}_2+M}{(p'-q_2)^2-M^2}(\tfrac{1}{2}\tau^e)i\not{q}_1+\epsilon^{cae}(\tfrac{1}{2}\tau^e)i\not{q}_1\frac{\not{q}_2-\bar{\not{p}}+M}{(q_2-\bar{p})^2-M^2}(\tfrac{1}{2}\tau^b)i\gamma_5\not{q}_2\Big]v(\bar{p})$$

$$\times\frac{1}{f_\pi{}^2}\left[\delta_{ad}\delta_{bc}(P-q_1)^2+\delta_{bd}\delta_{ac}(P-q_2)^2+\delta_{cd}\delta_{ab}(P-q_3)^2\right].$$

After taking traces and performing the Lorentz transformation as we did in Sec. IIC it turns out that the angular factor in the denominator cancels one in the numerator. The resulting angular integrations are simple and the result is

$$\operatorname{Im} K_{\hat{A}V}(t) \underset{t\to 0}{\sim} \frac{5 g t^2}{72(8\pi)^3 f_\pi{}^4}. \tag{2.11}$$

E. Im$K(t)$

If we now add the results of Secs. IIA–IID we obtain the final result, a threshold theorem valid in the chiral SU(2)⊗SU(2) limit:

$$\operatorname{Im} K(t) \underset{t\to 0}{\sim} \frac{g t^2}{3(8\pi)^3 f_\pi{}^4}\left[\tfrac{1}{3}g_A{}^2\left(\tfrac{5}{2}-\tfrac{17}{35}\pi^2\right)-\tfrac{7}{8}\right]. \tag{2.12}$$

The corrections to this result are expected to be of order $\mu^2 t$ and μ^4 (which we have dropped in this treatment). They would give rise to terms in Δ of order $\mu^4\ln\mu^2$ and μ^4, respectively, and are expected to be very small. The term we have calculated of order t^2 gives rise to a term in Δ of order μ^2 which from the cutoff dispersion relation gives a contribution which is very small.

III. CONCLUSIONS

So where are the corrections to the Goldberger-Treiman relation? Here we will discuss the three possibilities which in our opinion are the most likely candidates for countering our negative result.

A. Subtraction

If $\partial^\mu A_\mu^a(x)$ is not gentle then $D(t)$ requires a subtraction and we cannot calculate Δ by this method. If this is the case then one loses much of the rationale for the approximate validity of the Goldberger-Treiman relation. Of course this relation is exact in a chiral world, but in the real world for which the symmetry is broken one has no reason to believe that the corrections are small. The possible systematics of a subtraction have been examined by one of us.[14] It should be remarked that the axial-vector form factor seems to be falling like the nucleon form factor for spacelike momentum transfer. If the induced pseudoscalar form factor is likewise falling, then $D(t)$ is probably unsubtracted. If $D(t)$ needs a subtraction, however, then our game is over.

B. Experimental Values of g and g_A

In obtaining our value for

$$\Delta = 1 - \frac{M g_A}{g f_\pi} = 0.081 \pm 0.019$$

we have used the most accurate determinations of g_A and g published to date, $g_A = 1.226 \pm 0.011$, as given by the analysis of β-decay data of Blin-Stoyle and Freeman,[15] and $g^2/4\pi = 14.73 \pm 0.29$, as determined from the value of f^2 given by Samaranayake and Woolcock[16] $[g^2/4\pi = 4(M/\mu_{\pi^+})^2 f^2]$. If we use the slightly different value for the πp coupling constant given by MacGregor *et al.*,[17] $g^2/4\pi = 14.72 \pm 0.83$, then we get $\Delta = 0.081 \pm 0.037$. Should the value of g be too large by 4% or the value of g_A too small by 3–4%, then Δ would be consistent with our theoretical result. However, the analyses of experimental data mentioned above do not seem to leave room for such changes. The value of the other two parameters, the nucleon mass and f_π, are determined very accurately and are unlikely to be a problem. In particular f_π turns out to be, using the value of G_V given by Blin-Stoyle and Freeman,[15] $f_\pi = (0.932\,51 \pm 0.001\,44)\mu_{\pi^+}/\sqrt{2}$.

C. A New State – the Tripion

An exciting possibility is that our consideration of the hadron spectrum is not complete. If there existed a heavy pion[4, 18] with mass $3\mu < \mu_{\pi'} < 2M$ or even a large 3π enhancement it could easily account for the observed value of Δ. No such state has been seen,[19] but it could be looked for in $\pi + p \to \pi' + p \to 3\pi + p$, and $\gamma + p \to \pi' + p \to 3\pi + p$.

---

*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-3505.

†Graduate Fellow. Address after September 1, 1972: Centro de Investigacion del IPN, Apto. Postal 14-740, Mexico 14, D.F.

[1] M. L. Goldberger and S. B. Treiman, Phys. Rev. 111, 354 (1958).

[2] G. Ebel, H. Pilkuhn, and F. Steiner, Nucl. Phys. B17, 1 (1970).

[3] R. Dashen and M. Weinstein, Phys. Rev. 183, 1261 (1969). These authors have especially emphasized the role of symmetry in establishing the validity of the Goldberger-Treiman relation.

[4] H. Pagels, Phys. Rev. 179, 1337 (1969).

[5] With our conventions (1.2) and (1.3) $f_\pi \approx \mu_{\pi^+}/\sqrt{2}$.

[6] H. Pagels, Phys. Rev. D 3, 610 (1971).

[7] Z. Grossman and G. Preparata, Phys. Rev. D 5, 2069 (1972).

[8] A. Sirlin, Phys. Rev. D 5, 436 (1972).

[9] The expression for the $\rho\pi$ contribution to Im$K(t)$ in Ref. 4, Eq. (2.24), is wrong. One of us (H.P.) would like to thank Dr. J. Feigenbaum for pointing this out. The correct expression for vanishing pion mass is $\mathrm{Im}K(t) = (\frac{1}{2}g)(g_{\rho\pi}{}^2/4\pi)[(t - m_\rho{}^2)/t]^2\theta(t - m_\rho{}^2)$. Using this result in (1.14) we obtain $\Delta_\rho = 0.004$, in agreement with the conclusion of Ref. 4.

[10] T. P. Cheng and R. Dashen, Phys. Rev. Letters 26, 594 (1971).

[11] A. G. Höhler, H. P. Jakob, and R. Strauss, Phys. Letters 35B, 445 (1971).

[12] This corrects the remark in L. F. Li and H. Pagels, Phys. Rev. Letters 26, 1204 (1971), where it was asserted that Δ was nonanalytic to leading order. That Δ is analytic in leading order was independently observed by B. W. Lee (private communication).

[13] S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

[14] H. Pagels, Phys. Rev. 182, 1913 (1969).

[15] R. J. Blin-Stoyle and J. M. Freeman, Nucl. Phys. A150, 369 (1970).

[16] V. K. Samaranayake and W. S. Woolcock, in Proceedings of the Fifth International Conference on Elementary Particles, Lund, Sweden, 1969 (unpublished), as reported by G. Ebel *et al.* (Ref. 2). See also V. K. Samaranayake and W. S. Woolcock, Phys. Rev. Letters 15, 936 (1965).

[17] M. H. MacGregor, R. A. Arndt, and R. M. Wright, Phys. Rev. 169, 1128 (1968).

[18] C. Michael, Phys. Rev. 166, 1826 (1968); R. A. Coleman and J. W. Moffat, Phys. Rev. 186, 1635 (1969).

[19] C. Alff-Steinberger *et al.*, Phys. Rev. 145, 1072 (1966); P. R. Klein *et al.*, *ibid.* 150, 1132 (1966); A. W. Key *et al.*, *ibid.* 166, 1430 (1968).