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Hadronic Matter at High Density

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The density of hadron levels $\rho(m)$ has been derived from the statistical bootstrap model to have the asymptotic form $\rho(m) \sim cm^a e^{bm}$ with $a \leq -\frac{5}{2}$. We study the properties of ultradense hadronic matter characterized by level densities of this type. If $a < -\frac{5}{2}$ (as is preferred by the bootstrap model), we show that hadronic matter may be thermodynamically unstable at high densities. This instability is characterized by macroscopic fluctuations of the energy density and the existence of a negative specific heat. Implications of the instability are discussed with reference to high-energy collisions and astrophysics.

I. INTRODUCTION

Dual models,[1,2] the Veneziano model,[3-7] and the statistical bootstrap model[8-10] all indicate that the density of hadron levels, $\rho(m)$, has the asymptotic form

$$\rho(m) \sim c m^a e^{bm}. \tag{1.1}$$

In this paper we will utilize this level density to investigate the properties of hadronic matter at high densities. It is of course a straightforward matter to construct from Eq. (1.1) a partition function $Z(V, T)$ in the canonical ensemble and to derive from the partition function the thermodynamical properties of the system. Thus, for example, the pressure P and mean energy $\langle E\rangle$ are given by the expressions[11,12]

$$P = T\frac{\partial}{\partial V}\ln Z(V, T), \tag{1.2}$$

$$\langle E\rangle = T^2\frac{\partial}{\partial T}\ln Z(V, T). \tag{1.3}$$

This procedure is the one that other authors[13,7] have used in studying the properties of bulk matter characterized by the exponentially rising level density (1.1). We will review the details of this approach in Sec. II.

An important feature of the canonical ensemble for a system with a level density of the type (1.1) is the existence of a singularity in the partition function at a temperature $T_0 = b^{-1}$. For temperatures $T > T_0$, the partition function does not exist, and hence T_0 is said[8] to be the ultimate temperature of the system.

Since the singularity of $Z(V, T)$ at $T = T_0$ is a phenomenon without parallel in conventional statistical mechanics, it would be reassuring to verify any conclusions derived from $Z(V, T)$ by working directly with the density of states in the microcanonical ensemble. This is the principal objective of this paper. We find that the microcanonical ensemble is equivalent to the canonical ensemble only if the parameter a in Eq. (1.1) satisfies the constraint $a \geq -\frac{5}{2}$. In this case T_0 does define the ultimate temperature of the system, T approaching T_0 in the limit of infinitely large energy densities.

If, on the other hand, $a < -\frac{5}{2}$, the system may be characterized by macroscopic fluctuations of the energy density. Equivalence of the microcanonical and canonical ensembles is no longer guaranteed. Temperatures greater than T_0 are permitted in the microcanonical ensemble, and the system as a whole may be unstable, characterized by a negative specific heat.

In the following two sections we shall calculate the properties of matter described by the level density (1.1). The calculation in Sec. II is performed from the viewpoint of the canonical ensemble and that in Sec. III from the viewpoint of the microcanonical ensemble. In Sec. IV the nature of the instability encountered for the case $a < -\frac{5}{2}$ is explained. Finally in Sec. V we mention some implications of our findings for thermodynamical treatments of high energy scattering and of certain astrophysical problems.

II. THE CANONICAL ENSEMBLE

In this section we review the results that other authors[7,8,13] have obtained by constructing the partition function $Z(V, T)$ for a system characterized by the level density (1.1). For simplicity we consider a system of bosons; our results are nonetheless applicable for systems containing fermions if the energy density is large relative to the Fermi level. The partition function for a system of bosons enclosed in a volume V at a temperature T, and characterized by a level density $\rho(m)$, may be written in the form[8]

$$Z(V, T) = \exp\left\{\frac{-V}{(2\pi)^3}\int_{m_0}^{\infty} dm\,\rho(m)\right.$$

$$\times \int d^3p \ln\left[1-\exp\left(-\frac{(p^2+m^2)^{1/2}}{T}\right)\right]\bigg\}, \tag{2.1}$$

where m_0 is the mass of the lightest particle that can be produced. Expanding the logarithm, one can perform the d^3p integration to obtain

$$Z(V, T)=\exp\left[\frac{VT}{(2\pi)^3}\int_{m_0}^{\infty} dm\rho(m) \times\sum_{n=1}^{\infty} n^{-2} m^2 K_2\left(\frac{nm}{T}\right)\right], \tag{2.2}$$

where $K_2(z)$ is a modified Bessel function. The asymptotic expansion of $K_2(z)$ has the form

$$K_2(z)\sim\left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left(1+\frac{15}{8z}+\cdots\right). \tag{2.3}$$

Substituting the level density (1.1) in Eq. (2.2), then, one finds that the $n=1$ term is singular at $T=T_0=b^{-1}$,

$$\ln Z(V, T) = cV\left(\frac{T}{2\pi}\right)^{3/2}\left(\frac{TT_0}{T_0-T}\right)^{a+5/2}\Gamma\left(a+5/2, \frac{m_0(T_0-T)}{TT_0}\right) + \text{less singular terms}. \tag{2.4}$$

Here $\Gamma(x, y)$ denotes an incomplete Γ function. In general the nth term of the sum in Eq. (2.2) would produce singularities at $T=nT_0$. In this paper we will be interested only in the singularity at $T=T_0$. Note that if we were considering a system of fermions, the $n=1$ term in Eq. (2.2) would be unchanged (although terms with $n>1$ would be different). Therefore for a discussion of the singularity of $Z(V, T)$ at $T=T_0$, our restriction to boson systems is really immaterial.

We can now catalog the form of $Z(V, T)$ for $T\simeq T_0$ and various values of the parameter a,

$a\neq -\frac{5}{2}-n,\ n=0, 1, 2, \ldots,$

$$\ln Z(V, T)\simeq\frac{cV\Gamma(a+5/2)}{(2\pi)^{3/2}} T_0^{2a+13/2}(T_0-T)^{-a-5/2}, \tag{2.5}$$

$a=-\frac{5}{2}$,

$$\ln Z(V, T)\simeq cV\left(\frac{T_0}{2\pi}\right)^{3/2}\ln\left[\frac{T_0^2}{m_0(T_0-T)}\right], \tag{2.6}$$

$a=-\frac{7}{2}$,

$$\ln Z(V, T)\simeq\frac{cV}{(2\pi)^{3/2}T_0^{1/2}}(T-T_0)\ln\left[\frac{T_0^2}{m_0(T_0-T)}\right]. \tag{2.7}$$

In Eqs. (2.5)–(2.7) only the terms most singular at $T=T_0$ have been retained.

With these expressions for $\ln Z(V, T)$, it is a straightforward matter to calculate the behavior of the pressure P and mean energy $\langle E\rangle$ as T approaches the limiting value T_0. Equations (1.2) and (1.3) yield the following results for various values of a:

$a>-\frac{5}{2}$,

$$P\simeq\frac{c\Gamma(a+5/2)}{(2\pi)^{3/2}}T_0^{2a+15/2}(T_0-T)^{-a-5/2}, \tag{2.8}$$

$$\langle E\rangle\simeq\frac{cV\Gamma(a+7/2)}{(2\pi)^{3/2}}T_0^{2a+17/2}(T_0-T)^{-a-7/2}; \tag{2.9}$$

$a=-\frac{5}{2}$,

$$P\simeq c\frac{T_0^{5/2}}{(2\pi)^{3/2}}\ln\left[\frac{T_0^2}{m_0(T_0-T)}\right], \tag{2.10}$$

$$\langle E\rangle\simeq\frac{cT_0^{7/2}V}{(2\pi)^{3/2}(T_0-T)}; \tag{2.11}$$

$-\frac{5}{2}>a>-\frac{7}{2}$,

$$P\simeq \text{constant}, \tag{2.12}$$

$$\langle E\rangle\simeq\frac{cV\Gamma(a+7/2)}{(2\pi)^{3/2}}T_0^{2a+17/2}(T_0-T)^{-a-7/2}; \tag{2.13}$$

$a=-\frac{7}{2}$,

$$P\simeq \text{constant}, \tag{2.14}$$

$$\langle E\rangle\simeq cV\left(\frac{T_0}{2\pi}\right)^{3/2}\ln\left[\frac{T_0^2}{m_0(T_0-T)}\right]; \tag{2.15}$$

$a<-\frac{7}{2}$,

$$P\simeq \text{constant}, \tag{2.16}$$

$$\langle E\rangle\simeq \text{constant}\times V. \tag{2.17}$$

For $a\geq -\frac{7}{2}$, then, T_0 appears to define an ultimate temperature, with T approaching T_0 as the energy density $\langle E\rangle/V$ becomes infinitely large. For $a<-\frac{7}{2}$ the situation seems to be quite different. According to Eq. (2.17) the temperature T_0 is reached at a finite energy density E_0/V_0. Evidently systems with $a<-\frac{7}{2}$ and $\langle E\rangle/V$ above E_0/V_0 are thermodynamically unstable. The precise nature of this instability is, however, unclear in the present approach.

A clearer picture will be provided in the next section, where we study systems of the type (1.1) in the microcanonical ensemble. We find that for $a<-\frac{5}{2}$ instabilities may occur. The system is then characterized by a large internal inhomogeneity or, equivalently, by large fluctuations of the internal energy density.

III. THE MICROCANONICAL ENSEMBLE

We have seen that for matter described by the level density (1.1), the canonical ensemble is inadequate for the treatment of very dense systems – at least for $a< -\frac{7}{2}$. Thus it is necessary to study these systems by a direct construction of the microcanonical ensemble. This approach is straightforward enough, but the actual calculations are clumsy and inelegant by comparison with the construction of the canonical ensemble in Sec. II. For a system of energy E enclosed in a volume V the fundamental quantity we will need is the phase space density $\sigma(E, V)$. Thermodynamic properties of the system are all specified by $\sigma(E, V)$. In particular the temperature T and pressure P are given by the expressions[11]

$$T=\left[\frac{\partial}{\partial E}\ln\sigma(E, V)\right]^{-1} \tag{3.1}$$

and

$$P=T\frac{\partial}{\partial V}\ln\sigma(E, V)\,. \tag{3.2}$$

Heuristically, it is easy to understand why $\sigma(E, V)$ is an important quantity. A fundamental tenet of statistical mechanics is that all substates of a system have equal probability to be occupied. Hence the most probable state of a system is simply that state which occupies the largest phase volume. The thermodynamic properties calculated from $\sigma(E, V)$ correspond to the properties of the most probable state by virtue of the fact that this state provides the dominant contributions to $\sigma(E, V)$.

In constructing the partition function for a system of matter described by the level density $\rho(m)$, we were able to start from the simple expression (2.1). There is no simple analog for $\sigma(E, V)$. The density of states for n identical bosons of total energy E enclosed in a volume V is given by the rather complicated equation

$$\sigma(E, V)=\frac{1}{n!}\left[\frac{V}{(2\pi)^3}\right]^n\prod_{i=1}^{n}\int d^3p_i\delta\left(\sum_{i=1}^{n}E_i-E\right), \tag{3.3}$$

where $E_i=(p_i^2+m^2)^{1/2}$ and m is the boson mass. Equation (3.3) is simply the product of n integrals[12] $h^{-3}d^3x_i d^3p_i$, constrained by energy conservation and corrected for double counting by the factor $1/n!$.

For a system of bosons described by a level density $\rho(m)$, the phase-space density[9,14] is slightly more complex than Eq. (3.3):

$$\sigma(E, V)=\sum_{n=1}^{\infty}\left[\frac{V}{(2\pi)^3}\right]^n\frac{1}{n!}\prod_{i=1}^{n}\int_{m_0}^{\infty}dm_i\rho(m_i)\times\int d^3p_i\delta\left(\sum_{i=1}^{n}E_i-E\right). \tag{3.4}$$

This is a sum of n-particle terms of the form (3.3) generalized to permit each of the n particles to have masses in the range $m_0\leq m_i<\infty$. The most probable number of particles in the system corresponds to the value of n for which the summand of Eq. (3.4) is maximal.

Let us now proceed to calculate $\sigma(E, V)$ for the level density (1.1). Our procedure follows closely that developed by Frautschi[9,15] in his derivation of solutions of the statistical bootstrap equation. The reader may consult Sec. II of Frautschi's paper[9] for details we omit. There is a slight difference in our equations in that Frautschi neglects the contribution of 1-particle states to $\sigma(E, V)$. This omission was appropriate in formulating the bootstrap constraint but could not be justified in the present discussion.

A second difference lies in the fact that Frautschi imposes over-all momentum conservation by means of a factor $\delta^3(\sum_{i=1}^{n}p_i)$ in Eq. (3.4). The effects of this factor will be seen to be insignificant for the large energy densities in which we are interested.

With the level density (1.1), Eq. (3.4) reads

$$\sigma(E, V)=\sum_{n=1}^{\infty}\left[\frac{V}{(2\pi)^3}\right]^n\frac{1}{n!}\prod_{i=1}^{n}\int_{m_0}^{\infty}dm_i\,cm_i^{a}\times\int d^3p_i\,e^{bm_i}\,\delta\left(\sum_{i=1}^{n}E_i-E\right). \tag{3.5}$$

Following Frautschi,[9] rewrite the energy E_i of the ith particle in terms of its kinetic energy Q_i:

$$E_i=m_i+Q_i\,. \tag{3.6}$$

The factors e^{bm_i} in Eq. (3.5) may thus be written in the form

$$\prod_{i=1}^{n}e^{bm_i}=e^{bE}\prod_{i=1}^{n}e^{-bQ_i}\,. \tag{3.7}$$

The contributions of states with large kinetic energy to the integrals d^3p_i are damped by the factors e^{-bQ_i}. The Q_i are essentially limited to $Q_i<b^{-1}$. Hence particles with masses $m_i\gg b^{-1}$ are nonrelativistic, and their momenta

$$p_i\simeq(2m_iQ_i)^{1/2} \tag{3.8}$$

are limited to $p_i<(m_i/b)^{1/2}$. More specifically, one can estimate the ith momentum and mass integrals in Eq. (3.5):

$$I_i(\Lambda_i)=c\int_{m_0}^{\Lambda_i}dm_i m_i^{a}\int d^3p_i e^{-bQ_i}\simeq c\left(\frac{2\pi}{b}\right)^{3/2}\int_{m_0}^{\Lambda_i}dm_i m_i^{a+3/2}\,. \tag{3.9}$$

The cutoff Λ_i on the mass integral is imposed by the constraint of energy conservation. It is clear that one can distinguish three types of behavior for $I_i(\Lambda_i)$ depending on the value of a:

$$a>-5/2\,,\quad I_i(\Lambda_i)\simeq c\left(\frac{2\pi}{b}\right)^{3/2}\frac{\Lambda_i^{a+5/2}}{a+5/2}, \tag{3.10}$$

$$a=-5/2\,,\quad I_i(\Lambda_i)\simeq c\left(\frac{2\pi}{b}\right)^{3/2}\ln(\Lambda_i/m_0)\,, \tag{3.11}$$

$$a<-5/2\,,\quad I_i(\Lambda_i)\simeq c\left(\frac{2\pi}{b}\right)^{3/2}\frac{m_0^{a+5/2}-\Lambda_i^{a+5/2}}{-a-5/2}. \tag{3.12}$$

These three cases must be treated separately.

A. $a>-\frac{5}{2}$

According to Eqs. (3.9) and (3.10), the dominant part of the mass integral dm_i is contributed by states with large mass (unless these states are forbidden by the energy-conservation constraint). The n-particle contribution to $\sigma(E, V)$, denoted by $\sigma^n(E, V)$, has then the following approximate form:

$$\sigma^n(E,V)\simeq\frac{e^{bE}}{n!}\left[\frac{V}{(2\pi)^3}c\left(\frac{2\pi}{b}\right)^{3/2}\right]^n \times\prod_{i=1}^{n}\int_{m_0}^{\infty}dm_i m_i^{a+3/2}\delta\left(\sum_{i=1}^{n}m_i-E\right). \tag{3.13}$$

Define the quantity

$$\pi^n(E)=\left[c\left(\frac{2\pi}{b}\right)^{3/2}\right]^n \times\int_{nm_0}^{E}dx\prod_{i=1}^{n}\int_{m_0}^{\infty}dm_i m_i^{a+3/2}\delta\left(\sum_{i=1}^{n}m_i-x\right), \tag{3.14}$$

in terms of which $\sigma^n(E, V)$ may be written

$$\sigma^n(E,V)\simeq\frac{e^{bE}}{n!}\left[\frac{V}{(2\pi)^3}\right]^n\frac{d\pi^n(E)}{dE}. \tag{3.15}$$

With the help of Eq. (3.9), $\pi^n(E)$ may be written in the form

$$\pi^n(E)=\prod_{i=1}^{n}I_i(\Lambda_i)\,, \tag{3.16}$$

subject to the constraint

$$\sum_{i=1}^{n}\Lambda_i=E\,. \tag{3.17}$$

The maximum contribution to $\pi^n(E)$ is obtained when the Λ_i are all of order E/n. This provides the estimate

$$\pi^n(E)\simeq\left[\frac{c}{a+5/2}\left(\frac{2\pi}{b}\right)^{3/2}\left(\frac{E}{n}\right)^{a+5/2}\right]^n. \tag{3.18}$$

Note that the quantity $\pi^n(E)$ receives contributions from all n-particle states with energy less than or equal to E. Because each integral $I_i(\Lambda_i)$ is dominated by states with masses of order Λ_i, $\pi^n(E)$ is in fact dominated by states with energy of order E. Hence Eq. (3.18) approximates the contribution of these states quite well, and we can safely apply Eq. (3.15) to obtain an approximation for $\sigma^n(E, V)$:

$$\sigma^n(E,V)\simeq\frac{(a+5/2)e^{bE}}{E(n-1)!}\left[\frac{cV}{(a+5/2)(2\pi b)^{3/2}}\left(\frac{E}{n}\right)^{a+5/2}\right]^n. \tag{3.19}$$

The most probable number of particles is found by maximizing $\sigma^n(E, V)$ with respect to n. Following Frautschi[9] we write $\sigma^n(E, V)$ in the form

$$\sigma^n(E,V)=\frac{(a+5/2)}{E}e^{bE}e^{f(n,E,V)}. \tag{3.20}$$

By Stirling's approximation $f(n,E,V)$ may be written

$$f(n,E,V)\simeq-(n-1)\ln n+n+n\ln\left[\frac{cVE^{a+5/2}}{(a+5/2)(2\pi b)^{3/2}}\right]-(a+5/2)n\ln n. \tag{3.21}$$

The maximum of $\sigma^n(E, V)$ corresponds to the maximum of $f(n,E,V)$, i.e.,

$$n=N\simeq\left[\frac{cVe^{-a-5/2}E^{a+5/2}}{(a+5/2)(2\pi b)^{3/2}}\right]^{1/(a+7/2)}. \tag{3.22}$$

Note that for large E, N grows only like $E^{(a+5/2)/(a+7/2)}$. Therefore the mean energy per particle and hence the average mass are large when the energy density is sufficiently large. This justifies the approximations used to obtain Eq. (3.19). From Eqs. (3.19)–(3.22) one obtains the estimate

$$\sigma(E,V)\simeq\sigma^N(E,V)\simeq\frac{(a+5/2)}{E}e^{bE}e^{f(N,E,V)}, \tag{3.23}$$

where

$$f(N,E,V)\simeq(a+7/2)\left[\frac{cVe^{-a-5/2}E^{a+5/2}}{(a+5/2)(2\pi b)^{3/2}}\right]^{1/(a+7/2)}. \tag{3.24}$$

If in Eq. (3.4) we had imposed over-all momentum conservation, then there would have been one fewer d^3p_i integrations in each of the $\sigma^n(E, V)$, and $\sigma(E, V)$ would have picked up an additional factor $(2\pi b)^{3/2}V^{-1}E^{-3/2}$. Such a factor would have negligible effect on our calculations of the temperature and pressure given below.

The thermodynamic properties of the system (3.23) are extracted by application of Eqs. (3.1) and (3.2). Using Eq. (3.1) we find that for large volumes and a large energy density,

$$T^{-1} \simeq b + (a+5/2)\left[\frac{ce^{-a-5/2}V/E}{(a+5/2)(2\pi b)^{3/2}}\right]^{1/(a+7/2)}, \tag{3.25}$$

or

$$\frac{E}{V} = \frac{ce^{-a-5/2}}{(a+5/2)(2\pi b)^{3/2}}\left[\frac{(a+5/2)TT_0}{T_0 - T}\right]^{a+7/2}$$
$$\simeq \frac{c}{(2\pi)^{3/2}}\left(\frac{a+5/2}{e}\right)^{a+5/2} T_0^{2a+17/2}(T_0 - T)^{-a-7/2}. \tag{3.26}$$

Similarly, Eq. (3.2) defines the pressure,

$$P = T\left[\frac{ce^{-a-5/2}}{(a+5/2)(2\pi b)^{3/2}}\right]^{1/(a+7/2)}\left(\frac{E}{V}\right)^{(a+5/2)/(a+7/2)}. \tag{3.27}$$

With the help of Eq. (3.26), this may be written in the form

$$P = T\left[\frac{(a+5/2)TT_0}{T_0 - T}\right]^{a+5/2}\frac{ce^{-a-5/2}}{(a+5/2)(2\pi b)^{3/2}}$$
$$\simeq \frac{c}{(2\pi b)^{3/2}}\left(\frac{a+5/2}{e}\right)^{a+5/2} T_0^{2a+15/2}(T_0 - T)^{-a-5/2}. \tag{3.28}$$

Equations (3.28) and (3.26) are to be compared with the corresponding equations (2.8) and (2.9) derived from the canonical ensemble. Except for slight differences in the over-all coefficients, the expressions from the microcanonical ensemble agree with those from the canonical ensemble. This discrepancy is due to the roughness of our estimates of the various integrals in Eq. (3.13), and we may therefore conclude that for $a > -\frac{5}{2}$ the canonical and microcanonical ensembles are equivalent.

B. $a = -\frac{5}{2}$

For $a = -\frac{5}{2}$, Eq. (3.11) shows that $I_i(\Lambda_i)$ is again dominated by high mass states. As in the previous case, therefore, $\pi^n(E)$ receives its principal contributions for $\Lambda_i \simeq E/n$. This provides the estimate

$$\pi^n(E) \simeq \left[c\left(\frac{2\pi}{b}\right)^{3/2}\ln\left(\frac{E}{nm_0}\right)\right]^n. \tag{3.29}$$

Once again $\pi^n(E)$ is dominated by states with energy of order E, and we can apply Eq. (3.15) to obtain an approximation for $\sigma^n(E, V)$:

$$\sigma^n(E, V) \simeq \frac{cm_0 Ve^{bE}}{(2\pi b)^{3/2}E(n-1)!}\left[\frac{cV\ln(E/nm_0)}{(2\pi b)^{3/2}}\right]^{n-1}. \tag{3.30}$$

This expression is maximal for

$$n = N \simeq \frac{cV}{(2\pi b)^{3/2}}\ln\left(\frac{E}{Nm_0}\right). \tag{3.31}$$

The sum on n of $\sigma^n(E, V)$ may be performed approximately by replacing n in the argument of the logarithm in Eq. (3.30) by N. This replacement is justified by the fact that $\ln n$ is very slowly varying relative to $n!$. Summing $\sigma^n(E, V)$, then, we obtain the approximate result

$$\sigma(E, V) = \sum_{n=1}^{\infty}\sigma^n(E, V)$$
$$\simeq \frac{cm_0 Ve^{bE}}{(2\pi b)^{3/2}E}\exp\left[\frac{cV}{(2\pi b)^{3/2}}\ln\left(\frac{E}{Nm_0}\right)\right]$$
$$= \frac{cm_0 V}{(2\pi b)^{3/2}E}e^{bE}\left(\frac{E}{Nm_0}\right)^{cV/(2\pi b)^{3/2}}. \tag{3.32}$$

From Eq. (3.31) we see that the number density N/V is a function only of the energy density E/V:

$$\frac{N}{V} = \frac{c}{(2\pi b)^{3/2}}\ln\left(\frac{E/V}{m_0 N/V}\right). \tag{3.33}$$

In the limit of large energy density, the number density is also large, but N/V grows only as the logarithm of E/V. If we apply Eq. (3.1) to the phase space density (3.32), we find that

$$\frac{1}{T} \simeq b + \frac{1}{E}\left[-1 + \frac{cV}{(2\pi b)^{3/2}}\right]. \tag{3.34}$$

Terms involving $\partial N/\partial E$ are negligible by virtue of the slow growth of N relative to E. For comparison with Eq. (2.11), Eq. (3.34) may be rewritten in the approximate form

$$\frac{E}{V} \simeq \frac{cT_0^{7/2}}{(2\pi)^{3/2}(T_0 - T)}. \tag{3.35}$$

If we had imposed over-all momentum conservation on Eq. (3.4), then $\sigma(E, V)$ would acquire an extra factor $(2\pi b)^{3/2}V^{-1}E^{-3/2}$. This would change the -1 term in Eq. (3.34) to $-\frac{5}{2}$. This term, however, is negligible for the volumes in which we are interested.

According to Eq. (3.2) the pressure is given by the expression

$$P \simeq \frac{cT}{(2\pi b)^{3/2}}\left[\ln\left(\frac{E}{Nm_0}\right) - \frac{V}{N}\frac{dN}{dV}\right]. \tag{3.36}$$

Differentiating Eq. (3.31) with respect to V, we find that

$$\frac{\partial N}{\partial V} = \ln\left(\frac{E}{Nm_0}\right)\left[\frac{(2\pi b)^{3/2}}{c} + \frac{V}{N}\right]^{-1}. \tag{3.37}$$

Thus in the limit of large energy density [when V/N becomes negligible in Eq. (3.37)], $\partial N/\partial V$ grows like $\ln(E/Nm_0)$. Since in this limit V/N falls like $1/\ln(E/Nm_0)$, the term $(V/N)\partial N/\partial V$ may be neglected in Eq. (3.36). Rewriting this equation with the aid of Eqs. (3.35) and (3.33) we find

$$P \simeq \frac{cT_0^{5/2}}{(2\pi)^{3/2}}\ln\left[\frac{T_0^2}{m_0(T_0 - T)}\right], \tag{3.38}$$

where terms of order $\ln[\ln(E/Nm_0)]$ have been dropped. Equations (3.38) and (3.35) may now be compared with Eqs. (2.10) and (2.11). They are completely identical. Thus we conclude that the canonical and microcanonical ensembles are equivalent in the case $a=-\frac{5}{2}$.

C. $a<-\frac{5}{2}$

The state of affairs when $a<-\frac{5}{2}$ is considerably different from the cases previously considered. According to Eq. (3.12) the integral $I_i(\Lambda_i)$ receives its dominant contribution from the low mass region. Therefore, important contributions to $\sigma^n(E,V)$ in Eq. (3.13) can come from configurations in which $n-1$ particles have small masses – with a mean mass $\bar{m}$ – and the nth particle has a mass fixed by energy conservation to be of order $E-(n-1)\bar{m}$. Such configurations will be the dominant ones when E becomes sufficiently large,

$$E \gg n\bar{m}(E). \tag{3.39}$$

The mean mass $\bar{m}(\Lambda_i)$ in the integral $I_i(\Lambda_i)$ is given by the expression

$$\bar{m}(\Lambda_i)=\int_{m_0}^{\Lambda_i} dm\, m^{a+5/2} \Big/ \int_{m_0}^{\Lambda_i} dm\, m^{a+3/2}. \tag{3.40}$$

For various values of a, this yields the following values for $\bar{m}(\Lambda_i)$:

$$-\tfrac{5}{2}>a>-\tfrac{7}{2}, \quad \bar{m}(\Lambda_i)\simeq\frac{-a-5/2}{a+7/2}\, m_0^{-a-5/2}\Lambda_i^{a+7/2}, \tag{3.41}$$

$$a=-\tfrac{7}{2}, \qquad \bar{m}(\Lambda_i)\simeq m_0\ln(\Lambda_i/m_0), \tag{3.42}$$

$$a<-\tfrac{7}{2}, \qquad \bar{m}(\Lambda_i)\simeq\frac{a+5/2}{a+7/2}m_0. \tag{3.43}$$

In deriving Eq. (3.9) we assumed that all the particles are nonrelativistic – including those of low mass. This assumption is valid only if $m_0 \gg b^{-1}$.[16] Thus the numerical coefficients in Eqs. (3.41)–(3.43) may not be properly estimated. The Λ_i dependence, upon which our results primarily depend, is certainly valid, so we will ignore possible errors in the coefficients.

If the condition (3.39) is *not* met, one can impose energy conservation in Eq. (3.16) in the approximate form

$$\bar{m}(\Lambda_i)=E/n. \tag{3.44}$$

This provides the estimate

$$\pi^n(E)=[I_i(\Lambda)]^n, \tag{3.45}$$

with Λ_i given by Eqs. (3.41)–(3.44). In the case $-\frac{5}{2}>a>-\frac{7}{2}$, Λ_i is found to be

$$\Lambda_i=\left[\left(\frac{a+7/2}{-a-5/2}\right)m_0^{a+5/2}\frac{E}{n}\right]^{1/(a+7/2)}. \tag{3.46}$$

Thus, using Eqs. (3.12) and (3.15), we obtain the estimate

$$\sigma^n(E,V)\simeq\frac{e^{bE}}{n!}\left[\frac{cVm_0^{a+5/2}}{(-a-5/2)(2\pi b)^{3/2}}\right]^n \times\left[\left(\frac{a+7/2}{-a-5/2}\right)\frac{m_0^{a+5/2}}{n}\right]^{-1/(a+7/2)} \times[1-(\Lambda_i/m_0)^{a+5/2}]^{n-1}. \tag{3.47}$$

This expression is maximal for

$$n=N\simeq\frac{cm_0^{a+5/2}V}{(-a-5/2)(2\pi b)^{3/2}}. \tag{3.48}$$

Recall that the present discussion is valid only when condition (3.39) is *not* met, i.e., when

$$E^{-a-5/2}<\frac{cV}{(a+7/2)(2\pi b)^{3/2}} \quad (-5/2>a>-7/2). \tag{3.49}$$

For any fixed energy *density* Eq. (3.49) can always be satisfied if V is sufficiently large. In particular, for any macroscopic volume, Eqs. (3.47) and and (3.48) should be reliable. Estimating $\sigma(E,V)\simeq\sigma^N(E,V)$ and applying Eq. (3.1), one finds that

$$\frac{1}{T}\simeq b+\left[\frac{cV/E}{(a+7/2)(2\pi b)^{3/2}}\right]^{1/(a+7/2)}, \tag{3.50}$$

or

$$\frac{E}{V}\simeq\frac{c}{(2\pi)^{3/2}(a+7/2)}T_0^{2a+17/2}(T_0-T)^{-a-7/2}. \tag{3.51}$$

Similarly, applying Eq. (3.2) one obtains the result

$$P\simeq\frac{cm_0^{a+5/2}T_0^{5/2}}{(-a-5/2)(2\pi)^{3/2}}. \tag{3.52}$$

Except for slight discrepancies in the numerical coefficients the expressions (3.50) and (3.51) agree with those calculated in the canonical ensemble, Eqs. (2.13) and (2.12).

Next consider the case $a=-\frac{7}{2}$ and suppose that condition (3.39) is still *not* met. From Eqs. (3.42) and (3.44) the cutoffs Λ_i are of order

$$\Lambda_i\simeq m_0e^{E/nm_0}. \tag{3.53}$$

Using Eqs. (3.45), (3.12), and (3.15), then, we obtain the estimate

$$\sigma^n(E,V)\simeq\frac{e^{bE}e^{-E/nm_0}}{m_0n!}\left[\frac{cV}{m_0(2\pi b)^{3/2}}\right]^n \times[1-e^{-E/nm_0}]^{n-1}. \tag{3.54}$$

This is maximal for $n=N$ as given by Eq. (3.48). Note that the present discussion is valid as long as

$$\frac{E}{\ln(E/m_0)} < \frac{cV}{(2\pi b)^{3/2}} \quad (a=-7/2), \tag{3.55}$$

and in particular for any system of macroscopic volume. Taking $\sigma(E, V) \simeq \sigma^N(E, V)$ and applying Eqs. (3.1) and (3.2) we obtain the expressions

$$\frac{1}{T} \simeq b + m_0^{-1} \exp\left[\frac{-E(2\pi b)^{3/2}}{cV}\right] \tag{3.56}$$

or

$$\frac{E}{V} \simeq \frac{c}{(2\pi b)^{3/2}} \ln\left[\frac{1}{m_0(T^{-1} - b)}\right], \tag{3.57}$$

and

$$P = \frac{cT_0^{5/2}}{(2\pi)^{3/2} m_0}. \tag{3.58}$$

These expressions for P and E are seen to agree with those calculated in the canonical ensemble, Eqs. (2.14) and (2.15). Thus we have shown that if $-\frac{5}{2} > a \geq -\frac{7}{2}$, the microcanonical and canonical viewpoints are equivalent – provided that the volume of the system under consideration is sufficiently large.

Let us now return to the case where Eq. (3.39) is valid. As described earlier, the dominant contributions to $\sigma^n(E, V)$ arise in this case from configurations containing $n-1$ light particles of mean mass $\bar{m}(E)$ and a single heavy particle with mass of order $E - (n-1)\bar{m}(E)$. The contribution of such configurations may be estimated[9] to be

$$\sigma^n(E, V) \simeq \frac{cVE^{a+3/2}e^{bE}}{(2\pi b)^{3/2}(n-1)!}\left[\frac{cVm_0^{a+5/2}}{(-a-5/2)(2\pi b)^{3/2}}\right]^{n-1}. \tag{3.59}$$

The $n-1$ integrals over low-mass states have been estimated retaining only the dominant term $m_0^{a+5/2}$ in Eq. (3.12). The factor $n/n! = 1/(n-1)!$ arises from the fact that the high-mass particle may be associated with any of the n mass integrals. The mass of the heavy particle has been approximated by E using Eq. (3.39).

The most probable number of particles N is found by maximizing $\sigma^n(E, V)$ with respect to n. The result coincides with Eq. (3.48). The constraint (3.39) may thus be rephrased in the following forms for various values of a less than $-\frac{5}{2}$:

$$-5/2 > a > -7/2, \quad E^{-a-5/2} \gg \frac{cV}{(a+7/2)(2\pi b)^{3/2}}, \tag{3.60}$$

$$a = -7/2, \qquad \frac{E}{\ln(E/m_0)} \gg \frac{cV}{(2\pi b)^{3/2}}, \tag{3.61}$$

$$a < -7/2, \qquad E \gg \frac{cm_0^{a+7/2}}{(-a-5/2)(2\pi b)^{3/2}}. \tag{3.62}$$

For any fixed volume V the inequalities (3.60) and (3.61) can be met if E is taken to be sufficiently large. On the other hand, the larger V is chosen to be, the larger the *energy density* E/V will have to be for the inequality to hold. Hence for macroscopic volumes the inequalities (3.60) and (3.61) are virtually impossible to satisfy. In contrast the inequality (3.62) for the case $a < -\frac{7}{2}$ is a function only of the energy density E/V. If E/V is sufficiently large, Eq. (3.62) will be satisfied regardless of how large V may be.

One can perform the sum over n of $\sigma^n(E, V)$ as given by Eq. (3.59) to obtain

$$\sigma(E, V) = \sum_{n=1}^{\infty} \sigma^n(E, V) \simeq \frac{cVE^{a+3/2}e^{bE}}{(2\pi b)^{3/2}} \exp\left[\frac{cm_0^{a+5/2}V}{(-a-5/2)(2\pi b)^{3/2}}\right]. \tag{3.63}$$

Equations (3.1) and (3.2) may now be applied to extract the temperature and pressure of the system. One obtains the following expressions:

$$T = \left(b + \frac{a+3/2}{E}\right)^{-1} \tag{3.64}$$

or

$$E \simeq (a+3/2)\frac{T_0^2}{T_0 - T} \tag{3.65}$$

and

$$P \simeq \frac{cm_0^{a+5/2}T_0^{5/2}}{(-a-5/2)(2\pi)^{3/2}}. \tag{3.66}$$

Had we constrained the total momentum $\sum \vec{p}_i$ to vanish in Eq. (3.4), the density of states (3.63) would have acquired an additional factor $(2\pi b)^{3/2}V^{-1}E^{-3/2}$. The only effect of this factor would be to change the coefficient at $a + \frac{3}{2}$ in Eq. (3.65) to a. This difference will be insignificant in our discussion of this equation; the crucial feature of Eq. (3.65) is the negative sign of the coefficient and not its magnitude.

If now we compare Eqs. (3.66) and (3.65) with Eqs. (2.12)–(2.17), we see that the expressions for the pressure are compatible, but those for the energy are *not*. Hence the canonical and microcanonical ensembles are *not* equivalent when any of the inequalities (3.60)–(3.62) are valid. In the next section we will explain this discrepancy in detail.

IV. A SYSTEM WITH NEGATIVE SPECIFIC HEAT

We have shown in the previous sections that if $a < -\frac{5}{2}$ and any of the inequalities (3.60)–(3.62) are met, then the canonical and microcanonical ensembles for a system with the level density (1.1) are inequivalent. In general, equivalence of the canonical and microcanonical descriptions is obtained whenever fluctuations of the internal energy density are small. In the canonical ensemble these fluctuations are measured by the quantity

$$\langle E^2\rangle - \langle E\rangle^2 = T^4 \frac{\partial^2}{\partial T^2}\ln Z(V,T) + 2T^3\frac{\partial}{\partial T}\ln Z(V,T)\,. \tag{4.1}$$

Using Eqs. (2.5) and (2.7) we can calculate the energy fluctuations for the cases $-\frac{5}{2} > a > \frac{7}{2}$,

$$\langle E^2\rangle - \langle E\rangle^2 \simeq \frac{cV\Gamma(a+9/2)}{(2\pi)^{3/2}} T_0^{\,2a+21/2}(T_0 - T)^{-a-9/2} \tag{4.2}$$

and $a = -\frac{7}{2}$,

$$\langle E^2\rangle - \langle E\rangle^2 \simeq \frac{cV}{(2\pi)^{3/2}T_0^{\,1/2}}(T_0 - T)^{-1}\,. \tag{4.3}$$

The canonical and microcanonical descriptions will be inequivalent if the fluctuations are large, i.e., if

$$\frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2} \gg 1\,. \tag{4.4}$$

Using Eqs. (2.13) and (2.15) then, we can determine the conditions under which fluctuations are large:

$-5/2 > a > -7/2$,

$$\langle E\rangle^{-a-5/2} \gg \frac{c\Gamma(a+7/2)V}{(2\pi b)^{3/2}(a+9/2)^{a+7/2}}\,, \tag{4.5}$$

$a = -7/2$,

$$\langle E\rangle \Big/ \ln\left[\frac{(2\pi b)^{3/2}\langle E\rangle^2}{cVm_0}\right] \gg \frac{cV}{(2\pi b)^{3/2}}\,. \tag{4.6}$$

The conditions (4.5) and (4.6) are essentially the same as conditions (3.60) and (3.61), respectively.

The origin of large energy fluctuations when one of conditions (3.60)–(3.62) is met can be understood upon a detailed examination of the most probable states in the microcanonical ensemble. If $a \geq -\frac{5}{2}$ or if conditions (3.49) or (3.55) are met, we showed that for any n-particle state, the most probable distribution of masses has $m_i \simeq E/n$, $i = 1, 2, \ldots, n$. This distribution is clearly uniform. Suppose we were to divide an n-particle system of energy E and volume V into two subsystems, of volume $V/2$ each. The most probable distribution of particles between the two subsystems is such that each subsystem should have $n/2$ particles and a total energy of $E/2$. The most probable distribution of masses is still $m_i \simeq E/n$. In other words, one can say that the subsystems have the same thermodynamic properties as the whole system. This feature is essential for the construction of a canonical ensemble.

For the case $a < -\frac{5}{2}$, when one of conditions (3.60)–(3.62) is met, the system as a whole lacks this uniformity. The most probable configuration of n particles with total energy $E \gg n\overline{m}(E)$ is one in which there are $n-1$ particles of mass $m_i \simeq \overline{m}(E)$, $i = 1, 2, \ldots, n-1$, and one heavy particle with mass $m_n \simeq E - (n-1)\overline{m}(E)$. This is obviously an extremely nonuniform configuration, and the thermodynamic properties of various subsystems are inequivalent with the properties of the system as a whole.

In the statistical bootstrap model, all particles are assumed to occupy a common volume V_0. The bootstrap constraint relates this volume to the parameters a, b, c, and m_0 by the approximate expression[9, 20]:

$$V_0 = \frac{(-a - 5/2)(2\pi b)^{3/2}}{cm_0^{\,a+5/2}}\ln 2\,. \tag{4.7}$$

Therefore for $a < -\frac{5}{2}$, N may be written as

$$N \simeq (\ln 2)V/V_0\,, \tag{4.8}$$

and the "high-density" requirement (3.39) has the form

$$\frac{E}{V} \gg \frac{\overline{m}}{V_0}\ln 2\,. \tag{4.9}$$

The most probable high-density configuration for $a < -\frac{5}{2}$ can thus be described as follows: The volume V is filled with N particles. The particles are packed together so that they are almost touching. All but one of the particles are light, with masses of order $\overline{m}(E)$. There is a single heavy particle with mass of order $E - (n-1)\overline{m}(E)$.

Suppose now that we fill the volume V with n particles of mass E/n. The system will evolve away from this uniform distribution toward a nonuniform distribution in which most of the energy is carried by a single massive particle. Subsequently the system will fluctuate among states of this type. These fluctuations involve large changes in the local energy density which destroy the equivalence of the canonical and microcanonical ensembles.

In a sense, the behavior of a system with $a < -\frac{5}{2}$, when the energy density is raised above the critical value $\overline{m}(E)/V_0$, can be described as a sort of condensation. This behavior is easier to understand if we focus attention on the case $a < -\frac{7}{2}$ (so that $\overline{m} \simeq m_0$) and assume for the moment that the spectrum has an "ionization point" at some mass $M \gg m_0$, i.e., $\rho(m) = 0$ for $m > M$. For

$m < M$ the level density is described by $\rho(m)$; there are, however, no particles with masses greater than M. Consider a volume V initially characterized by a low energy density, $E/V \ll m_0/V_0$. At this stage the system will behave like a cold gas of $N \simeq E/m_0$ particles.

Suppose now that the energy of the system is gradually increased. When E/V becomes larger than m_0/V_0, the production of states with masses higher than m_0 begins to occur. Let E be increased to the point that E/V is much greater than m_0/V_0 but still small compared to M/V_0. The number of particles will be of order $N \simeq \ln 2\, V/V_0$. Most of the energy will be concentrated in particles of mass M, since production of massive states is favored statistically and M is the maximum possible mass. As the energy density increases, N remains constant and more and more particles with masses of order M are produced.

At low energies the energy density was uniform, but as the energy increased there was a condensation into massive particles and a subsequent nonuniform energy density. Eventually when $E/V \geq M/V$ the energy density will again be uniform. At this stage most of the particles in the system will have masses of order M. If E/V is increased above this point, the mean mass will remain at M and the system will behave like a gas of bosons of mass M. What we have just described is clearly a phase transition in which the original phase dominated by light particles (of mass m_0) condenses into a phase dominated by heavy particles (of mass M). While the condensation is taking place there can be fluctuations in the energy density of order M/V_0. In the limit $M \to \infty$ the fluctuations are limited only by the total energy of the system, and the present phase transition appears as a gross instability.

The phase transition we have just described is a rather peculiar sort of event. We started with a system of light particles and poured energy into it. As the energy density grew higher and higher the system did not boil, i.e., dissociate into lighter particles; it condensed into heavier particles. The clue to this astonishing behavior is provided by Eq. (3.64). Note how the temperature T behaves as the energy is increased. Since $a + \frac{3}{2} < 0$, T *decreases* as E increases.

The relevant quantity in the present discussion is clearly the specific heat,

$$C_V = \frac{\partial E}{\partial T}\bigg|_{V\ \text{fixed}}, \tag{4.10}$$

which measures the rate of change of the energy with the temperature. Ordinary systems, on which our intuition is founded, have $C_V > 0$. A dense hadron system, characterized by $a < -\frac{5}{2}$, and satisfying one of conditions (3.60)–(3.62), has a *negative specific heat*. Indeed, from Eq. (3.65) we have explicitly

$$C_V = (a + 3/2)\left(\frac{T_0}{T - T_0}\right)^2. \tag{4.11}$$

Systems with negative specific heat have been discussed by Lynden-Bell and Wood[21] and by Thirring.[22] The principal feature of interest to us is their instability. If two systems with negative specific heat are placed in thermal contact, they are unstable. In particular consider a dense system of hadrons characterized by the level density (1.1) and satisfying one of conditions (3.60)–(3.62). Any subsystem of this system has a negative specific heat, so the system as a whole is thermodynamically unstable.

If a system with negative specific heat is placed in a heat bath, equilibrium cannot be established. The system will exchange energy with the heat bath until it reaches a point where its specific heat is again positive. Therefore if there is some region of energies for which the microcanonical ensemble indicates the existence of a negative specific heat, this region will be jumped over by a phase transition in the canonical ensemble. Consequently the canonical ensemble never exhibits a negative specific heat. We shall illustrate this point graphically for the case where $a < -\frac{7}{2}$ and there is an "ionization point" at mass M. In Fig. 1 we show the temperature-energy density curve derived from the microcanonical ensemble for such a system. There is a region of densities between m_0/V_0 and M/V_0 for which $T - T_0$ falls like E^{-1} as required by Eq. (3.65). At large energy densities T grows like $(E/V)^{1/4}$, characteristic of a high temperature Bose gas. For $E/V < m_0/V_0$ the temperature falls to 0 at $E/V = 0$ as required by the third law of thermodynamics. If we were to construct[22] the canonical ensemble for this system and plot the internal energy density against

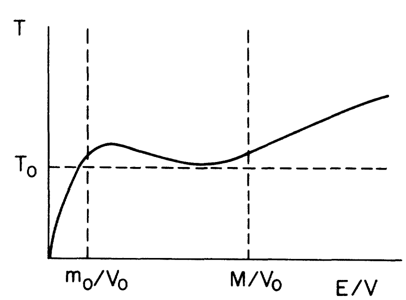

FIG. 1. The temperature is shown as a function of the energy density in the microcanonical ensemble. If the spectrum is cut off at mass M, there is a region of negative specific heat for densities between m_0/V_0 and M/V_0.

the temperature, we would obtain the curve of Fig. 2. The temperature of the system is defined by the heat bath. At low temperatures the system behaves normally, its internal energy rising continuously with the temperature. When the internal energy density is of the order of a few times m_0/V_0, copious production of particles of mass M can begin to take place. This condensation process continues at constant temperature until the internal energy density is of order M/V_0. At this point the condensation is complete and the temperature must be raised to raise the energy density further.

As discussed in Sec. II, the construction of the canonical ensemble for $a < -\frac{7}{2}$ fails if there is no mass cutoff; i.e., if the spectrum has no ionization point. The physical reason already stated is the existence of uncontrollable macroscopic fluctuations in the energy density. A related technical point may also be considered. In the microcanonical ensemble, the entropy may be identified with the logarithm of the density of states. Therefore for $a < -\frac{5}{2}$ when one of conditions (3.60)–(3.62) is met, Eq. (3.63) provides the following expression:

$$S = \ln\sigma(E, V) \simeq \ln\left[\frac{cVE^{a+3/2}}{(2\pi b)^{3/2}}\right] + bE + \frac{cm_0^{a+5/2}V}{(-a-5/2)(2\pi b)^{3/2}}. \tag{4.12}$$

For ordinary thermodynamics to be valid, S should be an extensive quantity. The logarithm in Eq. (4.12) does not exhibit this property. Thus, insofar as this term cannot be neglected, the system will not obey ordinary thermodynamics. It is of course precisely this term which provides the negative specific heat in Eq. (4.11). A calculation of C_V for systems with $a \geq -\frac{5}{2}$ or for the cases (3.49) and (3.55) would show that analogous terms are negligible.

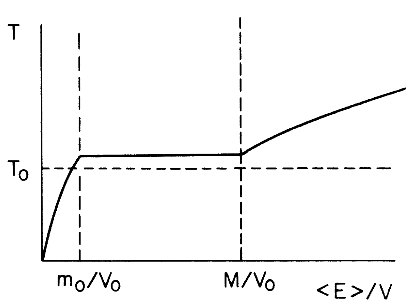

FIG. 2. The internal energy density is shown as a function of the temperature in the canonical ensemble. The spectrum is assumed to be cut off at mass M. The region of energy densities between m_0/V_0 and M/V_0 is bridged by a phase transition. In the limit $M\to\infty$ the region of phase transition corresponds to an instability.

V. IMPLICATIONS

We have shown that a system of dense hadronic matter characterized by the level density (1.1) can be thermodynamically unstable if any of the conditions (3.60)–(3.62) are met. In this section we shall consider some of the implications of this result. It is first useful to review precisely what the statistical bootstrap model–one of the sources of Eq. (1.1)–has to say about the level density. The bootstrap condition[8-10] may be phrased in the following way: (1) Take a volume V_0, which defines the size of all hadronic particles. (2) Assume some input level density $\rho(m)$. (3) Compute the density of states by means of the equation[9]

$$\bar\sigma(E, V_0) = \sum_{n=2}^{\infty}\left[\frac{V}{(2\pi)^3}\right]^{n-1}\prod_{i=1}^{n}\int dm_i\rho(m_i) \times\int d^3p_i\,\delta\left(\sum_{i=1}^{n}E_i - E\right)\delta^3\left(\sum_{i=1}^{n}\vec{p}_i\right). \tag{5.1}$$

[This represents a modification of Eq. (3.4) to exclude 1-particle contributions and to include the constraint of over-all momentum conservation.] (4) Require that at high energies $\bar\sigma(E, V_0)$ reproduce the input level density $\rho(E)$. The reader is referred to the original papers[8-10] for a detailed discussion of this program. Here we shall only note that the solution to the bootstrap is a level density of the form (1.1). The extent to which the parameter a is limited in the solution depends on just how the bootstrap constraint (4) is phrased. If one requires only that

$$\frac{\ln\bar\sigma(E, V_0)}{\ln\rho(E)} \underset{E\to\infty}{\longrightarrow} 1, \tag{5.2}$$

one has the constraint $a \leq -\frac{5}{2}$.[8] With the stronger condition

$$\frac{\bar\sigma(E, V_0)}{\rho(E)} \underset{E\to\infty}{\longrightarrow} 1, \tag{5.3}$$

one eliminates the possibility that $a = -\frac{5}{2}$. The constraint is then $a < -\frac{5}{2}$.[9] Finally if one requires that

$$E\left[\frac{\bar\sigma(E, V_0) - \rho(E)}{\rho(E)}\right] \underset{E\to\infty}{\longrightarrow} 0, \tag{5.4}$$

the constraint is sharpened to $a = -3$.[10]

We have presented the conditions (5.2)–(5.4) in the historical order in which they were proposed. Logically, of course, Eq. (5.4) is the strongest statement of the bootstrap, Eqs. (5.3) and (5.2) being progressivley weakened versions. It is a curious fact that the favored value, $a = -3$, lies in the region of thermodynamic instability. This curiosity has profound consequences for applications of the bootstrap spectrum to various physical problems.

The applications that have been proposed for the spectrum (1.1) are in two general areas: high-energy collisions and astrophysics. We shall discuss each of these in turn.

High-Energy Collisions

Hagedorn and Ranft[19] have constructed a thermodynamical model for the description of particle production in high-energy collisions. Since they have chosen to use the value $a=-\frac{5}{2}$ in the level density (1.1), their application of thermodynamics is not plagued by any instabilities occasioned by large energy densities. There is, however, one curious point worth noting. In a high-energy collision the interaction takes place in a small volume, namely V_0, which is approximately the size of the produced particles themselves. One might ask if this volume is so small as to invalidate a thermodynamical approach. The answer to this question may be found by examining Eq. (3.34). The volume V_0 may be considered large enough if the inequality

$$\frac{cV_0}{(2\pi b)^{3/2}} \gg 1 \tag{5.5}$$

is satisfied. [For large volumes the -1 term in Eq. (3.34) should be negligible.] In Hagedorn and Ranft's fits[19] to high-energy-collision data, their parameters b, c, and V_0 are such that $cV_0(2\pi b)^{-3/2} \simeq 9$. Thus Eq. (5.5) seems to be satisfied.

If, as the bootstrap would have one believe, $a<-\frac{5}{2}$, can the Hagedorn-Ranft approach be generalized? Substituting the expression (4.7) for V_0 in any of Eqs. (3.60)–(3.62) we see that the "high-density" constraint requires only that E be large relative to m_0. Since a system satisfying this "high-density" requirement is thermodynamically unstable, it seems very unlikely that the thermodynamical description of Hagedorn and Ranft could be appropriate for such a case. It is possible, however, to construct less sophisticated statistical models which incorporate the bootstrap spectrum and are able to reproduce some of the impressive results of the Hagedorn-Ranft model.

We will describe one such model as an example. Suppose that in most high-energy collisions a pair of massive resonances is produced. Details of the production process may involve dynamical details of the interaction which lie outside the scope of the statistical approach. The decays of the heavy resonances can, however, be described statistically.

Roughly speaking the information of a resonance of mass m corresponds to the concentration of an energy m in a volume $V=V_0$. This volume [see Eq. (4.7)] is determined at the time of the collision by the range of the strong interactions. As time passes the hadronic matter can expand to fill a larger volume. When this volume has reached a size of order $2V_0/\ln 2$, there is according to Eq. (4.8) an appreciable probability that the hadronic matter will consist of 2 particles – one heavy and one light. This corresponds to the decay of the original resonance into a second heavy resonance and a light particle. As the volume expands further the second resonance will also decay. In this fashion there will be produced a sequence of secondary resonances and a concomitant cascade of light particles. From the discussion of Sec. III it is clear that in the rest frame of each resonance, the momenta of its decay products will be bounded by $p_i<(m_i/b)^{1/2}$. If the production mechanism is such that the original resonance is produced with a limited transverse momentum, then the transverse momenta of its decay products will also be limited.

The experimentally observed bound on transverse momenta is thus achieved in a simple fashion. Just as in the model of Hagedorn and Ranft,[19] the production of states with large transverse momenta is damped by a factor of the form

$$\exp(-bE_i) \leq \exp[-b(p_{iT}{}^2+m_i{}^2)^{1/2}]\ , \tag{5.6}$$

where p_{iT} denotes the transverse momentum of the produced particle. Another feature[23] of the Hagedorn-Ranft model shared by the present model is the manner in which the production of massive pairs – such as $p\bar{p}$ – is suppressed. A resonance with mass greater than $2m_p$ can decay into $p\bar{p}$, but statistical competition reduces the probability of this decay mode by a factor $\exp(-2bm_p)$.

The model we have just outlined is very close to ones proposed by Hwa and Lam[24] and by Jacob and Slansky.[25] These authors are concerned primarily with the mechanism by which the original heavy resonances are produced. They assume a decay scheme similar to that predicted by our statistical model and successfully account for a number of features of one particle inclusive production experiments.

Hamer[26] has further tested the plausibility of a statistical decay scheme with a study of proton-antiproton annihilation at rest. He assumes that all annihilations proceed by resonance formation. The resonance decay chain is traced with the aid of a spectrum of states explicitly constructed[18] as a solution to the bootstrap constraint (5.4). The model successfully accounts for a large fraction of the observed production cross sections.

Astrophysical Problems

Applications of the level density (1.1) have been made in two areas of astrophysics: the structure of neutron stars[27–30] and the early history of the universe.[7,13] In both these problems one is dealing with macroscopic volumes of hadronic matter. We have shown that conventional thermodynamics may be inadequate for the treatment of such systems if $a < -\frac{5}{2}$. Therefore, if the possibility $a < -\frac{5}{2}$ is to be reckoned with, the results in Refs. 27–30, 7, and 13 must all be reexamined. There are a number of questions which must be answered in the course of such a study. We close with a brief list.

(1) What happens to $\rho(m)$ at extremely high masses? Is the spectrum cut off above a certain mass; that is, is there an "ionization point"? If not, what happens when m is so large that gravitational effects become significant?

(2) How does $\rho(m)$ behave at low masses? At what mass does the asymptotic form (1.1) first become appropriate?

(3) What are the dynamical properties of the energy fluctuations in dense hadronic matter? Statistical mechanics indicates the presence of large fluctuations but says nothing about the time scale on which they develop. Can these fluctuations be related to the formation of galaxies in the early universe?

(4) In a system of particles with baryon number $B \leq 1$, a large baryon number density implies the existence of a large particle number density. In this case, are particle interactions still adequately described by the formation of resonances with the level density (1.1)?

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[16]Actually b^{-1} seems to be the same order as m_0, which is presumably the pion mass. This evidence comes from three sources: a fit of the observed particle spectrum to $\rho(m)$ (Ref. 17), a detailed numerical solution of the bootstrap model (Ref. 18), and an application of the statistical model to high-energy collisions (Refs. 8 and 19).

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