Broken Scale Invariance and Asymptotic Behavior

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We show that broken-scale-invariance Ward identities place strong restrictions on the asymptotic behavior of perturbation-theory amplitudes.

I. INTRODUCTION

A by-product of the recent flurry of interest in scale invariance has been the recognition[1,2] that perturbation-theory amplitudes satisfy an equation which can be described as a "broken-scale-invariance Ward identity." This name is appropriate because, apart from anomaly terms of the general class made familiar by Adler[3] and by Bell and Jackiw,[4] the equation in question is just the Ward identity for the dilation current. Unlike most Ward identities, this one has some nontrivial consequences for the asymptotic behavior of amplitudes. Indeed, in Refs. 1 and 2, it was shown how the usual renormalization-group results could be obtained from it in a particularly expeditious fashion. There is, however, much more to be learned and this paper will be devoted to extracting the full consequences for high-energy behavior of the broken-scale-invariance Ward identity.

In order to simplify our presentation, we shall first guess a sensible-looking asymptotic expansion and then show how it can be proven with the help of the broken-scale-invariance Ward identity. The expansion we shall consider is very closely related to Wilson's operator-product expansion,[5] whose validity in perturbation theory has already been discussed by Brandt[6] and Zimmermann.[7] We hope to convince the reader that a proof based on the broken-scale-invariance Ward identity is especially simple. In order to avoid notational complications, we shall always deal with the eminently renormalizable theory of a massive scalar field having a quartic self-interaction ($\lambda\phi^4$ theory). The extension of our arguments to any renormalizable theory should be quite evident.

Let us first consider a nonlocal product $\phi(x)\phi(y)$ of the field operators of the theory. It is not unreasonable to suppose that in the limit $x\to y$ this nonlocal product can be expanded in terms of other local operators (local in a sense we shall shortly explain) as follows:

$$\phi(x)\phi(y)\to\sum_n B_n(x-y)\,O_n\big(\tfrac{1}{2}(x+y)\big)\,, \qquad (1)$$

where $B_n(x-y)$ are c-number functions and $\{O_n\}$ is a "complete" set of operators. (This, of course, is the assumption of Wilson's operator-product expansion.)

In a free-field theory, the set $\{O_n\}$ is easy to identify. It consists of all the monomials one can form out of powers of derivatives and powers of fields: ϕ^2, ϕ^4, $\partial_\mu\phi\partial_\nu\phi$, etc. [These operators are conveniently classified in terms of their dimension d in powers of energy: $d(\phi)=1$, $d(\phi^2)=2$, $d(\phi^4)=4$, $d(\partial_\mu\phi\partial_\nu\phi)=4$, etc.] In the presence of interactions, we can still consider this same set of operators; subtractions are now required to render their matrix elements finite, and the number of subtractions needed grows with the dimension d of the operator considered. Nevertheless, if we make some universal choice of subtraction point (for example, all four-momenta equal to zero), a perfectly definite meaning attaches to each monomial we can form out of powers of derivatives and powers of fields. The operators defined by this perturbation-theory prescription are presumably local in the strict sense of the word (their matrix elements satisfy dispersion relations, and so on), but we shall not attempt to prove that they are. We regard this as an independent question about the general properties of renormalized perturbation theory.

Something can be said about the c-number functions $B_n(x)$ as well. Dimensional analysis tells us that if O_n has dimension d_n, then B_n must have dimension $2-d_n$. But for the presence of the mass μ of the field, this would mean that $B_n(x)$ must be homogeneous of degree d_n-2 in x. On the other hand, the dependence of $B_n(x)$ on μ must be such that in the limit $\mu\to 0$ only the usual logarithmic infrared singularities are encountered. Therefore, apart from powers of $\ln(\mu^2x^2)$, only positive powers of μ^2 may occur and they must actually be powers of the dimensionless combination μ^2x^2 in order to keep the over-all dimension of $B_n(x)$ fixed. In the limit $x\to 0$, the leading contribution to $B_n(x)$ is therefore of the form (function of x homogeneous of degree d_n-2) times [sum of powers of $\ln(\mu^2x^2)$]. Since logarithmic singularities are associated with loop integrations, we expect only a finite number of powers of $\ln(\mu^2x^2)$ if we consider amplitudes expanded out to a finite order in perturbation theory. In the future, we shall refer to a function of the form (function of x homogeneous of degree n) times [polynomial in $\ln(x^2\mu^2)$] as being $O(x^n)$. Therefore the functions $B_n(x)$ in Eq. (1) should be $O(x^{d_n-2})$ in the limit $x\to 0$, that is to say, less and less singular as d_n increases.

Now, since our primary concern is the high-energy behavior of scattering amplitudes, it is convenient to transform Eq. (1) into a statement about the asymptotic behavior of momentum-space amplitudes. We shall use the same notation as in Ref. 2, where $\Gamma^{(n)}(p_1\cdots p_n)$ stood for the connected, one-particle irreducible amplitude for n particles (having ingoing momenta $p_1\cdots p_n$) with external-line propagators divided out, $\Gamma_A^{(n)}(q;\, p_1\cdots p_n)$ stood for a similar amplitude with an insertion of the operator A carrying ingoing momentum q, and so on. Obviously, the momentum-space equivalent of letting $x_1-x_2\to 0$ in the product $\phi(x_1)\phi(x_2)$ is to let the difference p_1-p_2 of the momenta carried by $\phi(x_1)$ and $\phi(x_2)$ go to infinity. This suggests that the proper momentum-space transcription of Eq. (1) is

$$\Gamma^{(n)}(p_1\cdots p_n)\underset{q\to\infty}{\sim}\sum_m C_m(q)\Gamma_{O_m}^{(n-2)}(\Delta;\, p_3\cdots p_n),\qquad (2)$$
$$p_1=q+\tfrac{1}{2}\Delta\,,\quad p_2=-q+\tfrac{1}{2}\Delta\,,$$

where the $C_m(q)$ are just Fourier transforms of the corresponding $B_m(x)$ [and therefore asymptotically of $O(q^{2-d_m})$ in the sense explained in the previous paragraph]. Because of the symmetry of the theory under the operation $\phi\to-\phi$, it is apparent that only operators O_m involving even powers of ϕ can appear in the expansion. The allowed dimensions are therefore 2 (corresponding to ϕ^2), 4 (corresponding to ϕ^4, $\partial_\mu\phi\partial_\nu\phi$, etc.), and so on. Since $C_n(q)$ is $O(q^{2-d_n})$, the leading term in the expansion, corresponding to the operator of lowest dimension, is

$$\Gamma^{(n)}(p_1\cdots p_n)\underset{q\to\infty}{\sim}C_{\phi^2}(q)\Gamma_{\phi^2}^{(n-2)}(\Delta;p_3\cdots p_n)+O(q^{-2}),\qquad (3)$$
$$p_1=q+\tfrac{1}{2}\Delta\,,\quad p_2=-q+\tfrac{1}{2}\Delta\,,$$

with C_{ϕ^2} of $O(q^0)$ [i.e., a polynomial in $\ln(q^2/\mu^2)$]. The terms of $O(q^{-2})$ will involve operators of dimension four and so on.

So far, of course, this is but an educated guess as to the asymptotic behavior of momentum-space amplitudes. One should not, however, be misled into thinking that a statement such as Eq. (3) is in any sense self-evident. For example, this equation asserts that the leading asymptotic behavior is a function only of the magnitude of the large momentum, and *not* of its "direction cosines" with respect to the momenta which are held fixed. For individual Feynman diagrams this is not necessarily the case, so that if Eq. (3) is correct, it is so by virtue of some nontrivial cancellations between different diagrams.

This is where the broken-scale-invariance Ward identity comes in. In the next sections of this paper, we shall show that such Ward identities do two things: They provide a very simple scheme for demonstrating that Eq. (2) is actually correct (i.e., they guarantee that the requisite cancellations do occur) and they impose conditions on the expansion functions $C_n(q)$ which are similar to what one might expect from renormalization-group arguments. Another useful feature of this approach is that it can easily be generalized to other renormalizable theories and to expansions of more interesting quantities, such as current matrix elements. Once Eqs. (2) and (3) have been disposed of, the generalization to more interesting amplitudes will be evident. It is worth mentioning that although our heuristic arguments started from the configuration-space expansion of Eq. (1), we shall not be able to prove that it is true. For technical reasons which will soon be apparent, we must content ourselves with the momentum-space expansion of Eq. (2).

II. PROOF OF THE LEADING TERM

The expansion in Eq. (2) consists of terms of well-defined orders in large momentum which can evidently be studied individually. Accordingly, in this section we shall concentrate on the leading term [written explicitly in Eq. (3)], since the proof is especially simple and clearly illustrates the general procedure for dealing with further terms in the expansion.

As a first step, we shall use Weinberg's theorem[8] to show that if Eq. (3) is true for $n=4$, it is automatically true for $n>4$. The relevant part of Weinberg's theorem can be stated as follows: Suppose we have a graph with n external lines of momenta $p_1\cdots p_n$, and suppose that the momenta $p_1\cdots p_n$ are taken to infinity as follows: $p_i=\hat{p}_i+\lambda e_i$, with e_i spacelike and $\lambda\to\infty$. Then, in the limit, the graph is $O(\lambda^{4-m-\overline{m}})$, where $\overline{m}$ is the minimum number of internal lines which, when cut, completely separate the external lines carrying $p_1\cdots p_m$ from those carrying $p_{m+1}\cdots p_n$ (see Fig. 1). The instruction that the large momenta be asymptotically spacelike is absolutely crucial,[8] and we must in the future understand that in Eqs. (2) and (3) (and in any similar asymptotic expansions) the momentum q is large and spacelike. It

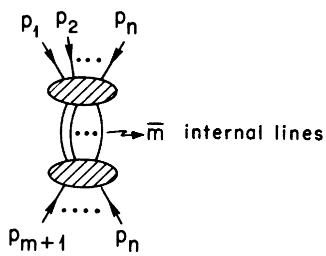

FIG. 1. Example of a graph such that cutting $\bar{m}$ internal lines completely separates the sets of external momenta $\{p_1, \ldots, p_m\}$ and $\{p_{m+1}, \ldots, p_n\}$.

is this restriction on q which makes it difficult to go from Eq. (2) to Eq. (1).

Let us now consider Eq. (3) for $n>4$. By virtue of Weinberg's theorem, it is clear that only Feynman diagrams which can be decomposed as in Fig. 2 can be asymptotically $O(q^0)$. Therefore, in the skeleton expansion of $\Gamma^{(n)}$ for $n>4$, only those skeletons in which the two external lines carrying large momentum are attached to the same vertex can be $O(q^0)$. If the four-particle vertex is assumed to satisfy Eq. (3), these skeletons undergo the transformation summarized in Fig. 3, yielding graphs which are easily recognized as the skeleton for $\Gamma^{(n-2)}_{\phi^2}$. In making the transformation, we assumed that the limit could be taken inside the integration over internal momentum. This is justified by the convergence of all the relevant integrals before and after taking the limit.

A related result, which will prove useful, follows from similar arguments. The amplitude $\Gamma^{(n)}_{\phi^2}$ has a skeleton expansion for $n \geq 4$. Therefore, if we study its $O(q^0)$ asymptotic behavior along the lines of the previous paragraph, we find that if $\Gamma^{(4)}$ satisfies Eq. (3) then

$$\Gamma^{(n)}_{\phi^2}(\Delta; p_1 \cdots p_n) \underset{q\to\infty}{\sim} C_{\phi^2}(q)\Gamma^{(n-2)}_{\phi^2\phi^2}(\Delta, \Delta'; p_1 \cdots p_n) + O(q^{-2}), \tag{4}$$

$p_1 = q + \frac{1}{2}\Delta'$, $p_2 = -q + \frac{1}{2}\Delta'$,

for $n \geq 4$. An important point is that if $\Gamma^{(n)}_{\phi^2}$ is

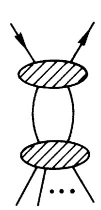

FIG. 2. All diagrams with $O(q^0)$ asymptotic behavior must be decomposable in this fashion. The arrowed lines are the external lines carrying large momentum.

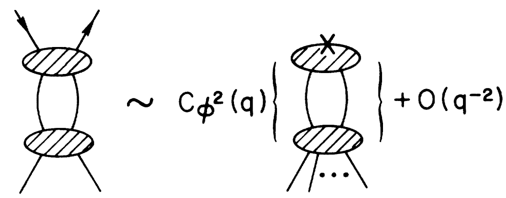

FIG. 3. Taking the limit of the diagrams of $O(q^0)$ in $\Gamma^{(n)}$. The cross represents an insertion of the operator ϕ^2.

needed to order r in the coupling constant, it is only necessary to insert primitive vertices correct to order $r-1$ in the skeleton expansion. Therefore Eq. (4) is guaranteed correct to order r if one assumes $\Gamma^{(4)}$ to satisfy Eq. (3) only to order $r-1$.

It remains to show that $\Gamma^{(4)}$ actually satisfies Eq. (3). Since $\Gamma^{(4)}$ is primitively divergent, it has no skeleton expansion and some more powerful technique is needed. We shall now show that an effective procedure is to argue by induction with the help of the broken-scale-invariance Ward identity. The explicit form of this Ward identity was found in Refs. 1 and 2 to be

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + n\gamma\right)\Gamma^{(n)}(p_1 \cdots p_n) = -i\Gamma^{(n)}_\theta(0; p_1 \cdots p_n),$$

where β, γ are power series in the coupling constant λ beginning with λ^2, λ, respectively, and θ is proportional to ϕ^2 [more precisely, $\theta = \mu^2 f(\lambda)\phi^2 = \alpha\phi^2$]. Since matrix elements of ϕ^2 arise naturally in our argument, we prefer to write this equation in the form

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + n\gamma\right)\Gamma^{(n)}(p_1 \cdots p_n) = -i\alpha\Gamma^{(n)}_{\phi^2}(0; p_1 \cdots p_n). \tag{5}$$

A similar equation for $\Gamma^{(n)}_{\phi^2}$, of which we shall make use, was derived in Ref. 2:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + n\gamma + \gamma_{\phi^2}\right)\Gamma^{(n)}_{\phi^2}(\Delta; p_1 \cdots p_n) = -i\alpha\Gamma^{(n)}_{\phi^2\phi^2}(0\Delta; p_1 \cdots p_n), \tag{6}$$

where α, β, γ are the same functions as they appear in Eq. (5), and γ_{ϕ^2} is a power series in λ beginning $O(\lambda)$.

The first thing to observe is that the proposed expansion for $\Gamma^{(n)}$ [Eq. (3)] and its consequence for $\Gamma^{(n)}_{\phi^2}$ [Eq. (4)] cannot be consistent with Eqs. (5) and (6) for an arbitrary C_{ϕ^2}: Inserting the leading terms of the expansions for $\Gamma^{(n)}$ and $\Gamma^{(n)}_{\phi^2}$ into Eq. (5) gives

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+n\gamma\right)[C_{\phi^2}(q)\Gamma^{(n-2)}_{\phi^2}(\Delta;p_3\cdots p_n)] = -i\alpha C_{\phi^2}(q)\Gamma^{(n-2)}_{\phi^2\phi^2}(0\Delta;p_1\cdots p_n).$$

Simplifying this with Eq. (6) yields the condition

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+2\gamma-\gamma_{\phi^2}\right)C_{\phi^2}(q)=0. \tag{7}$$

The definition of C_{ϕ^2} in Eq. (3) shows that it is a power series in $\ln(-q^2/\mu^2)$ with dimensionless coefficients. Equation (7) therefore guarantees that if $\beta(\lambda)=0$, the logarithms sum up to the power $(-q^2/\mu^2)^{\gamma'}$, where $\gamma'=-2\gamma+\gamma_{\phi^2}$. This is reminiscent of Wilson's operator-product expansion[5]; one is tempted to identify the "scaling dimensions" of ϕ and ϕ^2 as $\delta_\phi=1+\gamma$ and $\delta_{\phi^2}=2+\gamma_{\phi^2}$ (in the absence of interactions, γ and γ_{ϕ^2} go to zero and the scaling dimension becomes identical with the ordinary dimension). Then C_{ϕ^2} is proportional to $(-q^2)^{\delta'}$, where $\delta'=\delta_{\phi^2}-2\delta$, which is, for this case, precisely the relation proposed by Wilson between asymptotic behavior and scaling dimensions. When we study the higher-order terms in Eq. (2), however, we shall see that C_{ϕ^2} has $O(q^{-2})$ (and smaller) contributions which might sum up to a power just as well as the $O(q^0)$ contribution. Therefore, we cannot attach too much weight to the fact that the $O(q^0)$ term can sum up to the power in Wilson's relation.

We are now ready to proceed with the proof by induction that $\Gamma^{(4)}$ satisfies Eq. (3). Let us therefore suppose that, to $O(\lambda^{n-1})$, $\Gamma^{(4)}$ satisfies Eq. (3), and that as a consequence C_{ϕ^2} is known to $O(\lambda^{n-1})$ and satisfies Eq. (7) to that order. [The notation $O(\lambda^n)$ has the usual meaning, as opposed to the somewhat different meaning we attach to the symbol $O(q^n)$ when applied to asymptotic behavior. No confusion should arise.] To keep track of orders in the coupling constant, we shall provisionally adopt the notation ${}^{[a]}\Gamma^{(n)}$ to indicate the expansion of $\Gamma^{(n)}$ out to $O(\lambda^a)$ only (and a corresponding notation for other matrix elements which appear in our equations). Our problem then is to show that ${}^{[n]}\Gamma^{(4)}$ satisfies Eq. (3).

To do this, we study the broken-scale-invariance Ward identity for ${}^{[n]}\Gamma^{(4)}$, which can be written

$$\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma^{(4)}(p_1\cdots p_4) = -i\alpha\,{}^{[n]}\Gamma^{(4)}_{\phi^2}(0;p_1\cdots p_4) - \left(\beta\frac{\partial}{\partial\lambda}+4\gamma\right){}^{[n-1]}\Gamma^{(4)}(p_1\cdots p_4) \tag{8}$$

if we let it be understood that the right-hand side is expanded out only to $O(\lambda^n)$. The reason that ${}^{[n-1]}\Gamma^{(4)}$, rather than ${}^{[n]}\Gamma^{(4)}$, appears on the right-hand side is that, as mentioned in connection with Eq. (5), $\beta\partial/\partial\lambda$ and γ are both effectively of $O(\lambda)$. If we use Eq. (4) and the remarks which follow it, plus our assumption on the asymptotic behavior of ${}^{[n-1]}\Gamma^{(4)}$, Eq. (8) transforms into the following statement about the asymptotic behavior of ${}^{[n]}\Gamma^{(4)}$:

$$\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma^{(4)}(p_1\cdots p_4) \underset{q\to\infty}{\sim} -i\alpha\,{}^{[n-1]}C_{\phi^2}(q)\,{}^{[n]}\Gamma^{(2)}_{\phi^2\phi^2}(0\Delta;p_1\cdots p_4) - \left(\beta\frac{\partial}{\partial\lambda}+4\gamma\right){}^{[n-1]}C_{\phi^2}(q)\,{}^{[n-1]}\Gamma^{(2)}_{\phi^2}(\Delta;p_3p_4)+O(q^{-2}),$$
$$p_1=q+\tfrac{1}{2}\Delta,\quad p_2=-q+\tfrac{1}{2}\Delta. \tag{9}$$

The right-hand side of this equation can be recast, using Eq. (6), into the form

$$\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma^{(4)}(p_1\cdots p_4)\sim {}^{[n-1]}C_{\phi^2}(q)\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma^{(2)}_{\phi^2}(\Delta;p_3p_4) + {}^{[n]}\Gamma^{(2)}_{\phi^2}(\Delta;p_3p_4)\left(-\beta\frac{\partial}{\partial\lambda}-2\gamma+\gamma_{\phi^2}\right){}^{[n-1]}C_{\phi^2}(q)+O(q^{-2}). \tag{10}$$

Notice that in the first term on the right-hand side, it is legitimate to replace ${}^{[n-1]}C_{\phi^2}$ by ${}^{[n]}C_{\phi^2}$ since $\mu(\partial/\partial\mu)\Gamma^{(2)}_{\phi^2}$ has an expansion which begins at $O(\lambda)$ [the $O(\lambda^0)$ piece of $\Gamma^{(2)}_{\phi^2}$ is dimensionless and therefore is annihilated by $\mu\partial/\partial\mu$] and we are only keeping track of terms of $O(\lambda^n)$ or less in the equation.

So far, of course, the term of $O(\lambda^n)$ in C_{ϕ^2} is undetermined. On the other hand, we know that if $\Gamma^{(4)}$ is to satisfy Eq. (3) to $O(\lambda^n)$, then C_{ϕ^2} *must* satisfy Eq. (7) to the same order. If we look only at terms of $O(\lambda^n)$ or less, that equation can be written

$$\mu\frac{\partial}{\partial\mu}\,{}^{[n]}C_{\phi^2}(q)=\left(-\beta\frac{\partial}{\partial\lambda}-2\gamma+\gamma_{\phi^2}\right){}^{[n-1]}C_{\phi^2}(q), \tag{11}$$

where ${}^{[n-1]}C_{\phi^2}$ rather than ${}^{[n]}C_{\phi^2}$ appears on the right-hand side because the operator in large parentheses is $O(\lambda)$. It is not hard to see that, since C_{ϕ^2} is a joint power series in $\ln(-q^2/\mu^2)$ and λ with dimensionless coefficients, this equation *determines* ${}^{[n]}C_{\phi^2}$, apart from an additive constant of $O(\lambda^n)$, once ${}^{[n-1]}C_{\phi^2}$ is given. If we *construct* the function ${}^{[n]}C_{\phi^2}$ in this manner, Eq. (10) can finally be rewritten as

$$\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma^{(4)}(p_1\cdots p_4)\underset{q\to\infty}{\sim}\mu\frac{\partial}{\partial\mu}[{}^{[n]}C_{\phi^2}(q)\,{}^{[n]}\Gamma_{\phi^2}(\Delta;p_1\cdots p_4)]+O(q^{-2}),$$
$$p_1=q+\tfrac{1}{2}\Delta\,,\quad p_2=-q+\tfrac{1}{2}\Delta\,. \tag{12}$$

For the moment, ${}^{[n]}C_{\phi^2}$ is arbitrary to the extent of an additive constant of $O(\lambda^n)$. Since $\Gamma^{(4)}$, C_{ϕ^2}, and $\Gamma^{(2)}_{\phi^2}$ are all dimensionless quantities, the "integral form" of this relation is

$${}^{[n]}\Gamma^{(4)}(p_1\cdots p_4)\sim{}^{[n]}C_{\phi^2}(q)\,{}^{[n]}\Gamma^{(2)}_{\phi^2}(\Delta;p_3p_4)+F(q\Delta p_3p_4)+O(q^{-2}), \tag{13}$$

where F is $O(\lambda^n)$ [since $\Gamma^{(4)}$ satisfies Eq. (3) to $O(\lambda^{n-1})$] and satisfies $\mu\partial F/\partial\mu=0$. F is of course a dimensionless function so that the latter condition implies that F is a function only of ratios of its momentum arguments and not of μ. However, any singularities F may have in the variables Δ, p_2, and p_3 must, by the usual principles of analyticity, occur at locations determined by the mass. Therefore, F is actually entire in these arguments. The only way it can do this and still be a function only of ratios of momenta is for it to be just a constant [necessarily of $O(\lambda^n)$]. However, in Eq. (13), we are free to adjust ${}^{[n]}C_{\phi^2}$ by a constant of $O(\lambda^n)$, and since the $O(\lambda^n)$ piece of $\Gamma^{(2)}_{\phi^2}$ is itself a constant, the product ${}^{[n]}C_{\phi^2}\,{}^{[n]}\Gamma^{(2)}_{\phi^2}$ is, to $O(\lambda^n)$, adjustable to the extent of a constant of $O(\lambda^n)$. Therefore, F can actually be absorbed into C_{ϕ^2} in Eq. (13), thereby completely determining C_{ϕ^2} to $O(\lambda^n)$ and guaranteeing that $\Gamma^{(4)}$ satisfies Eq. (3) to $O(\lambda^n)$. The resulting C_{ϕ^2} of course has been constructed to satisfy Eq. (7). The induction step of the proof is therefore complete. Since the lowest-order contribution to $\Gamma^{(4)}$ automatically satisfies Eq. (3), we are therefore guaranteed that, to any finite order in λ, $\Gamma^{(4)}$ satisfies Eq. (3), and that the C_{ϕ^2} so defined satisfies Eq. (7) to any finite order.

The crucial feature of Eq. (3) is its assertion that the various powers of $\ln(-q^2/\mu^2)$ coming from various orders in the perturbation expansion all multiply the same function (and of course that that function is a matrix element of a specific local operator). The interesting feature of our argument is that the use of the broken-scale-invariance Ward identity allows one to demonstrate that this actually happens without ever having to look in detail at the behavior of specific diagrams. Presumably this is possible because, as was emphasized in Refs. 1 and 2, the Ward identity embodies the constraints of the renormalization group, while the renormalization group is precisely a tool for organizing asymptotic logarithms.

III. HIGHER-ORDER TERMS

Next, we must consider the higher-order contributions to Eq. (2). Actually, the steps one must go through to prove any given order in the expansion are completely analogous to the steps one goes through in proving the leading order. The only difference is that the number of independent terms one has to juggle increases rapidly as one goes to higher and higher orders. For this reason, a general proof requires the development of a compact and efficient notation. Rather than trying to develop such a notation, we prefer to illustrate how things go in nonleading orders by sketching the proof of the $O(q^{-2})$ terms in Eq. (2). It should then be clear how the proof of a general term would go.

According to our heuristic argument, the $O(q^{-2})$ terms in the expansion are determined by operators of dimension four. There are five independent such operators:

$$\phi^4,\quad \partial_\alpha\phi\partial^\alpha\phi,\quad \Box\phi^2$$

of spin zero and

$$(\partial_\mu\partial_\nu-\tfrac{1}{4}g_{\mu\nu}\Box)\phi^2,\quad (\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{4}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi)$$

of spin two. As mentioned earlier, we agree to define matrix elements of these operators by using the standard subtraction prescription[9] with the subtraction point taken at zero four-momentum. It is worth noting that, as a consequence, to zeroth order in the coupling constant, these operators have very simple matrix elements: $\Gamma^{(4)}_{\phi^4}=1$, all other matrix elements zero;

$$\Gamma^{(2)}_{\partial_\alpha\phi\partial^\alpha\phi}(q;p_1p_2)=-p_1\cdot p_2\,,$$

all other matrix elements zero, and so on. If then, for ease of writing, we denote the five operators of dimension four by B_i, $i=1,\ldots,5$, the expansion of $\Gamma^{(n)}$ out to $O(q^{-2})$ becomes

$$\Gamma^{(n)}(p_1\cdots p_n)\underset{q\to\infty}{\sim} C_{\phi^2}(q)\Gamma^{(n-2)}_{\phi^2}(\Delta;p_3\cdots p_n)+\sum_{i=1}^{5} C_{B_i}(q)\Gamma^{(n-2)}_{B_i}(\Delta;p_3\cdots p_n)+O(q^{-4}),$$
$$p_1=q+\tfrac{1}{2}\Delta,\quad p_2=-q+\tfrac{1}{2}\Delta, \tag{14}$$

where C_{B_i} are of $O(q^{-2})$ [that is to say, a function of appropriate Lorentz tensor structure homogeneous of degree -2 in q times a polynomial in $\ln(-q^2/\mu^2)$ with dimensionless coefficients] and C_{ϕ^2} is now expanded out to $O(q^{-2})$. By analogy with the proof of the leading term, we expect to need the expansion to $O(q^{-2})$ of $\Gamma^{(n)}_{\phi^2}$ as well. A natural guess for this would be

$$\Gamma^{(n)}_{\phi^2}(\Delta;p_1\cdots p_n)\sim C_{\phi^2}(q)\Gamma^{(n-2)}_{\phi^2\phi^2}(\Delta\Delta';p_3\cdots p_n)+\sum_{i=1}^{5} C_{B_i}(q)\Gamma^{(n-2)}_{\phi^2 B_i}(\Delta\Delta';p_3\cdots p_n)+O(q^{-4}).$$

In fact, it turns out that we must add to this a term which looks very much like a contact term. The corrected expansion is

$$\Gamma^{(n)}_{\phi^2}(\Delta;p_1\cdots p_n)\underset{q\to\infty}{\sim} C_{\phi^2}(q)\Gamma^{(n-2)}_{\phi^2\phi^2}(\Delta\Delta';p_3\cdots p_n)$$
$$+\sum_{i=1}^{5} C_{B_i}(q)\Gamma^{(n-2)}_{\phi^2 B_i}(\Delta\Delta';p_3\cdots p_n)+\hat{C}_{\phi^2}(q)\Gamma^{(n-2)}_{\phi^2}(\Delta+\Delta';p_3\cdots p_n)+O(q^{-4}),$$
$$p_1=q+\tfrac{1}{2}\Delta',\quad p_2=-q+\tfrac{1}{2}\Delta', \tag{15}$$

where $\hat{C}_{\phi^2}$ is $O(q^{-2})$. We shall shortly see the significance of, and understand the necessity for, the term involving $\hat{C}_{\phi^2}$.

Our experience with the leading term tells us that we shall also need the broken-scale-invariance Ward identity satisfied by the matrix elements of the B_i. Since these Ward identities can readily be derived via the methods of Refs. 1 and 2, we shall simply list the relevant results:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+n\gamma\right)\Gamma^{(n)}_{B_i}(\Delta;p_1\cdots p_n)+\sum_{j=1}^{5}\gamma^{B}_{ij}\Gamma^{(n)}_{B_j}(\Delta;p_1\cdots p_n)=-i\alpha\Gamma^{(n)}_{\phi^2 B_i}(0\Delta;p_1\cdots p_n),$$
$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+n\gamma+\gamma_{\phi^2}\right)\Gamma^{(n)}_{\phi^2 B_i}(\Delta\Delta';p_1\cdots p_n)+\sum_{j=1}^{5}\gamma^{B}_{ij}\Gamma^{(n)}_{\phi^2 B_j}(\Delta\Delta';p_1\cdots p_n)+\gamma_{\phi^2 B_i}\Gamma^{(n)}_{\phi^2}(\Delta+\Delta';p_1\cdots p_n)$$
$$=-i\alpha\Gamma^{(n)}_{\phi^2\phi^2 B_i}(0\Delta\Delta';p_1\cdots p_n) \tag{16}$$

where α is the same constant as in Eq. (5) and all the γ's are at least of $O(\lambda)$.

We are now ready to proceed with the proof, which we will model as closely as possible on the proof of the leading term. First of all, according to Weinberg's theorem, the only diagrams contributing to $\Gamma^{(n)}$ which can have asymptotic behavior of $O(q^{-2})$ or higher are those which can be decomposed as in Fig. 4. This means that the asymptotic behavior to $O(q^{-2})$ of an arbitrary $\Gamma^{(n)}$ is determined by the asymptotic behavior to $O(q^{-2})$ of $\Gamma^{(4)}$ and $\Gamma^{(6)}$. However, if we insert Eq. (14) into Fig. 4 for the upper blobs, the internal-loop integrations will converge only if the number of external lines attached to the lower blob is greater than four (because four- and two-particle matrix elements of operators of dimension four are in general primitively divergent). As a result, one can easily discover, by studying appropriate skeleton expansions, that if $\Gamma^{(4)}$ and $\Gamma^{(6)}$ are assumed to satisfy Eq. (14), then $\Gamma^{(n)}$ for $n>6$ automatically satisfies Eq. (14). By the same token, one finds that if $\Gamma^{(4)}_{\phi^2}$ is assumed to satisfy Eq. (15) and if $\Gamma^{(4)}$ and $\Gamma^{(6)}$ are assumed to satisfy Eq. (14), then $\Gamma^{(n)}_{\phi^2}$ for $n>4$ automatically satisfies Eq. (15). Indeed, since insertions in $\Gamma^{(n)}_{\phi^2}$ of $\Gamma^{(4)}$, $\Gamma^{(6)}$, and $\Gamma^{(4)}_{\phi^2}$ are necessarily of lower order in coupling constant than $\Gamma^{(n)}_{\phi^2}$ itself, it is only necessary to insist that these fundamental Green's functions satisfy Eqs. (14) and (15) to $O(\lambda^{r-1})$ in order to

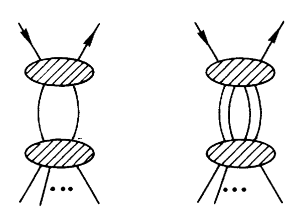

FIG. 4. All diagrams with asymptotic behavior of $O(q^{-2})$ or higher must be decomposable in either of these two fashions.

guarantee that $\Gamma_{\phi^2}^{(n)}$ satisfies Eq. (15) to $O(\lambda^r)$. It therefore remains to be demonstrated that $\Gamma^{(4)}$, $\Gamma^{(6)}$, and $\Gamma_{\phi^2}^{(4)}$ satisfy the appropriate expansions.

We note also that the expansions of Eqs. (14) and (15) are not consistent with the broken-scale-invariance Ward identities of Eqs. (5) and (16) for arbitrary expansion functions B_i, $\hat{C}_{\phi^2}$, and C_{ϕ^2}. Arguments completely analogous to those which led to Eq. (7) yield the conditions

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+2\gamma-\gamma_{\phi^2}\right)C_{\phi^2}=-i\hat{C}_{\phi^2}, \tag{17a}$$

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+2\gamma\right)C_{B_i}+\sum_{j=1}^{5}\gamma_{ji}^{B}C_{B_j}=0, \tag{17b}$$

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+2\gamma\right)\hat{C}_{\phi^2}+\sum_{i=1}^{5}\gamma_{\phi^2 B_i}C_{B_i}=0. \tag{17c}$$

Equation (16) is just the extension to the next order in the expansion of Eq. (7): to $O(q^{-2})$, C_{ϕ^2} satisfies an inhomogeneous equation with $\hat{C}_{\phi^2}$ serving as the inhomogeneous term. In fact, were the "contact term" missing from Eq. (15), so that $\hat{C}_{\phi^2}=0$, then Eq. (17a) would imply that C_{ϕ^2} is *purely* $O(q^0)$, with no $O(q^{-2})$ piece. This is one reason for the necessity of the "contact term." These equations are also analogous to Eq. (7) in the sense that, up to simple integration constants, they allow one to determine the expansion functions C_{ϕ^2}, C_{B_i}, and $\hat{C}_{\phi^2}$ to $O(\lambda^n)$ if they are given to $O(\lambda^{n-1})$.

Let us now sketch how the induction step of the proof goes. We assume that $\Gamma^{(4)}$, $\Gamma^{(6)}$, and $\Gamma_{\phi^2}^{(4)}$ satisfy Eqs. (14) and (15) to $O(\lambda^{n-1})$ and that the corresponding expansion functions satisfy Eqs. (17a)–(17c) out to the same order in the coupling constant. Then we consider the broken-scale-invariance Ward identity for $\Gamma^{(6)}$ expanded out to $O(\lambda^n)$:

$$\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma^{(6)}(p_1\cdots p_6)=-i\alpha\,{}^{[n]}\Gamma_{\phi^2}^{(6)}(0;p_1\cdots p_6)-\left(\beta\frac{\partial}{\partial\lambda}+6\gamma\right){}^{[n-1]}\Gamma^{(6)}(p_1\cdots p_6), \tag{18}$$

where we use the same notation as in Eq. (8) to indicate the expansion of a matrix element out to a given order in λ. Again, ${}^{[n-1]}\Gamma^{(6)}$ appears on the right-hand side, rather than ${}^{[n]}\Gamma^{(6)}$, because the operator in parentheses is effectively $O(\lambda)$ and the whole equation is understood to be expanded out only to $O(\lambda^n)$. Our assumption that $\Gamma^{(4)}$, $\Gamma^{(6)}$, and $\Gamma_{\phi^2}^{(4)}$ satisfy Eqs. (14) and (15) now determines the expansion to $O(q^{-2})$ of the right-hand side of the Ward identity (18):

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma^{(6)}(p_1\cdots p_6)\underset{q\to\infty}{\sim}&-i\alpha\,{}^{[n-1]}C_{\phi^2}(q)\,{}^{[n]}\Gamma_{\phi^2\phi^2}^{(4)}(0\Delta;p_3\cdots p_6)\\&-i\alpha\sum_{i=1}^{5}{}^{[n-1]}C_{B_i}(q)\,{}^{[n]}\Gamma_{\phi^2B_i}^{(4)}(0\Delta;p_3\cdots p_6)-i\alpha\,{}^{[n-1]}\hat{C}_{\phi^2}(q)\,{}^{[n]}\Gamma_{\phi^2}^{(4)}(\Delta;p_3\cdots p_6)\\&-\left(\beta\frac{\partial}{\partial\lambda}+6\gamma\right)\left[{}^{[n-1]}C_{\phi^2}(q)\,{}^{[n-1]}\Gamma_{\phi^2}^{(4)}(\Delta;p_3\cdots p_6)+\sum_{i=1}^{5}{}^{[n-1]}C_{B_i}\,{}^{[n-1]}\Gamma_{B_i}^{(4)}(\Delta;p_3\cdots p_6)\right]+O(q^{-4}),\\&p_1=q+\tfrac{1}{2}\Delta,\quad p_2=-q+\tfrac{1}{2}\Delta.\end{aligned} \tag{19}$$

This can be simplified with the help of the broken-scale-invariance Ward identities of Eqs. (6) and (16) to

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma^{(6)}(p_1\cdots p_6)\sim&\,{}^{[n-1]}C_{\phi^2}(q)\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma_{\phi^2}^{(4)}(\Delta;p_3\cdots p_6)\\&+{}^{[n]}\Gamma_{\phi^2}^{(4)}(\Delta;p_3\cdots p_6)\left[-\left(\beta\frac{\partial}{\partial\lambda}+2\gamma-\gamma_{\phi^2}\right){}^{[n-1]}C_{\phi^2}(q)-i\,{}^{[n-1]}\hat{C}_{\phi^2}(q)\right]\\&+\sum_{i=1}^{5}{}^{[n-1]}C_{B_i}(q)\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma_{B_i}^{(4)}(\Delta;p_3\cdots p_6)\\&+\sum_{i=1}^{5}{}^{[n]}\Gamma_{B_i}^{(4)}(\Delta;p_3\cdots p_6)\left[-\left(\beta\frac{\partial}{\partial\lambda}+2\gamma\right){}^{[n-1]}C_{B_i}(q)-\sum_{j=1}^{5}\gamma_{ji}^{B}\,{}^{[n-1]}C_{B_j}(q)\right]+O(q^{-4}),\\&p_1=q+\tfrac{1}{2}\Delta,\quad p_2=-q+\tfrac{1}{2}\Delta.\end{aligned} \tag{20}$$

The expressions in square brackets can then be simplified with the help of Eqs. (17a)–(17c) as follows. Since $\Gamma_{\phi^2}^{(4)}$ has an expansion beginning at $O(\lambda^2)$, the square bracket multiplying it is needed only to $O(\lambda^{n-2})$;

since C_{ϕ^2} is guaranteed to satisfy Eq. (17a) to $O(\lambda^{n-1})$, this square bracket may therefore be replaced by $\mu(\partial/\partial\mu)^{[n-2]}C_{\phi^2}$. Similar remarks are true for the square brackets multiplying the $\Gamma^{(4)}_{B_i}$, with one exception: $\Gamma^{(4)}_{\phi^4}$ has an expansion in λ beginning at λ^0, so that the square bracket multiplying it is needed to $O(\lambda^n)$. If we use Eq. (17b) to *determine* ${}^{[n]}C_{\phi^4}$, the square bracket can be replaced by $\mu(\partial/\partial\mu)^{[n]}C_{\phi^4}$. There will then be an undetermined additive constant of integration in C_{ϕ^4} which can easily be seen, because C_{ϕ^4} is $O(q^{-2})$ and has dimension -2, to be of the form constant $\times\lambda^n q^{-2}$. The net effect of this is that

$$\mu\frac{\partial}{\partial\mu}\,{}^{[n]}\Gamma^{(6)}(p_1\cdots p_6)\sim\mu\frac{\partial}{\partial\mu}\left(C_{\phi^2}(q)\Gamma^{(4)}_{\phi^2}(\Delta;p_3\cdots p_6)+\sum_{i=1}^{5}C_{B_i}(q)\Gamma^{(4)}_{B_i}(\Delta;p_3\cdots p_6)\right)+O(q^{-4}), \tag{21}$$

where the expression in large parentheses is expanded out to $O(\lambda^n)$. At this point we can undo the differentiation with respect to μ, getting

$${}^{[n]}\Gamma^{(6)}(p_1\cdots p_6)\sim C_{\phi^2}(q)\Gamma^{(4)}_{\phi^2}(\Delta;p_3\cdots p_6)+\sum_{i=1}^{5}C_{B_i}(q)\Gamma^{(4)}_{B_i}(\Delta;p_3\cdots p_6)+F(q,\Delta,\ p_3\cdots p_6)+O(q^{-4}), \tag{22}$$

where, again, the expression in large parentheses is expanded out to $O(\lambda^n)$ and F is $O(q^{-2})$, $O(\lambda^n)$, and satisfies $(\partial/\partial\mu)F=0$. The same sort of argument as was made in the sequel to Eq. (13) allows one to conclude that F is actually of the form constant $\times\lambda^n q^{-2}$. But this is just the right form to be eliminated by adjusting the integration constant in our determination of ${}^{[n]}C_{\phi^4}$. Therefore, we have at the same time shown that ${}^{[n]}\Gamma^{(6)}$ does satisfy Eq. (14) and determined ${}^{[n]}C_{\phi^4}$ so that it satisfies Eq. (17b).

To complete the induction step, it is necessary to show that $\Gamma^{(4)}$ and $\Gamma^{(4)}_{\phi^2}$ satisfy the proper equations. The argument is just the same as the one we have gone through, with minor modifications, so we shall not bother to write it out. An important point, which is easy to check, is that the various integration constants, analogous to F in the last equation, can consistently be absorbed by adjusting the integration constants in the expansion functions C_{B_i}, etc. It is at this point that the necessity for the "contact term" proportional to $\hat{C}_{\phi^2}$ in Eq. (15) becomes fully apparent: A term of that structure is necessary to absorb one of the integration constants which arises. Again, to complete the proof, it is necessary to find a starting point for induction, and this is trivially supplied by lowest-order perturbation theory.

Although the detailed steps of the proof become quite cumbersome in $O(q^{-2})$, it is clear that basically the same operations must be gone through as in the proof of the expansion to $O(q^0)$. It is also clear how the proof of a general order would go, although in the absence of a supercompact notation, writing out the explicit steps would be very difficult.

IV. CONCLUSION

The most important thing to notice about these arguments is that the essential steps in the proof of the $O(q^0)$ and $O(q^{-2})$ terms in the asymptotic expansion are identical and logically determined only by the structure of the broken-scale-invariance Ward identities. For this reason, we are convinced that our argument can be extended to prove the general term in Eq. (2). It also seems clear that, since broken-scale-invariance Ward identities are not peculiar to $\lambda\phi^4$ theory, but can be derived in any renormalizable theory, expansions such as Eq. (2) can be proven in the context of other theories. This equation is, of course, not of much interest in itself, since there is no way of measuring off-mass-shell particle Green's functions. However, since broken-scale-invariance Ward identities exist for more interesting quantities, such as amplitudes involving currents, we expect to be able to derive for them expansions quite similar to Eq. (2). We have concentrated our attention on particle amplitudes since we wanted to make our point about the connection between broken-scale-invariance Ward identities and asymptotic behavior in the simplest possible, if not the most realistic, context. We hope that in this context, the general power of these Ward identities for studying asymptotic behavior has been made very clear.

It should be noted that the limit we have studied is not entirely without practical interest, at least when applied to virtual Compton scattering amplitudes. On the one hand, the behavior of a matrix element of two currents, for large spacelike momenta carried by the currents, determines whether or not radiative corrections to a given process are finite. On the other hand, the same asymptotic behavior can be used to derive sum rules for inelastic weak production and electroproduction in the manner of the original Bjorken limit. The sum rules one would derive in this way are weaker than the classic Bjorken-limit sum rules, but have the virtue of being correct in perturbation theory.

There are, of course, other asymptotic limits of more direct phenomenological interest. In particular, it would be interesting to know what happens when the particles carrying large momentum are kept on their mass shell. It is conceivable, but not obvious, that methods similar to those we have discussed can be applied usefully to this case. It would also be interesting to know if these methods have anything to say about the so-called light-cone expansion,[10] which is of direct interest in studies of high-energy electroproduction.

Although we have not given an exhaustive discussion, we hope that we have shown that the broken-scale-invariance Ward identities are a powerful tool for investigating at least certain kinds of high-energy behavior. Since these Ward identities are a consequence only of the renormalization structure of perturbation theory, they must incorporate essentially the same information as the renormalization group. However, the expression of this information in the form of Ward identities seems technically superior to the old-fashioned renormalization-group arguments since it enables us to discuss very simply quite complicated situations. Therefore, although we have been discussing what is in a certain sense old physics, it is interesting to see how a new point of view allows us to extract new consequences.

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Light-Cone Structures and Sum Rules as Seen in the Parton Dual-Resonance Model*

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We propose a simple and unique asymptotic symmetry scheme to incorporate spins, SU(3), *CPT* invariance, and the generalized Bose statistics into the parton dual-resonance model for deep-inelastic lepton-hadron reactions. The scheme embodies Harari-Rosner quark duality diagrams, adopts Chan-Paton SU(3)-symmetry factors, identifies Mandelstam, Bardakci, and Halpern's multiplicative quark model in the Bjorken limit, and generates Fritzsch and Gell-Mann's light-cone structures identically. The dynamical (parton-dual-resonance-model) approach is therefore unified with the symmetry (light-cone) approach, and the complementarity between these two is established. Light-cone sum rules are derived, and a new sum rule is suggested. As a further application of this scheme, we calculate the complete sets of bilocal light-cone structures occurring in the two-heavy-boson processes.

I. INTRODUCTION

It is generally believed that symmetries may become more exact in reactions at very high energies, since then the symmetry-breaking effects due to the masses of the external particles may be neglected. The light-cone approach[1] to the deep-inelastic processes is a beautiful example manifesting this belief. Fritzsch and Gell-Mann's abstractions[2] from the free quark model of the light-cone commutators, and of the bilocal light-cone algebra, are in fact based on this point of view. However, the weakness (also the beauty) of this approach, is the inability in dealing with the dynamical aspects of these processes under discussion.

Entirely at the other extreme, the parton dual-resonance model[3] for deep-inelastic lepton-hadron