[1]P. A. M. Dirac, Proc. Roy. Soc. (London) A133, 60 (1931).
[2]A near complete list of references to the theoretical and experimental literature on monopoles can be found in H. V. McIntosh and A. Cisneros, J. Math. Phys. 11, 896 (1970).
[3]L. I. Schiff, Phys. Rev. 160, 1257 (1967).
[4]A. Peres, Phys. Rev. Letters 18, 551 (1967).
[5]J. Schwinger, Science 165, 757 (1969).
[6]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).
[7]A brief and simple discussion of this point can be found in Ref. 4.
[8]G. Wentzel, Progr. Theoret. Phys. (Kyoto), Suppl. 37 and 38, 163 (1966).
[9]C. R. Hagen, Phys. Rev. 140, B804 (1965).
[10]See, for example, V. Fock, *Theory of Space-Time and Gravitation* (Pergamon, London, 1964), Chap. II, Sec. 26, pp. 77–83.
[11]R. Utiyama, Phys. Rev. 101, 1597 (1956).
[12]For the mathematics which underlies a global description of the Yang-Mills field see, for example, S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, London, 1963), Vol. I, Chap. III, pp. 113–153.
[13]This is an adaption of an argument due to A. Peres, Phys. Rev. 167, 1449 (1968); see also H. J. Lipkin, W. I. Weisberger, and M. Peshkin, Ann. Phys. (N.Y.) 53, 203 (1969).
[14]W. Miller, Jr., *Lie Theory and Special Functions* (Academic, New York, 1968), Chap. 6, Theorem 6.3, p. 259. These quantum numbers refer to the unitary representations of the simply connected covering group of E(3). For E(3) itself, s must be an integer. It is not obvious from the outset which choice is required physically and indeed this depends on the choice of $\vec{A}$.
[15]Schwinger's quantization in which half-integer values of eg are excluded [J. Schwinger, Phys. Rev. 144, 1087 (1966)] is obtained if the additional requirement that the magnetic field be a pseudovector be imposed. For a monopole field this requires g to be a pseudoscalar and the remaining part of $\vec{A}$ to be antisymmetric under parity. Thus, with the properly symmetrized form of (3.12) exponentiation of the infinitesimal generators yields the Euclidean group rather than its covering group and consequently half-integer values of eg are excluded from the representation.
[16]See, for example, M. A. Naimark, *Normed Rings* (Noordhoff, The Netherlands, 1964), Chap. I, p. 102. In particular, a self-adjoint operator is required to have a dense domain in the underlying Hilbert space.
[17]The explicit calculation can be found in Ref. 2.
[18]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, London, 1965), Chap. V, Sec. 36, pp. 123–125.
[19]M. Berrondo and H. V. McIntosh, J. Math. Phys. 11, 125 (1970).
[20]A. Joseph, Progress Report, Wave Mechanics Group, Oxford, Report No. 15, 1969 (unpublished).

PHYSICAL REVIEW D VOLUME 5, NUMBER 2 15 JANUARY 1972

Space-Time Code. II

David Finkelstein*
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033
(Received 8 February 1971)

Quantum concepts can be applied to space-time processes to make a quantum (q) theory that is free of the possibility of divergencies inherent in classical continuum theories, yet causal, Lorentz-invariant, and asymptotically Poincaré-invariant for large times. A general technique, algebraic quantization, is provided for going from classical (c) paradigms, typically discrete logical structures, to q analogs. Applied to the two-dimensional c checkerboard, algebraic quantization gives a q theory of time and space asymptotic to the four-dimensional Minkowski c theory in the limit of large time. Applied to the simplest dynamics on such a checkerboard, a piece that makes the same move again and again, algebraic quantization gives a q dynamics asymptotic to a massless spin-$\frac{1}{2}$ two-component dynamics in the same limit. The quantum of time, if it exists, must have spin $\frac{1}{2}$. Some features of general relativity such as curvature seem plausible consequences of a quantum theory of space-time processes.

I. INTRODUCTION

We turn now from quantum geometry to quantum dynamics.[1] The development of physics, if we see it right, looks like

$$\cdots c \cdots cq \cdots q.$$

The c period was the epoch of Newton, Faraday, Maxwell, Einstein. In it, time and matter were both classical systems.

The cq period is now, the time of Heisenberg, Dirac, Tomonoga, Schwinger, and Feynman; a brief interregnum during which vigorous hybrids of c (classical) time and space with q (quantum) matter have been created. In the cq period we have learned to specify the kinematics of matter M

no longer by a phase space but by the algebra M^A of physical quantities belonging to M. (We take an algebra to have +, ×, *, and the complex numbers $\mathbb{C}$. The algebra S^A of a set S: = the collection of complex functions $S \to \mathbb{C}$.) In the cq period dynamics is still given, as in the epoch c, by a map of $T \times M \to M$, where T is time, the real-number axis in the simplest theories, the future timelike Minkowski cone in more relativistic ones. In cq, T is c and M is q. The × in $T \times M$ is the continental divide of present-day physics. The flow of theories of T brings us to general relativity, that of M to quantum mechanics. Discovering the unity of these two bodies of knowledge is still the main problem before physics today.

The q period toward which we have been feeling our way of late is one in which both time and matter are q systems, if it will still be worthwhile to regard them as separate entities at all.[2]

For I believe that the separation between T and M must be an artificial one, indeed, we never meet T without M or M without T. There must be one point of view from which both T and M are seen as part of one deeper concept. Here we attempt to express everything in terms of one object we call the dynamical *process*. Then time emerges as the number of elementary *processes* in sequence, and matter is to be constructed from the *processes* of our interaction with it. The processes of a c dynamical system form a c system. Those of a cq dynamical system form a mixed or cq system. Here we consider a q dynamical system, whose processes form a q system.

Such a dynamical process is an ensemble of many quanta and has a complex q logical structure. However, the procedures of q logic, those of one system dealt with by Birkhoff-von Neumann and those of many systems developed in the physics of q ensembles, all are analogs of familiar c paradigms. For each q logical structure these paradigms form a "g theory," a general logical skeleton which we flesh with q meat.

We have now spoken of four kinds of theory we call c, cq, q, and g. c is the approximate theory usually called the classical limit; cq is the approximate theory of the present-day quantum-mechanical type; q is supposed to be the true theory sought; g is a c theory of the general structure of q. We must provide the following paths between these worlds:

(1) g → q. We go from g paradigms to q physical theories by an algorithm described below called *algebraic quantization*.

(2) q → cq. From q to cq is a limiting process in which the number of quanta of time – chronons – is permitted to go to ∞, or more conveniently a constant τ, the quantum of time, is permitted to go to 0. This we call the *limit of classical time*.[3]

(3) cq → c. From cq to c is a limiting process in which the constant of action $\hbar$ is allowed to go to 0. This is the *limit of classical mechanics*.

Canonical quantization and its various forms due to Dirac, Schwinger, and Feynman[4] are ways to undo the limiting process $\hbar \to 0$, to reconstruct cq from c. There can be no confusion between canonical quantization c → cq and algebraic quantization g → q. If we are right, moreover, algebraic quantization is sufficient in itself, and there is never any need or sense for a canonical quantization following it.

The model of time given here is possibly wrong, certainly incomplete. We have more confidence in the general direction: For how could the one world be half classical, half quantum? Here we only correct some definite errors in STC (Space-Time Code),[1] express the essential concept of a q dynamics, and give some models. Nevertheless, some conclusions do seem to stand out.

II. DYNAMICS

What is a dynamical system?

In most present-day physical theories, the laws of dynamics are expressed by a one-parameter group of automorphisms of the physical system. The parameter is time and is a c quantity from the start. The action of the passage of time T upon the system M is defined by a map

$$T \bigcirc M:\ T \times M \to M. \tag{1}$$

Instead we consider a dynamical system as one that is subject to and defined by a system of *processes*. We express the laws of dynamics by giving the system of all kinematically possible processes Π; an associative composition (where the subscripts merely order the Π's)

$$\Pi_2 \bigcirc \Pi_1:\ \Pi_2 \times \Pi_1 \to \Pi \tag{2}$$

giving the resultant of two processes performed sequentially; and a preferred class D of processes, those that are dynamically allowed.

We call such a triple $(\Pi, \bigcirc, D)$ (and improperly the system Π itself) a *dynamical process*. The pair $(\Pi, \bigcirc)$ alone are said to constitute a *kinematical* process. The kinematics defines what *might* happen, the dynamics, what *does*.

For a c example, consider a point particle moving in 3-space with the passage of a discrete time $t = 0, \tau, 2\tau, \ldots$. The *path* π is then a sequence of points,

$$\pi = (x_0, x_1, x_2, \ldots).$$

However, the point x_1 can be regarded as obtained from x_0 by the process of adding the vector displacement $\Delta x_0 \equiv x_1 - x_0$. The path π itself can equally be defined by x_0 and the sequence of displacements or *process*

$$\Pi = (\Delta x_0, \Delta x_1, \ldots, \Delta x_N),$$

where $\Delta x_n \equiv x_{n+1} - x_n$. The path π abstracted from its initial point x_0 is the process Π, the sequence of displacements to which the particle is subject. The law of composition $\Pi_2 \circ \Pi_1$ of processes is *concatenation*, the stringing together of two sequences of displacements to form one. The class D of dynamically allowed processes for a free particle is the class of all sequences with

$$\Delta x_0 = \Delta x_1 = \cdots = \Delta x_N.$$

We call sequences whose terms are all equal *diagonal*, and then in this example D is the class of diagonal sequences. This dynamics resembles the nonrelativistic dynamics of a free point particle.

For a simpler example of a c dynamical process we take one appropriate to a man in the game of checkers, which is known to be relevant to the Dirac equation.

We take Π to be the set of binary sequences $\chi_1, \chi_2, \ldots$ with $\chi_i = 0, 1$, and define the product $\Pi_2 \circ \Pi_1$ by concatenation of sequences:

$$(00) \circ (111) = 00111.$$

Any such process we call a c *binary process*. This defines the kinematics.

We take for the dynamics the class D defined by

$$\chi_{i+1} = \chi_i, \quad i = 0, 1, \ldots$$

which singles out as allowed all diagonal (bishop's) paths.

We wish now to consider dynamical theories in which Π is a q system. What shall be meant by a composition $\Pi_2 \circ \Pi_1$ in that case? And by the associative law? These are purely logical concepts, and call for a further development of q logic. We give next a general procedure for such developments.

III. ALGEBRAIC QUANTIZATION

Here we give an algebraic technique for going from c paradigms to q analogs, from g to q. It is intended for the construction of a q-process dynamics from a c one, but it also provides q counterparts for many concepts of c physics that can be expressed in terms of sets such as phase spaces belonging to the systems, and maps between these sets. The generalization rules are, in the order of their performance:

(1) Coordinatize. Each set is replaced by its algebra of quantities (coordinates), each map is reversed in direction.

(2) Generalize. Replace each algebra, commutative in virtue of its construction, by a general algebra, not necessarily commutative.

(3) Specialize. Replace each algebra by an irreducible one to single out the pure quantum.

A system, regarded kinematically, may be identified with the algebra of its quantities, except that the two map contragrediently. This algebra is commutative for c systems, general for general systems, irreducible for q systems.

The idea of defining a system by its algebra rather than a phase space or Hilbert space stems from Heisenberg and Dirac who in the earliest days of quantum mechanics defined the kinematics of a q particle on a line by the commutation relations

$$pq - qp = \hbar / i,$$

which is to say by the abstract algebra generated by p and q subject to this relation.[5]

This general quantization algorithm cannot be confused with the more special canonical quantization of Hamiltonian dynamical systems. If we try to apply this rule to Hamiltonian systems, we only bring back to consciousness the grotesque internal logical structure buried in any continuum-based physical theory.

Some results of algebraic quantization follow. Most of these results have long been used informally in q physics, but to familiarize ourselves with the present terminology we may work one out here.

Consider for example the concept of the product of two systems S_1, S_2, which we designate by $S_1 \times S_2$. (Imagine this concept appears in some classical form of theory, a paradigm, and we are to provide a corresponding q form, its analog.) In a c theory S_1 and S_2 are presented as sets, and $S_1 \times S_2$ is the set of (ordered) pairs (m, n), for all $m \in S_1$, $n \in S_2$. What shall $S_1 \times S_2$ mean in a q theory, where S_1 and S_2 may be given by Hilbert spaces? We compute as follows:

(1). Coordinatize. Question: How is the algebra of $S_1 \times S_2$ formed from those of S_1, S_2 in c theory? Answer: $(S_1 \times S_2)^A = S_1^A \times S_2^A$ in the c theory. We multiply (commutative) algebras when we multiply sets.

(2) Generalize. Question: What operation (functor) on general algebras reduces to the product for commutative algebras? Answer: The "commutative product," the product in which the elements of the two algebras are taken to commute with each other.

(3) Specialize. Question: What does this commutative product of algebras reduce to when the factors in it are the algebras of two Hilbert spaces H_1, H_2? Answer: The algebra of the direct or tensor product $H_1 \times H_2$. Thus we multiply Hilbert

spaces in q theory when we multiply sets in the c paradigm. Algebraic quantization of the product of sets gives the (direct or tensor) product of Hilbert spaces. This is one entry in the tabulation that follows. The others are obtained similarly.

It is required that the steps 1, 2, and 3 be intrinsic or natural. Natural is a concept of categorical algebra. We begin with the map $S \to S^{\mathrm{A}}$ from sets to their algebras. This is natural in the following sense: When a map of S into another set T is carried out, a map of algebras (a unit preserving, $*$ preserving homomorphism) is induced (from T^{A} into S^{A}). A natural map such as this map from sets to their algebras is called a functor. We always enter step 1 with some functor of sets, and leave it with a functor of commutative algebras. In the example, we entered with $S_1 \times S_2$, a functor from two sets to one, and left with $S_1^{\mathrm{A}} \times S_2^{\mathrm{A}}$, a functor from two commutative algebras to one.

Thus in step 1 we seek a correspondence of functors. This correspondence too is to be natural. It is then called a functor map (natural transformation, functorial morphism).

In step 2 we extend a functor from commutative algebras to algebras.

In step 3 we restrict a functor from algebras to irreducible algebras. The given correspondence from Hilbert spaces to their algebras is again natural, a functor.

We may seek also a functor of Hilbert spaces, the product in the above example, with a natural map (functor map) to a specified functor of algebras, that resulting from step 2, if we wish finally to express our concepts in Hilbert space rather than algebraic terms.

The use of the categorical algebra of Eilenberg and Maclane made here, though rudimentary, was essential. It would seem that categorical algebra is for metaphysics (as it is for metamathematics) much what group theory is for physics. The use of categorical algebra was suggested by a remark of Giles and Kummer[2] and saved much trial and error.

Let us now apply algebraic quantization to the systematic construction of a q logic adequate for q dynamics.

IV. A SYSTEM OF q LOGIC

The extensions of q logic needed for q dynamics are summarized here. Section IV A gives the calculus of classes for a *simple* system pretty much as in the work of Birkhoff-von Neumann, and some ways to make a new system out of an old one. Section IV B deals with the calculus of classes of a *compound* system, a theory of binary relations, and ways to make compound systems out of a simple one. Section IV C gives ways to make *complex* systems out of a simple one.

Our format tabulates each concept followed by: its general algebraic form; (c) its c form; (q) its q form; and possibly remarks.

A. One System

A *system* S to which all concepts defined in this part are implicitly relative is given in general by an algebra S^{A} (always with $*$ operation and the complex numbers $\mathbb{C}$ understood), the *quantities* of S. A system map $m: S_1 \to S_2$ is given by a contragredient algebra map $m_{\mathrm{A}}: S_2^{\mathrm{A}} \to S_1^{\mathrm{A}}$. For c systems, S^{A} is commutative, the algebra of all functions on an *underlying set* $I(S)$. For q systems S^{A} is irreducible, the algebra of all maps of an *underlying inner-product space* $\hat{I}(S)$. In this part all concepts defined are relative to one implicit system S. Purely q concepts are shown under the caret.

Class (of a system S): =projection (quantity equal to its $*$ and square) in S^{A}; (c) a subset $P, Q, \ldots$ of the underlying set of S; (q) subspace $\hat{P}, \hat{Q}, \ldots$ of the underlying linear space of S^{A}.

$P \subset Q$, P is *included* in Q (of classes P, Q): =the basic eigenvalue equation $PQ=P$; (c) the subset inclusion $P \subset Q$; (q) the subspace inclusion $\hat{P} \subset \hat{Q}$.

I and $\emptyset$, ***universal*** and ***null*** class: =quantities 1 and 0; (c) $I(S)$ and $\emptyset$; (q) $\hat{I}(S)$ and the 0 vector, as subspaces of $\hat{I}(S)$.

$P \cup Q$, P or Q (*adjunction*): $=\sup(P, Q)$; (c) set join $P \cup Q$; (q) span $\hat{P} \cup \hat{Q}$ (the set join of two subspaces never being required).

$P \cap Q$, P and Q (*conjunction*): $=\inf(P, Q)$; (c) set meet $P \cap Q$; (q) subspace meet $\hat{P} \cap \hat{Q}$.

Q is a ***complement*** of P: $=P \cup Q = I$, $P \cap Q = \emptyset$; (c) is the set complement of P; (q) $\hat{Q}$ is a complementary subspace to $\hat{P}$.

$\sim P$, the ***negation*** of P: $=1-P$; (c) same as complement of P; (q) orthogonal complement of subspace $\hat{P}$.

$P \perp Q$, P ***excludes*** Q: $=PQ=0$; (c) $P \cap Q = 0$; (q) $\hat{P}$ and $\hat{Q}$ are orthogonal subspaces.

$P \operatorname{com} Q$, P is ***compatible*** or ***commutes*** with Q: $=P \cap (Q \cup \sim Q) = (P \cap Q) \cup (P \cap \sim Q)$, or $PQ = QP$; (c) trivially true for all classes; (q) a basis exists for $\hat{I}(S)$ adapted to both subspaces $\hat{P}$ and $\hat{Q}$.

$f(S)$, a ***coordinate*** f of S: =map $f: S \to \mathbb{C}$; (c) complex function on $I(S)$; (q) spectral family $d\hat{P}_f(z)$ of subspaces, z a complex variable. Any coordinate f may be represented by a coordinate quantity $f = \int z\, d\hat{P}_f(z)$, where the projection-valued measure $d\hat{P}_f(z)$ is defined by the algebra map $f_{\mathrm{A}}: \mathbb{C}^{\mathrm{A}} \to S^{\mathrm{A}}$.

$P \subset_1 Q$, P is *just included* in Q: $= P \subset X \subset Q$ if and only if $P = X$ or $X = Q$; (c) $Q = P \cup$ one additional point; (q) $\hat{Q} = \hat{P} \cup$ one additional 1-space.

$|P|$, the *measure* of P: = the length m of a chain $0 \subset_1 P_1 \subset_1 \cdots \subset_1 P_m = P$.

σ, a *singlet*: = projection σ with $|\sigma| = 1$; (c) point of $I(S)$; (q) a ray or 1-space of $\hat{I}(S)$.

If G is any group of maps $g: S \to S$ and G_A is the group of induced algebra maps, we then define as follows:

S/G, S *over* G: = the algebra $S^A \backslash G_A$, the collection of those quantities of S^A invariant under G_A; (c) the point set of equivalence classes of $I(S)$ modulo G. (q) the algebra of operators on $\hat{I}(S)$ commuting with all members of the (unitary) group G. Even if S is a q system, S/G generally is not.

$S \backslash G$, S *under* G: = the algebra S^A/G_A resulting from S^A by identification with respect to G_A; (c) the subset of $I(S)$ consisting of all fixed points under G; (q) no simpler expression available.

Let P be a class of S:

$S \backslash P$, S *under* P, the *restriction* of S to P: = the algebra PS^AP taken with the $+$, $\times$, $*$ of S^A but with the new unit P; (c) the subsystem defined by a subset $P \subset I(S)$; (q) the subsystem defined by a subspace $\hat{P} \subset \hat{I}(S)$.

The system 1: = the system whose algebra is $\mathcal{C}$; (c) phase space with just one point; (q) system with a one-dimensional Hilbert space. The system 1 is both a c system (commutative) and a q system (irreducible).

B. Two Systems

$S + T$, the *sum* of (systems) S and T: = the direct-sum algebra $S^A + T^A$; (c) the disjoint union $I(S) + I(T) = I(S+T)$; (q) the direct-sum Hilbert space $I(S) + I(T) = I(S+T)$ with projections on $I(S)$, $I(T)$ as superselections. For q systems there is also a *coherent sum* $S \hat{+} T$, represented by the direct-sum Hilbert space (without superselection). $\sum S_i$ and $\hat{\sum} S_i$ are defined similarly.

The system n : $= 1 + \cdots + 1$ (n terms).

The system $\hat{n}$: $= \hat{1} \hat{+} \cdots \hat{+} \hat{1}$ (n terms).

$S \times T$, ST, the *product* of (systems) S and T: = the direct-product algebra $S^A T^A$, in which the two subalgebras S^A, T^A commute; (c) the Cartesian product $I(S) \times I(T)$; (q) the direct-product Hilbert space $\hat{I}(S) \times \hat{I}(T)$. Similarly for $\prod S_i$. Associative and distributive laws hold for $\times$ and $+$, and for $\times$ and $\hat{+}$.

$S \mathcal{R} T$, a binary relation $\mathcal{R}$ between systems S, T: = a class of ST; (c) subset of the Cartesian product $I(S) \times I(T)$; (q) subspace of the direct product $\hat{I}(S) \times \hat{I}(T)$.

$S \sim T$, *similar* systems S, T: = two systems S, T provided with an equivalence map $e: S \to T$ (map with inverse); (c) two sets with a 1-1 map onto e: $\hat{I}(S) \to \hat{I}(T)$; (q) two Hilbert spaces with a unitary e: $\hat{\underline{I}}(S) \to \hat{\underline{I}}(T)$. We designate corresponding projections in S, T by $P(S) \sim P(T)$. *Replicas* of a system S are similar systems obtained from S by attaching labels, e.g., $S_1 \sim S_2$.

$S \equiv T$: = for similar systems $S \sim T$, the class $\cup_\sigma \sigma(S)\sigma(T)$, the union extending over all singlets $\sigma(S) \sim \sigma(T)$; (c) diagonal subset of the Cartesian product; (q) symmetric subspace of the direct product.[6]

Reflexive relation: = relation $S \mathcal{R} T$ with $(S \equiv T) \subset (S \mathcal{R} T)$; (c) subset of $I(S) \times I(T)$ including the diagonal; (q) subspace of $\hat{I}(S) \times \hat{I}(T)$ including the symmetric subspace.

R^T, the *transpose* of R: $= e \times e^{-1}(R)$ where $e: S \to T$ is the equivalence map of $S \sim T$ and $\mathcal{R} = S \mathcal{R} T$.

Symmetric relation: = relation $S = S^T$.

Transitive relation: = relation $\mathcal{T}$ with $S_1 \mathcal{T} S_2 \quad S_2 \quad S_3 \subset S_1 \mathcal{T} S_3$.

Functional relation: = relation $S \mathfrak{F} T = \cup_\sigma \sigma f_A(\sigma)$, where σ ranges over the singlets of S, and $f: S \to T$ is a map; (c) the graph $\cup_\sigma (f(\sigma) \times \sigma)$ of a point map $f: I(S) \to I(T)$; (q) the graph $\cup_\sigma (\hat{\sigma} \times f_A(\hat{\sigma}))$ of an algebra map f_A: $T^A \to S^A$.

$\mathrm{seq}_2 S$, the 2-*sequence* of S's: = the product $S_1 S_2$ of two replicas $S_1 \sim S_2$ of S; the ordered pair of two S's.

$\mathrm{dia}_2 S$, the *diagonal* 2-sequence of S's: $= \mathrm{seq}_2 S \backslash [S_1 \equiv S_2]$, the restriction of $S_1 S_2$ to the class $[S_1 \equiv S_2]$. (c) the diagonal of $I(S_1) \times I(S_2)$; (q) the subspace of symmetric tensors in $\hat{I}(S_1) \times \hat{I}(S_2)$; $\mathrm{dia}_2 S \sim S$ for c but not q systems.

Let G be the symmetric group on two similar systems, $S_1 \sim S_2$.

$\mathrm{ser}_2 S$, the 2-*series* of S's: $= \mathrm{seq}_2 S/G$ with G as above; (c) the set of unordered pairs of points of $I(S)$; (q) the subalgebra of $S^A \times S^A$ invariant under transposing; the direct sum of the subalgebras of the symmetric and antisymmetric subspace of $\hat{I}(S) \times \hat{I}(S)$.

C. Many Systems

$\mathrm{seq}_n S$, the n-*sequence* of S's: $= \prod_m S_m$ ($m = 1, \ldots, n$), where $S_m \sim S$ are similar systems; (c) the Cartesian product of n replicas of $I(S)$; (q) the direct product of n replicas of $\hat{I}(S)$.

seq$S$, the *sequence* of S's : $=\sum_n \text{seq}_n S$ ($n=0, 1, \ldots$); (c) the disjoint union of the $\text{seq}_n S$; set of all terminating sequences of points of $I(S)$. (q) the Maxwell-Boltzmann Fock space over $\hat{I}(S)$, with the number operator N as superselection rule. For q systems we also define the coherent sequence $\hat{\text{s}}\text{eq}S$: $=\hat{\sum}_n \text{seq}_n S$ ($n=0, 1, \ldots$).

$\text{dia}_n S$, the *diagonal* n-sequence of S's : $=\text{seq}_n S \backslash [S_1 \equiv \cdots \equiv S_n]$; (c) set of sequences of n repetitions of one point of $I(S)$; (q) the space of symmetric tensors of degree n over $\hat{I}(S)$. For c systems but not q, $\text{dia}_n S \sim S$.

dia$S$, the *diagonal* sequence of S's : $=\sum_n \text{dia}_n S$ ($n=0, 1, \ldots$); (c) set of terminating sequence of repetitions of one point of $I(S)$; (q) the Bose-Einstein Fock space over $\hat{I}(S)$, with the number of systems N as superselection. For c systems but not q, dia$S \sim NS$, where N is the system of the non-negative integers. For q systems we also define $\hat{\text{d}}\text{ia}S$, the *coherent* diagonal sequence of S's : $=\hat{\sum}_n \text{dia}_n S$, represented by the Bose-Einstein Fock space over $\hat{I}(S)$ without superselection.

Let G be the symmetric group on the systems in a sequence seq$S$. Then we define as follows[7]:

ser$S$, the series of S's : $=\text{seq}S/G$. (c) the unordered sequence of S's; (q) the subalgebra of seqS^A invariant under G.

V. A CATEGORICAL DUALITY

If S is a system and G is a group of maps of $S \to S$, we have noted two G-invariant systems within S, the quotient system S/G (for which G is ignored), and the invariant subsystem $S\backslash G$ (for which G is ignorable), with $S\backslash G \subset S/G$. For example, if S is the plane and G is its rotation group, then $S\backslash G$ is the origin 0, and S/G is the set of all concentric circles and 0.

The processes $/G$ and $\backslash G$ are dual. The subsystem has the quotient algebra; the quotient system has the subalgebra:

$$(S/G)^A = S^A \backslash G_A,$$

$$(S\backslash G)^A = S^A / G_A,$$

where $G_A : S^A \leftarrow S^A$ is the map of quantities induced by the map $G: S \to S$, $S^A\backslash G_A$ is the subalgebra of S^A made up of all quantities invariant under G_A, and S^A/G_A is the algebra made from S^A by adjoining all the relations $q = gq$ for all quantities q of S^A and all group elements g of G_A. This kind of duality is familiar in categorical algebra.

VI. q DYNAMICS

A principal difficulty in q dynamics has been to understand the concept of one quantum system acting upon another, as T on M or Π on Π, in the present primitive state of q logic. We understand this now as follows. Consider two general systems X and Y. They are typically specified by their algebras of physical quantities, X^A and Y^A, respectively: commutative algebras for c systems, irreducible for q systems. We wish to define a map $X \to Y$. The algebra X^A is a contravariant functor of the system X. Therefore a map $X \to Y$ is given by a map $Y^A \to X^A$.

Even the classical dynamics (1) is equivalently and familiarly expressed by a map

$$M^A \to T^A \times M^A \qquad (1^A)$$

contragredient to (1), mapping each quantity q of the system M into its time-dependent form $q(t)$. Here q is in M^A, $q(t)$ in $T^A \times M^A$. And the familiar cq dynamical law $q(t) = e^{iHt} q e^{-iHt}$ is of the form (1^A).

Likewise a process dynamics (2) may be understood as a contragredient map $\Pi_2 \bigcirc \Pi_1$:

$$\Pi^A \to \Pi^A \times \Pi^A \qquad (2^A)$$

and a projection D in Π^A. This is our concept of a general process dynamics. The specialization to q-process dynamics is clear.

The c-process dynamics includes Newtonian dynamics as special case, evidently. The cq-process dynamics includes the Heisenberg dynamics as a special case, as may be shown from the work of Feynman and Dyson. There the projection D is constructed from the Feynman-path amplitude. It seems to us most natural to describe the world by a q-process dynamics.

It is straightforward but instructive to discover by algebraic quantization what it means for a q product $\Pi_2 \bigcirc \Pi_1$ to obey such laws as closure, associativity, and commutativity. It is a matter of setting down the map diagrams that express these notations for a c system Π, replacing Π by its algebra of quantities Π^A, and reversing all the arrows. When all this is done, associativity is expressed by $(\Pi_3 \bigcirc \Pi_2) \bigcirc \Pi_1 = \Pi_3 \bigcirc (\Pi_2 \bigcirc \Pi_1)$ and commutativity by $\Pi_2 \bigcirc \Pi_1 = \Pi_1 \bigcirc \Pi_2$. It is simply a matter of learning to read backwards; for example, remembering that $\Pi_2 \bigcirc \Pi_1$ is a map $\Pi^A \to \Pi^A \times \Pi^A$, not $\Pi^A \times \Pi^A \to \Pi^A$.

VII. q BINARY PROCESS

The model of time we now take up has a classical prototype in the game of checkers. Each man is confronted by a binary decision: forward to the left or forward to the right. We designate these by 0 or 1, the elements of the system we denote by 2. A process Π is a sequence of moves such as 010111:

$$\Pi = \text{seq}2,$$

and processes are multiplied to form products $\Pi_2 \bigcirc \Pi_1$ by concatenation. The possible displacements p undergone by a man are defined by processes Π with the order of moves ignored and form the system

$$p = \text{ser}2 = (\text{seq}2)/G,$$

where G is the group of move interchanges. This is the *future cone* of the theory. The *null cone* is the system

$$n = \text{dia}2$$

of paths which are diagonal sequences and appear as bishop's moves.

In the picture we started from these descriptions of Π, p, n, constitute the g theory, the paradigm. Now by algebraic quantization we form the corresponding q theory, the q *binary process*. We assume the alternatives 0 and 1 are coherent in the quantum sense by replacing 2 with $\hat{2}$. The system $\hat{2}$ is that described by the two-dimensional complex linear space $\{\chi\}$ with inner product $\chi^* \chi'$. We call the system $\hat{2}$ in this application the *chronon* χ. A process is a sequence of chronons,

$$\Pi = \hat{\text{s}}\text{eq}\chi.$$

The future displacement, the resultant of Π, is

$$p = \hat{\text{s}}\text{er}\chi = \Pi/G,$$

and the null cone is

$$n = \hat{\text{d}}\text{ia}\chi.$$

The algebras formed by the quantities of these three systems are

$$\Pi^A = \hat{\text{s}}\text{eq}\chi^A,$$

$$p^A = \hat{\text{s}}\text{er}\chi^A = \Pi^A/G_A,$$

$$n^A = \hat{\text{d}}\text{ia}\chi^A.$$

The process has as quantities the maps (linear operators) of the Maxwell-Boltzmann Fock space of an indefinite number of chronons. The quantities of p are the quantities of Π that are invariant under exchange of chronons. The quantities of n are the maps of the space $\hat{\text{d}}\text{ia}\{\chi\}$, the symmetric tensors over χ space, a Bose-Einstein Fock space for an indefinite number of chronons.

We take $p = \hat{\text{s}}\text{er}\chi$ and compute the limit of classical time. The classical limit is the algebra generated by the additive quantities of p, with commutators neglected in comparison to products. The additive quantities of p are the same as those of Π, and have the form

$$\Sigma(\alpha) = \sum_1^N \alpha_n,$$

where α_n is an isomorph for the nth chronon of a one-chronon quantity α. Since $\Sigma(\alpha)$ is linear in α, it can be expressed in the form

$$\Sigma(\alpha) = \text{tr}\rho\alpha = \sum \rho_{A^*B}\alpha^{A^*B},$$

where tr is a trace over χ space and the statistical operator ρ is a map of $\{\chi\} \times \text{ser}\{\chi\}$. With respect to any basis $\{\chi_A\}$ for the χ's, ρ is represented by a form (ρ_{A^*B}) whose components are additive quantities of $\hat{\text{s}}\text{er}\chi$, with the properties

$$\rho = \rho^*$$

$[\alpha = \alpha^*$ implies $\Sigma(\alpha) = \Sigma(\alpha)^*]$ and

$$\rho \geq 0$$

$[\alpha \geq 0$ implies $\Sigma(\alpha) \geq 0]$.

In the classical limit p^A is generated by the four quantities ρ_{A^*B} taken commutative. The conditions on ρ become

$$\rho = \rho^*, \quad \det\rho \geq 0, \quad \text{tr}\rho \geq 0.$$

These are more recognizable if we introduce four real combinations x^μ of the ρ_{A^*B}, taking

$$x^\mu = \text{tr}\rho\sigma^\mu,$$

where the σ^μ are, say, the 2×2 Pauli matrices, with $\sigma^0 = 1$. The conditions on ρ become

$$x^\mu = x^{\mu *}, \quad (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \geq 0, \quad x^0 > 0.$$

Therefore the displacements have the four-dimensional structure appropriate to special relativity. Spatial rotations are induced by unitary transformations of χ, which transforms according to the spin-$\frac{1}{2}$ representation: Chronons have spin $\frac{1}{2}$. To determine the composition $p \bigcirc p'$ of displacements we must consider the law of composition $\Pi_2 \bigcirc \Pi_1 = \Pi_3$ of processes. In the limit of many chronons the statistical operators ρ_1 and ρ_2 add for such a composition just in case the processes are uncorrelated, incoherent:

$$\rho_1 + \rho_2 = \rho_3. \tag{3}$$

This implies the addition of the four-component objects $x_1^\mu + x_2^\mu = x_3^\mu$. The system p in this limit is invariant under

$$\rho \to \Lambda^H \rho \Lambda$$

for arbitrary unimodular Λ, because the trace $\text{tr}\rho\alpha$ is a unimodular invariant as well as a unitary invariant. Under this transformation, x^μ undergoes the isochronous Lorentz group. Thus Poincaré invariance emerges at least with future translations.

It remains to provide the class D of dynamically allowed processes. The class D is a projection in Π^A. In theories with interaction D will typically be recursively defined, as it is defined by difference or differential equations in c theories. In the simplest theories it is possible to give D directly in closed form. For example, the simplest dynamical law in recursive form is, "make the same move again," a primitive expression of Newton's first law of motion. The g form of this recursive law is

(1) $\varnothing \in D, \quad \chi \in D$,

(2) $(\Pi \in D) \cap (\chi = X(\Pi)) \Rightarrow (\chi\Pi \in D)$,

where $X(\Pi)$ is the last binary decision or chronon in the sequence Π. The integrated form of this law is

$D = \mathrm{dia}\chi$,

where χ is the c system 2. By algebraic quantization we obtain the q dynamical theory

$D = \widehat{\mathrm{dia}}\chi$,

where χ is the chronon $\hat{2}$. It is easy to see that this simplest q dynamics describes a spin-$\frac{1}{2}$ particle moving on the q null cone. In the limit of classical time the quantum travels with the speed of light and is therefore described as having mass 0. This similarity between $\widehat{\mathrm{dia}}\chi$ and the two-component neutrino is being studied further, as well as the natural inference from the electron theories of Dirac and Feynman that paths traveling backwards in time, ensembles of "antichronons," should be needed to describe massive or interacting quanta. Moreover, de Broglie's fusion theory of light, depicting a photon as two neutrinos traveling together, is difficult to express without self-contradiction in the continuum space-time of cq dynamics, but is a self-consistent and simple concept in q dynamics.

VIII. DISCUSSION AND PROSPECTS

We showed a system subject to a quantum binary process will in the limit of many chronons appear to be moving in the space-time of special relativity. If our picture is to be trusted, time is the number of chronons, and the chronon has spin $\frac{1}{2}$.

It is possible that this asymptotic agreement between quantum binary process and special relativity is fortuitous. It is also possible that chronons really exist and do have spin $\frac{1}{2}$. In territory this wild, we are lost as soon as we move ahead of our formal theory, but as we plod along it is sound to watch for encouraging signs. Two marks that we may be approaching the Great Divide come in sight during the present work.

Mark one is the multitime nature of this q dynamics. If Π_1 and Π_2 are dynamical systems, then there is always a natural sum $\Pi_1 + \Pi_2$ (in the usual sense that the nucleon is the sum $n+p$ of the neutron and proton) and a natural product $\Pi_1 \times \Pi_2$ (in the usual sense that the deuteron is np). The generators of $\Pi_1 \times \Pi_2$ are those of Π_1 together with those of Π_2, each system bringing along its own time. In c dynamics we go back to one time T by equating T_1 and T_2, selecting the diagonal $\mathrm{dia}_2 T \leftrightarrow T$ in the product $T_1 T_2$, but in q dynamics this is not an invariant procedure, $\widehat{\mathrm{dia}}_2 T$ is not isomorphic to T, and some kind of interaction seems necessary in order to compare T_1 and T_2. It is good to meet even this familiar relativistic nuisance when we are so far from home.

Mark two is the noncommutation of world paths, and the nonadditivity of displacements. Quantum mechanics and general relativity are both theories of a nonclassical noncommutation; of quantities in the one case, temporal displacements in the other. In q dynamics the two kinds merge into one, the noncommuting of Π. This quantum noncommutation in the microscopic world will survive in the c limit of many quanta only for the right statistics and then only to an extent that depends on the coherence of the constituent quanta. We were led to describe null rays as Bose-Einstein ensembles of chronons on purely logical grounds. But now this seems to open the possibility that gravity is a quantum effect, somewhat similar to superfluidity in this respect, but with a much smaller quantum coherence, judging from the weakness of gravitational interactions, and a coherence of chronons, not quanta of matter. It is just in case of coherence that the concatentation of paths does not obey the law of vector addition (3), and the phenomenon of curvature seems to arise.

Before anything more decisive can be said about these questions, the concept of interactions must be added to our scheme of things. The problem of doing this in the q domain so as to get a correct correspondence with the established results of the cq domain in the limit of classical time seems to be a definite one. It leads to developments of notation and concept outside the scope of this paper.

ACKNOWLEDGMENTS

I must thank Peter G. Bergmann for greatly improving the presentation through his kind criticism. It has been a great help to work these ideas over and over with Graham Frye. And, all in all, I am still being faithful in my way to an idea once expressed in conversation by R. P. Feynman. But the responsibility for this work remains mine.

*Young Men's Philanthropic League Professor of Physics. Supported in part by the National Science Foundation.

[1]Quantum space-time geometry was considered in STC [D. Finkelstein, Phys. Rev. 184, 1261 (1969)] which should not have been written, let alone read, before the present work. In STC as in many other world systems including those of Ptolemy, Newton, and Einstein (special relativity) the natural order for the construction of the theory is L, G, D: first a world Logic, then a pure world Geometry, and finally a Dynamics [cf. D. Finkelstein, in *Boston Studies in the Philosophy of Science,* edited by R. S. Cohen, Vol. 4 (1968)]. But when we start from a q logic we find it impractical to maintain this order of development. In particular, the formulation that particles may interact when they are at the same point becomes hard to express when the particles move in a q space, which in one sense has no points. We seem driven now to the inverse formulation, that particles are said to be at the same point when they interact. Our order of development is now L, D, G.

Operationally speaking too, the order L, D, G is more natural, for nine parts out of ten of the geometric information imparted by the metric tensor at a point may be distilled from observations of the dynamics of many test objects. We now tend to consider that geometrical concepts such as time and distance are secondary and statistical, much like temperature, and that certain dynamical concepts are the primary and real ones.

There are also differences between the q logic of STC and that given here. In order not to penalize a reader of STC, the differences will be footnoted as they arise.

[2]Chronons have been considered at least since 1913 (Poincaré); see Milič Čapek, *Philosophical Impact of Contemporary Physics* (Van Nostrand, New York, 1961), especially Chap. 13, for discussion and many early references. Also related to efforts towards q theory are H. Snyder, Phys. Rev. 79, 38 (1947); such work of C. F. von Weizsäcker, Naturwiss. 20, 545 (1955); E. J. Zimmerman, Am. J. Phys. 30, 97 (1962); R. Giles and H. Kummer, Queens University, Kingston Report No. 1970-12 (unpublished). But the distinction between q time and c discrete time (Zeno) is crucial.

The mixed c and q elements of present-day quantum mechanics are admirably dissected in C. Piron, Helv. Phys. Acta 42, 330 (1969).

[3]In STC (See Ref. 1) the limit of classical time is worked out for some space-time q geometries. We have not presented here the corresponding limiting process for q dynamics.

[4]Collected in *Selected Papers on Quantum Electrodynamics,* edited by J. Schwinger (Dover, New York, 1958).

[5]I. E. Segal, *Mathematical Problems of Relativistic Physics* (Am. Math. Soc., 1963), and other references there given. For us the algebra of a system is the analytic expression of the same body of information whose synthetic expression is the ortholattice of the system. See D. Finkelstein, in *Paradoxes and Paradigms,* edited by R. G. Colodney (Pittsburgh Univ. Press, Pittsburgh, Pa., 1972); Ref. 1; and Trans. N. Y. Acad. Sci. 25, 621 (1963).

[6]We hesitated to adopt this definition in STC (see Ref. 1) because, setting $|S| = n$,

$$|S_1 \equiv S_2| = n \quad \text{for c systems } S$$
$$= \tfrac{1}{2} n(n+1) \quad \text{for q systems } S\,.$$

Now we regard this difference as just another fact of life.

[7]In STC (See Ref. 1) we confused a sequence whose order is ignored (a series) with a sequence whose order is immaterial (a diagonal sequence). The Bose-Einstein ensemble is actually the latter, and we were misled by its measure to identify the Bose-Einstein ensemble with the former.

Now it is clear why the model called serχ in STC (see Ref. 1) led only to a model of the Minkowski null cone and not the entire future cone as the checkerboard paradigm suggested. What was in fact computed, we can see now, was $\widehat{\mathrm{dia}}\hat{2}$, and that a diagonal sequence of null vectors should have null resultant is gratifying, not puzzling.

This confusion between a constant sequence of quanta and an unordered sequence of quanta also appears in D. Finkelstein, in *Fundamental Interactions at High Energy I,* based on the proceedings of the 1969 Coral Gables Conference on Fundamental Interactions at High Energy, edited by T. Gudehus, G. Kaiser, and A. Perlmutter (Gordon and Breach, New York, 1969), p. 324.

In STC (see Ref. 1) where we attempted to model space-time points, the persistent appearance of conical "space-time" in the simplest models required cosmological interpretation. Now that we are modeling space-time processes instead, these conical structures are identified with the future cone of a generic point, and the cosmological question can be put off to a more reasonable stage of the work.

I am indebted to M. Aizenman for the distinction between dia$S$ and seq$S\backslash G$. The latter vanishes in the q case and coincides with dia$S$ in the c case.