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Regge Poles for Large Coupling Constants. II*

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Since we define the Pomeranchon to be the leading singularity of the "potential," and find that a high-energy particle resembles an expanding black disk if this singularity is located at $J>1$, we study as a model the generation of such a singularity by the ladder diagrams in ϕ^3 theory. We find that, when the coupling constant is sufficiently large, there are Regge poles located at $J>1$. Thus the Pomeranchon in this model is a Regge pole with $\alpha(0)>1$. This conclusion is reached by solving the Bethe-Salpeter equation to determine, in the strong coupling limit, the positions of the Regge (or Toller) poles at $t=0$. We find a family of approximately equally spaced Toller poles. The position of the leading singularity, but not the spacing, depends critically on whether the exchanged particle is massless or massive. Some numerical results are presented.

1. IMPACT PICTURE AND REGGE POLES

About a year ago, we obtained[1] definite predictions on several fundamental quantities in high-energy hadron physics on the basis of a physical picture which emerges from the high-energy behavior of massive quantum electrodynamics. This physical picture has more recently been confirmed also from ϕ^3 theory[2] provided that the coupling constant is sufficiently large. More specifically, at extremely high energies an electron in quantum electrodynamics and a scalar particle in ϕ^3 theory both behave like a Lorentz-contracted disk with a black core that slowly expands with increasing energy. The presence of this black core is due to the production, in the c.m. system or more generally in any C system,[3] of slow electron-positron pairs in quantum electrodynamics or slow scalar particles in the ϕ^3 theory. Because of the production of such slow particles, a target becomes more and more absorptive as energy increases, and scattering necessarily becomes appreciable at larger and larger transverse distances from the target. Thus[1] the total and the elastic cross sections both rise indefinitely with the energy, the diffraction peak shrinks, and the imaginary part of the forward elastic amplitude dominates over its real part.

Experimentally, it is much more interesting to deal with hadronic scattering processes at very high energies rather than electromagnetic interactions of electrons. In the c.m. system, the production of slow hadrons due to hadronic scattering appears to follow the same pattern as that of slow electron-positron pairs in quantum electrodynamics and that of slow scalar particles in ϕ^3 theory. This is supported by the preliminary detection of pionization products.[4] Possible further experimental tests have been discussed elsewhere.[3,5] If pionization products indeed exist in the predicted manner,[3,6] it follows that a target must become more and more absorptive as energy increases. Thus a hadron must also act as a Lorentz-contracted disk with a slowly expanding black core. Therefore, all the conclusions stated in the previous paragraph about the total and elastic cross section, the diffraction peak, and the ratio of the imaginary part to the real part of the forward elastic amplitude must hold for hadronic processes as well as in quantum electrodynamics and ϕ^3 theory.

Consider the proton-proton scattering amplitude in the impact-factor representation[2,7]:

$$\mathfrak{M}\sim \tfrac{1}{2}is\int d\vec{x}_\perp\, e^{i\vec{\Delta}\cdot\vec{x}_\perp}\langle \mathcal{J}^P(\vec{x}_\perp)|(1-e^{-A(\vec{x}_\perp,s)})|\mathcal{J}^P(\vec{x}_\perp)\rangle, \tag{1.1}$$

where $\mathcal{J}^P(\vec{x}_\perp)$ is the proton impact factor in the position space $\vec{x}_\perp$, s is the square of the c.m. energy, $\vec{\Delta}$ is the momentum transfer, and $(1-e^{-A(\vec{x}_\perp,s)})$ is an operator. We remark parenthetically that this exponential form $e^{-A(\vec{x}_\perp,s)}$ should not be taken too literally, as discussed previously.[2,8] This situation with exponentiation or eikonal approximation is particularly transparent in the case of high-energy potential scattering.[9] Equation (1.1) is a generalization of Eq. (9) of Ref. 1. Let us write down the power-series expansion

$$1-e^{-A}=\sum_{n=1}^{\infty} c_n A^n \,. \tag{1.2}$$

If $A(\vec{x}_\perp, s)$ is of the order of s^a as in quantum electrodynamics and ϕ^3 theory (with possible additional factors of ln$s$ ignored), then the nth term on the right-hand side of (1.2) is of the order of s^{na}. We shall define the term on the right-hand side of (1.1) proportional to A^n as the amplitude of n-Pomeranchon exchange. The fact that the amplitude for n-Pomeranchon exchange is of the order s^{1+na} suggests that the Pomeranchon, as a singularity in the angular momentum plane or the J plane, is located at $J=1+a$. This definition of the Pomeranchon as the J-plane singularity of the "potential," rather than the scattering amplitude itself, is different from the conventional one, but we believe that it is the correct way to extend the Regge theory[10] to relativistic processes.

From this point of view, the restriction that $\alpha(0)\leq 1$ for the Pomeranchon is unnecessary.[11,12] Instead, the Pomeranchon is located at $J>1$; it is most likely either a *moving* Regge pole or a *fixed* branch point. In either case, the leading singularities of the scattering amplitude in the J plane are always two moving branch points: For $t<0$ (spacelike), these two branch points form a complex conjugate pair on the line Re$J=1$; at $t=0$, both reach the point $J=1$; for $t>0$ (timelike), they move on the real J axis, one to the right and the other to the left.[13] For processes involving the exchange of quantum numbers, the additional exchange of Pomeranchons leads to a suppression of the amplitude, as well as the existence of a dip related to the width of the diffraction peak.

2. STATEMENT OF THE PROBLEM

Since the restriction $\alpha(0)\leq 1$ no longer holds, it becomes interesting to study the extreme case where $\alpha(0)$ is large. In a preceding paper,[14] we have treated the case of potential scattering. This case is well defined and the solution for large coupling constant is completely straightforward. It is the purpose of the present paper to study in detail the ladder diagrams with scalar particles, again in the limit of large coupling constants. Although we do not believe that the other diagrams are unimportant in this case, we nevertheless can gain some knowledge from the ladder diagrams alone. In particular, we find that the two cases where the mass of the exchanged particles is zero or nonzero are vastly different. When this mass is nonzero, the Regge pole is almost independent of the other masses involved.

The mathematics employed in treating the ladder diagrams is rather complicated. Roughly, the procedure is as follows. Wick[15] and Cutkosky[16] have reduced to differential equations the case where the mass of the exchanged particle is zero. It is thus fairly easy to obtain, for this case, the solution for large coupling constants g. With this knowledge, we can study is some detail the effect of introducing a mass. It is found that the Wick-Cutkosky case is greatly modified already when this mass is proportional to some inverse power of g. In this way we can trace the effect of increasing this mass systematically up to a value of order 1. In Secs. 3, 5, and 6, this procedure is carried out for forward scattering.

Throughout this paper, we consider only the case of forward scattering where $t=0$. The procedure used here can be easily extended to determine asymptotically, for $t\neq 0$, the mother (or parent) trajectories, but there may be difficulties with the daughter trajectories.[17-19]

3. WICK-CUTKOSKY LADDER DIAGRAMS

A. Exact Solution

We wish to consider the simple ladder diagrams in the t channel when the coupling constant is large. The mass λ of the exchanged particle is in general taken to be different from the masses m_1 and m_2 of the scattering particles. Except for brief excursions such as Sec. 4, m_1 and m_2 are taken to be equal: $m_1=m_2=m$. All the particles are assumed to be scalar particles.

In this section, we consider the simplest possible case where

$$\lambda=t=0\,. \tag{3.1}$$

In general, for the scattering of two particles of masses M_i and M_i' into two of masses M_f and M_f', the Regge residue factors:

$$\beta(t;M_i^2,M_f^2;M_i'^2,M_f'^2)=b(t,M_i^2,M_f^2)b(t,M_i'^2,M_f'^2)\,. \tag{3.2}$$

When (3.1) is satisfied, the function of one variable

$$b(z)=b(0,-z,-z) \tag{3.3}$$

satisfies the integral equation[20,21,2]

$$b(z)=\tau\left(\int_0^z dz'b(z')(z'+1)^{-2}(z'/z)^\gamma + \int_z^\infty dz'b(z')(z'+1)^{-2}\right) \tag{3.4}$$

for $z\geq 0$. In (3.4)

$$\gamma=\alpha(0)+1\,, \tag{3.5}$$

$$\tau=g^2/(16\pi^2\gamma)\,, \tag{3.6}$$

and we have taken $m=1$. Note that the γ here is not the same as γ used in Ref. 2.

The exact solutions of (3.4) are

$$\gamma=\tfrac{1}{2}(1+\tfrac{1}{4}g^2/\pi^2)^{1/2}-\tfrac{1}{2}-n\,, \tag{3.7}$$

with $n=0,1,2,\ldots,$ and

$$b(z)=(z+1)^{-\gamma-n}F(-n,-\gamma-n;\gamma+1;-z)\,, \tag{3.8}$$

where F is the hypergeometric function.[22] The particular hypergeometric function in (3.8) is actually a polynomial of order n.

B. Approximate Solution

The Wick-Cutkosky solution[15,16] of the Bethe-Salpeter equation[23,24] is possible only for $\lambda=0$. For $\lambda\neq0$, we must resort to approximate methods, making use of the large coupling constant g. Since there is no general method of solving integral equations approximately, special procedures must be devised for each particular problem. As a most useful guide, we first treat this solvable case $\lambda=0$ of Wick and Cutkosky.

Since γ is large for g large, assume a solution of the form

$$b(z)=A(z)\,e^{-\gamma\phi(z)}\,. \tag{3.9}$$

From the exact solution (3.8), we know that this form (3.9) is indeed correct. When (3.9) is substituted into the integral Eq. (3.4), the right-hand side is of the order τ/γ if the integrand peaks near $z'=z$. Thus $\tau/\gamma=O(1)$. We need to expand to two terms in the form

$$\tau/\gamma=\tau_0+\tau_1/\gamma+\cdots\,, \tag{3.10}$$

where τ_0 and τ_1 are numbers independent of γ.

Let

$$\xi=\gamma|z-z'|\,. \tag{3.11}$$

Then the explicit evaluation of (3.4) with (3.9) gives that

$$\begin{aligned}(\tau_0+\tau_1/\gamma)^{-1}A(z)=&\int_0^\infty d\xi[A(z)-\gamma^{-1}\xi A'(z)](z+1)^{-2}[1+2\gamma^{-1}\xi(z+1)^{-1}]e^{-\xi/z}(1-\tfrac{1}{2}\gamma^{-1}\xi^2/z^2)e^{\xi\phi'(z)}[1-\tfrac{1}{2}\gamma^{-1}\xi^2\phi''(z)]\\&+\int_0^\infty d\xi[A(z)+\gamma^{-1}\xi A'(z)](z+1)^{-2}[1-2\gamma^{-1}\xi(z+1)^{-1}]e^{-\xi\phi'(z)}[1-\tfrac{1}{2}\gamma^{-1}\xi^2\phi''(z)]\\&=(z+1)^{-2}A(z)\{[\phi'(z)-z\phi'(z)^2]^{-1}+\gamma^{-1}B(z)\}+\gamma^{-1}(z+1)^{-2}A'(z)[1-2z\phi'(z)][\phi'(z)-z\phi'(z)^2]^{-2}\,,\end{aligned} \tag{3.12}$$

where

$$B(z)=z[1-z\phi'(z)]^{-3}\{2z(z+1)^{-1}[1-z\phi'(z)]-1-z^2\phi''(z)\}-[\phi'(z)]^{-3}[2(z+1)^{-1}\phi'(z)+\phi''(z)]\,. \tag{3.13}$$

In writing down (3.12), we have assumed

$$z\phi'(z)\leqslant1 \tag{3.14}$$

for all z. The leading terms of (3.12) can be used to determine $\phi'(z)$:

$$(z+1)^2[\phi'(z)-z\phi'(z)^2]=\tau_0\,, \tag{3.15}$$

or more explicitly

$$\phi'(z)=\{1+z\pm[(1+z)^2-4\tau_0 z]^{1/2}\}/[2z(1+z)]\,. \tag{3.16}$$

The appearance of the $\pm$ sign in (3.16) is of crucial importance. Suppose first that $\tau_0<1$ so that the square root in (3.16) is positive for all z. In this case, we have the following behaviors for z either large or small:

$$\phi'(z)\sim z^{-1} \tag{3.17}$$

if the plus sign is used, and

$$\phi'(z)\sim\tau_0(1+z)^{-2} \tag{3.18}$$

if the minus sign is used. From (3.17) we see that $e^{-\gamma\phi(z)}$ is unbounded near $z=0$, while from (3.18) we see that $e^{-\gamma\phi(z)}$ fails to approach zero as $z\to\infty$. Thus both choices of sign are unacceptable, and the conclusion is reached that there is no nontrivial solution for $\tau_0<1$.

This lack of solution does not happen for $\tau_0\geqslant1$. Since we are interested in the leading Regge pole or the nearby Regge poles in the sense of Sec. 1, we must have

$$\tau_0=1\,. \tag{3.19}$$

With (3.19), the two solutions (3.16) are

$$\phi'(z)=(1+z)^{-1} \tag{3.20}$$

and

$$\phi'(z)=z^{-1}(1+z)^{-1}\,. \tag{3.21}$$

The second solution (3.21) is unacceptable for two reasons: $e^{-\gamma\phi(z)}$ is unbounded near $z=0$ and also fails to approach zero as $z\to\infty$. We therefore get

$$\phi(z) = \ln(1+z) . \tag{3.22}$$

The addition of a constant to $\phi(z)$ merely changes the normalization of the solution.

It remains to determine τ_1 and $A(z)$. By (3.19), (3.20), and (3.13), Eq. (3.12) reduces to

$$(1-z^2)A'(z) + (\tau_1 - 1)A(z) = 0 . \tag{3.23}$$

Accordingly

$$A(z) = \text{const}[(1-z)/(1+z)]^{(\tau_1 - 1)/2} . \tag{3.24}$$

In order that $A(z)$ is not singular at $z=1$, it is necessary that

$$\tau_1 = 1 + 2n , \tag{3.25}$$

with $n = 0, 1, 2, \ldots$. Thus

$$A(z) = \text{const}[(1-z)/(1+z)]^n , \tag{3.26}$$

and, by (3.22) and (3.9),

$$b(z) = \text{const}(1+z)^{-\gamma-n}(1-z)^n . \tag{3.27}$$

Furthermore, by (3.6), (3.10), (3.19), and (3.25), we have

$$\tau = \gamma + 1 + 2n \tag{3.28}$$

or

$$\tau = \tfrac{1}{4} g/\pi - \tfrac{1}{2} - n , \tag{3.29}$$

which agrees with the exact solution (3.7) for large g.

We therefore have a procedure of locating the first few Regge poles approximately when the coupling constant g is larger. In Sec. 5, we shall apply this procedure to the case where λ is small but not zero.

C. Critical Point

By assuming that the solution is of the form (3.9), we have found that in general there are two possible $\phi'(z)$'s as given by (3.16). When τ/γ is fixed at a value less than unity, both solutions are unacceptable, either because of the behavior as $z \to 0$ or due to the failure to vanish as $z \to \infty$. We have determined τ/γ to the zeroth approximation by requiring the two solutions to be real and intersecting. The intersection occurs at $z=1$, as seen from (3.20) and (3.21). We shall call this point $z=1$ of intersection the critical point.

The critical point is interesting for many reasons. First, if τ_1 is not chosen as specified by (3.25), the approximate solution (3.24) has a singularity at the critical point. But we know that the solution of the integral equation (3.4) cannot have such a singularity. Thus the presence of the singularity in (3.24) must mean that the approximation is not valid. How, then, can we use the absence of a singularity to determine τ_1?

Secondly, we can see the failure of our approximation near the critical point more explicitly since the exact solution is known for this Wick-Cutkosky case.[15,16] More precisely, let us compare the approximate solution (3.27) with the exact solution (3.8). If the constant in (3.27) is chosen to be unity, then (3.27) and (3.8) agree exactly when $n=0$ and 1. Since this agreement is absent when $n \geq 2$, it is instructive to consider in some detail the case $n=2$. In this case, the exact solution is

$$\begin{aligned} b(z) &= (z+1)^{-\gamma-2} F(-2, -\gamma-2; \gamma+1; -z) \\ &= (z+1)^{-\gamma-2}[(1-z)^2 - 2z/(\gamma+1)] . \end{aligned} \tag{3.30}$$

Thus (3.27) gives a good approximation except in the vicinity of the critical point. More precisely, it is not correct when

$$|z-1| = O(\gamma^{-1/2}) . \tag{3.31}$$

More generally, by quadratic transformation of the hypergeometric function,[25] the solution (3.8) can be written in the form

$$b(z) = (z+1)^{-\gamma} F(-\tfrac{1}{2}n, \gamma + \tfrac{1}{2}n + \tfrac{1}{2}; \gamma+1; 4z/(z+1)^2) , \tag{3.32}$$

or alternatively[26]

$$\begin{aligned} b(z) &= (z+1)^{-\gamma}(-\tfrac{1}{2})^N (2N-1)!![\Gamma(\gamma+1)/\Gamma(\gamma+N+1)] \\ &\quad \times F(-N, \gamma+N+\tfrac{1}{2}; \tfrac{1}{2}; (1-z)^2/(1+z)^2) \end{aligned} \tag{3.33}$$

when $n = 2N$ is even, and

$$\begin{aligned} b(z) &= (z+1)^{-\gamma-1}(1-z)(-\tfrac{1}{2})^N \\ &\quad \times (2N+1)!![\Gamma(\gamma+1)/\Gamma(\gamma+N+1)] \\ &\quad \times F(-N, \gamma+N+\tfrac{3}{2}; \tfrac{3}{2}; (1-z)^2/(1+z)^2) \end{aligned} \tag{3.34}$$

when $n = 2N+1$ is odd. These are the convenient forms for the vicinity of the critical point. Let

$$z = 1 + \bar{z}/\sqrt{\gamma} , \tag{3.35}$$

and consider N and $\bar{z}$ to be fixed as $\gamma \to \infty$. In this limit, when n is even, it follows from (3.33) that

$$\begin{aligned} b(z) &= (z+1)^{-\gamma}(-2\gamma)^{-N}(2N-1)!!\Phi(-N, \tfrac{1}{2}; \tfrac{1}{4}\bar{z}^2) \\ &= (z+1)^{-\gamma}(4\gamma)^{-N} H_{2N}(\tfrac{1}{2}\bar{z}) , \end{aligned}$$

and similarly for n odd from (3.34)

$$\begin{aligned} b(z) &= -(z+1)^{-\gamma}(-2\gamma)^{-N}(2N+1)!!(\tfrac{1}{2}\bar{z}/\sqrt{\gamma}) \\ &\quad \times \Phi(-N; \tfrac{3}{2}; \tfrac{1}{4}\bar{z}^2) \\ &= -(z+1)^{-\gamma}(4\gamma)^{-N-1/2} H_{2N+1}(\tfrac{1}{2}\bar{z}) . \end{aligned}$$

In other words, near the critical point, $b(z)$ is given approximately by

$$b(z) = (z+1)^{-\gamma}(4\gamma)^{-n/2} H_n(\tfrac{1}{2}\gamma^{1/2}(1-z)) . \tag{3.36}$$

In this derivation, Φ is the confluent hypergeometric function[27] while H_n is the Hermite polynomial.[28] Thus (3.27) is not a valid approximation if $n \geq 2$ and $1 - z = O(\gamma^{-1/2})$.

We therefore raise the question how (3.36) can be obtained from the integral Eq. (3.4) without using the exact solution. We must use a procedure similar to that of Sec. 3 B so that the step there from (3.24) to (3.25) can be justified. Indeed the proper procedure is strongly suggested by this answer (3.36): Since Hermite polynomials satisfy second-order ordinary differential equations, we must keep terms proportional to $A''(z)$ in addition to those proportional to $A(z)$ and $A'(z)$ as shown in (3.12) and (3.23). With these additional terms, (3.12) is revised to be

$$(\tau_0 + \tau_1/\gamma)^{-1}A(z) = \int_0^\infty d\xi\,[A(z) - \gamma^{-1}\xi A'(z) + \tfrac{1}{2}\gamma^{-2}\xi^2 A''(z)](z+1)^{-2}[1 + 2\gamma^{-1}\xi(z+1)^{-1}] \times e^{-\xi/z}(1 - \tfrac{1}{2}\gamma^{-1}\xi^2/z^2)\, e^{\xi\phi'(z)}[1 - \tfrac{1}{2}\gamma^{-1}\xi^2\phi''(z)] + \int_0^\infty d\xi\,[A(z) + \gamma^{-1}\xi A'(z) + \tfrac{1}{2}\gamma^{-2}\xi^2 A''(z)](z+1)^{-2}[1 - 2\gamma^{-1}\xi(z+1)^{-1}]\, e^{-\xi\phi'(z)}[1 - \tfrac{1}{2}\gamma^{-1}\xi^2\phi''(z)]. \tag{3.37}$$

Therefore, the extra terms to be added to the right-hand side of (3.12) are

$$\gamma^{-2}(z+1)^{-2}A''(z)\{[z^{-1} - \phi'(z)]^{-3} + [\phi'(z)]^{-3}\} = \gamma^{-2}(z+1)(z^3+1)A''(z) \tag{3.38}$$

by (3.20). Instead of (3.23), we get

$$\gamma^{-1}(z+1)(z^3+1)A''(z) + (1 - z^2)A'(z) + (\tau_1 - 1)A(z) = 0\,. \tag{3.39}$$

For large γ, this additional term is indeed negligible except when z is close to unity. When z is close to this critical point, we use the change of variable (3.35) to get

$$4\, d^2A/d\bar{z}^2 - 2\bar{z}\, dA/d\bar{z} + (\tau_1 - 1)A = 0\,. \tag{3.40}$$

This is indeed the differential equation for Hermite polynomials. Since we do not want exponential growth for both signs of $\bar{z}$, we again get (3.25) and (3.36).

We can now justify writing down (3.25) from (3.23) as follows. Equation (3.23) is not valid near the critical point $z = 1$, and an additional term of the form const $A''(z)$ needs to be added. No matter what this constant is, (3.25) follows from the resulting differential equation for Hermite polynomials. In other words, the coefficient 4 for the first term of (3.40) does not affect (3.25), but of course enters in the solution (3.36). Therefore, if we are only interested in locating the Regge pole for large g but not the details of the Regge residue for z close to the critical point, we do not need to know the coefficient of $A''(z)$, and the procedure of Sec. 3 B is sufficient.

4. MOTHER AND DAUGHTERS

We have found only a small subset of solutions for the Wick-Cutkosky case of the ladder diagrams. As seen from the exact solution (3.7), these Regge poles are one unit apart for $t = 0$. It is therefore natural to ask whether the solution for $n = 0$ is the mother (or parent) trajectory, while the others are daughter trajectories.[17-19] We shall show in this section that this is not the case, i.e., we do not have any daughter trajectory at all. This is accomplished by finding that the spacing is no longer unity if the masses m_1 and m_2 are unequal.

When m_1 and m_2 are not necessarily equal, the integral equation (3.4) is replaced by, still with $t = \lambda = 0$,

$$b(z) = \tau\left(\int_0^z dz'\, b(z')(z' + m_1^2)^{-1}(z' + m_2^2)^{-1}(z'/z)^\gamma + \int_z^\infty dz'\, b(z')(z' + m_1^2)^{-1}(z' + m_2^2)^{-1}\right). \tag{4.1}$$

Thus $b(z)$ satisfies the differential equation

$$z b''(z) + (\gamma + 1)b'(z) + \tau\gamma(z + m_1^2)^{-1}(z + m_2^2)^{-1}b(z) = 0\,. \tag{4.2}$$

If $m_1^2 \neq m_2^2$, this equation (4.2) has four regular singularities at 0, $-m_1^2$, $-m_2^2$, and ∞. Since functions of the Fuchsian type are too complicated, we resort immediately to approximate methods for g large.

Let $z = e^x$, then by (3.6)

$$[d^2/dx^2 - \tfrac{1}{4}\gamma^2 + (4\pi)^{-2}g^2 e^x(e^x + m_1^2)^{-1}(e^x + m_2^2)^{-1}]e^{\gamma x/2}\, b = 0\,. \tag{4.3}$$

The situation is therefore similar to the one treated before for potential scattering.[14] Since the potential

$$-e^x\,(e^x + m_1^2)^{-1}(e^x + m_2^2)^{-1} \tag{4.4}$$

has a minimum at

$$x=\ln(m_1 m_2)\,, \tag{4.5}$$

we use the approximation

$$(4.4)\sim -(m_1+m_2)^{-2}\left[1-(x-x_0)^2 m_1 m_2 (m_1+m_2)^{-2}\right]. \tag{4.6}$$

With (4.6), the problem is again reduced to that of the harmonic oscillator and the result is

$$\tfrac{1}{4}\gamma^2=(4\pi)^{-2}g^2(m_1+m_2)^{-2} - (n+\tfrac{1}{2})(4\pi)^{-1}g(m_1m_2)^{1/2}(m_1+m_2)^{-2}\,, \tag{4.7}$$

or

$$\gamma=(2\pi)^{-1}g/(m_1+m_2)-(n+\tfrac{1}{2})[4m_1m_2(m_1+m_2)^{-2}]^{1/2}\,. \tag{4.8}$$

When $m_1=m_2=1$, (4.8) reduces correctly to (3.29).

Since n is a non-negative integer, the spacing of the Regge poles (4.8) is given by

$$[4m_1m_2(m_1+m_2)^{-2}]^{1/2}. \tag{4.9}$$

Unless $m_1=m_2$, this spacing is not unity. Therefore, by continuity, there is no mother-daughter relation between the various poles.

It is interesting to note that the spacing (4.9) is always less than unity unless $m_1=m_2$.

5. CASE OF SMALL λ

A. Magnitude of λ

So far we have given essentially no new result for the ladder diagrams, since all the answers for the Wick-Cutkosky case can be obtained without the elaborate approximation scheme of Sec. 3. This approximation scheme will now be applied to the case where $\lambda\neq 0$ in this and the next sections.

The approximation scheme of Sec. 3 must be valid when λ is sufficiently small, provided that g and hence γ are large. The first question is: What is the order of magnitude of this small λ when significant deviation from the case $\lambda=0$ first appears?

This question is easily answered. From Sec. 3 B, we know that we must consider the region

$$z=O(1)\,. \tag{5.1}$$

From (3.11) we get

$$z'-z=O(\gamma^{-1})\,. \tag{5.2}$$

In the presence of λ but still with $t=0$, the integral equation satisfied by $b(z)$ is[20, 21]

$$b(z)=\tau\int_0^\infty dz'b(z')(z'+1)^{-2}\times\left(\frac{(z+z'+\lambda^2)-[(z+z'+\lambda^2)^2-4zz']^{1/2}}{2z}\right)^\gamma, \tag{5.3}$$

where m is still taken to be unity. Let the square root in (5.3) be written in the form

$$[(z-z')^2+2\lambda^2(z+z')+\lambda^4]^{1/2}.$$

Then we see that, in the region specified by (5.1) and (5.2), λ^2 cannot be neglected when

$$\lambda=O(\gamma^{-1})\,. \tag{5.4}$$

This is the desired answer. We accordingly define

$$\Lambda=\lambda\gamma\,, \tag{5.5}$$

and consider Λ to be a fixed positive number while $\gamma\to\infty$.

B. Approximate Differential Equation

For the present case, we want to obtain an approximate differential equation like (3.23). Even though the exact solution is no longer available, the procedure of Sec. 3 B can be followed.

In the region (5.1), (5.2), and (5.4), the last factor in the integral Eq. (5.3) may be approximated as follows:

$$\left(\frac{(z+z'+\lambda^2)-[(z+z'+\lambda^2)^2-4zz']^{1/2}}{2z}\right)^\gamma=\{1+\tfrac{1}{8}\,\xi\gamma^{-1}z^{-2}(\xi^2+4z\Lambda^2)^{-1/2}[(\xi^2+4z\Lambda^2)^{1/2}-\xi]^2\}\times\exp\{-\tfrac{1}{2}z^{-1}[(\xi^2+4z\Lambda^2)^{1/2}-\xi]\}, \tag{5.6}$$

where

$$\xi=\gamma(z'-z)\,. \tag{5.7}$$

When the form (3.9) is used, (5.3) becomes

$$(\tau_0+\tau_1/\gamma)^{-1}A(z)=\int_{-\infty}^{\infty}d\xi[A(z)+\gamma^{-1}\xi A'(z)](z+1)^{-2}[1-2\gamma^{-1}\xi(z+1)^{-1}]\times\{1+\tfrac{1}{8}\,\xi\gamma^{-1}z^{-2}(\xi^2+4z\Lambda^2)^{-1/2}[(\xi^2+4z\Lambda^2)^{1/2}-\xi]^2-\tfrac{1}{2}\gamma^{-1}\xi^2\phi''(z)\}\times\exp(-\{\xi\phi'(z)+\tfrac{1}{2}z^{-1}[(\xi^2+4z\Lambda^2)^{1/2}-\xi]\})\,. \tag{5.8}$$

Because of the form of the exponent, it is appropriate to use the change of variable

$$\xi = 2\Lambda z^{1/2}\sinh(\theta - \tfrac{1}{2}\ln\{\phi'(z)/[z^{-1}-\phi'(z)]\}), \tag{5.9}$$

and in this way we get a rather large number of modified Bessel functions. The result is

$$\begin{aligned}\tfrac{1}{2}(z+1)^2\phi'\frac{1}{\tau_0}\left(1-\frac{\tau_1}{\tau_0\gamma}\right)A(z) = A(z)\Big[&K_1(2\Lambda(\phi'-z\phi'^2)^{1/2})\left(\frac{\Lambda^2\phi'}{1-z\phi'}\right)^{1/2}\left(1-\frac{\Lambda^2}{2\gamma\phi'}\,\frac{1-2z\phi'}{1-z\phi'}\,(\phi''-2z\phi'\phi''-\phi'^2)\right)\\ &+\frac{1}{\gamma}K_2(2\Lambda(\phi'-z\phi'^2)^{1/2})\frac{\Lambda^2\phi'}{1-z\phi'}\left(\frac{3z\phi''}{\phi'}-\frac{\phi''}{\phi'^2(1-z\phi')}-\frac{3\phi'}{1-z\phi'}-\frac{2}{(z+1)\phi'}+\frac{4z}{z+1}\right)\Big]\\ &+\frac{1}{\gamma}A'(z)K_2(2\Lambda(\phi'-z\phi'^2)^{1/2})\,\frac{\Lambda^2(1-2z\phi')}{1-z\phi'}\,.\end{aligned} \tag{5.10}$$

The leading terms of (5.10) give that

$$(z+1)^2 = 2\Lambda\tau_0(\phi'-z\phi'^2)^{-1/2}K_1(2\Lambda(\phi'-z\phi'^2)^{1/2}), \tag{5.11}$$

while the terms of order γ^{-1} give the desired differential equation

$$\begin{aligned}-\tfrac{1}{2}(z+1)^2\tau_0^{-2}\tau_1\Lambda^{-2}(\phi'-z\phi'^2)A(z)\\ = A(z)\{&-\tfrac{1}{2}\Lambda(\phi'-z\phi'^2)^{-1/2}(1-2z\phi')(\phi''-2z\phi'\phi''-\phi'^2)K_1(2\Lambda(\phi'-z\phi'^2)^{1/2})\\ &+[3z\phi''-\phi''\phi'^{-1}(1-z\phi')^{-1}-z\phi'^2(1-z\phi')^{-1}-2(z+1)^{-1}+4z\phi'(z+1)^{-1}]K_2(2\Lambda(\phi'-z\phi'^2)^{1/2})\}\\ &+(1-2z\phi')A'(z)K_2(2\Lambda(\phi'-z\phi'^2)^{1/2})\,.\end{aligned} \tag{5.12}$$

C. Critical Point

We apply the argument of Sec. 3B, and consider (5.11) as an equation for determining $\phi' = \phi'(z)$. If τ_0 is too small, the resulting $\phi'(z)$ are not acceptable. Therefore, for the leading Regge poles, we must require

$$\phi' - z\phi'^2 = \text{const} \tag{5.13}$$

to have a double root somewhere. Hence $2z\phi' = 1$ and

$$\tau_0^{-1} = \operatorname*{Max}_{z>0} 4\Lambda z^{1/2}(z+1)^{-2}K_1(\Lambda z^{-1/2}), \tag{5.14}$$

or equivalently

$$\tau_0^{-1} = \operatorname*{Max}_{\zeta>0} 4\Lambda^2\zeta^3(\Lambda^2+\zeta^2)^{-2}K_1(\zeta)\,. \tag{5.15}$$

Note that, for given Λ, this maximum is reached at a unique value of z (and ζ).

As in Sec. 3C, we define the critical point z_0 as the value where the coefficient of the $A'(z)$ term vanishes. Thus

$$2z_0\phi_0' = 1, \tag{5.16}$$

where we use the shorthand notation $\phi_0' = \phi'(z_0)$. At (5.16),

$$(\partial/\partial\phi')\text{ right-hand side of (5.11)} = 0 \tag{5.17}$$

for fixed z, simply because this derivative always has a factor $1-2z\phi'$. Therefore, at this critical point z_0,

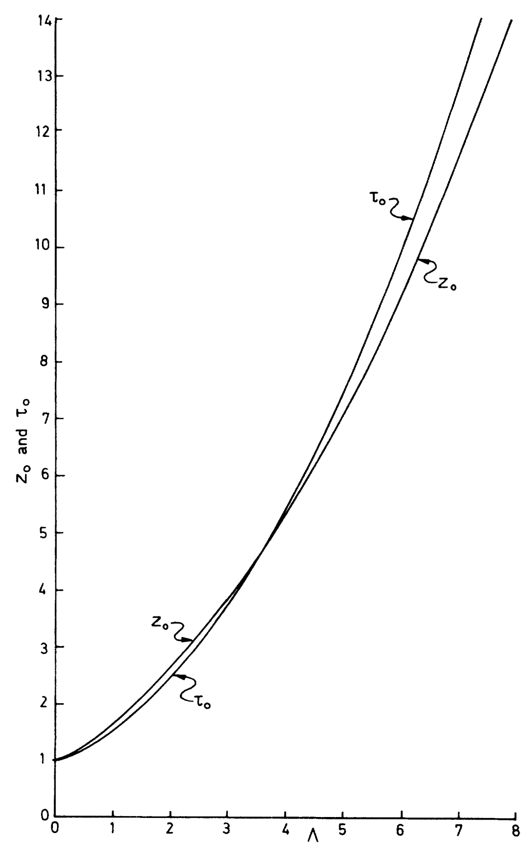

FIG. 1. Plots of z_0 and τ_0 as functions of Λ.

$$(\partial/\partial z)[(z+1)^{-2}(\phi'-z\phi'^2)^{-1/2}K_1(2\Lambda(\phi'-z\phi'^2)^{1/2})]=0 \tag{5.18}$$

with fixed ϕ'. But, for $z=z_0$ and any differentiable function f,

$$\begin{aligned}(\partial/\partial z)f(\phi'-z\phi'^2)&=-\phi'^2 f'(\phi'-z\phi'^2)\\&=-(4z^2)^{-1}f'(\tfrac{1}{4}z^{-1})\\&=(d/dz)f(\tfrac{1}{4}z^{-1}).\end{aligned} \tag{5.19}$$

It therefore follows from (5.18) that

$$(d/dz)[(z+1)^{-2}z^{1/2}K_1(\Lambda z^{-1/2})]=0, \tag{5.20}$$

and a comparison with (5.14) shows that

$$\tau_0^{-1}=4\Lambda z_0^{1/2}(z_0+1)^{-2}K_1(\Lambda z_0^{-1/2}). \tag{5.21}$$

In other words, the maximum is reached at the critical point. If we define

$$\zeta_0=\Lambda z_0^{-1/2}, \tag{5.22}$$

then by (5.15)

$$\tau_0^{-1}=4\Lambda^2\zeta_0^3(\Lambda^2+\zeta_0^2)^{-2}K_1(\zeta_0). \tag{5.23}$$

Given Λ, ζ_0 is also the positive solution of the transcendental equation

$$2(\Lambda^2-\zeta_0^2)K_1(\zeta_0)-(\Lambda^2+\zeta_0^2)\zeta_0K_0(\zeta_0)=0. \tag{5.24}$$

Since explicit solution is not possible for (5.24), we plot in Fig. 1 both the value of τ_0 and the position z_0 of the critical point as functions of Λ. It is interesting that these two curves are close to each other.

D. Positions of Regge Poles

It remains to find the generalization of (3.28) to the present case $\Lambda\neq 0$. For this purpose we must study (5.12) in the neighborhood of the critical point:

$$\begin{aligned}-\tfrac{1}{2}(z_0+1)^2\tau_0^{-2}\tau_1\Lambda^{-2}(\phi_0'-z_0\phi_0'^2)A(z)=\{&A(z)[3z_0\phi_0''-\phi_0''\phi_0'^{-1}(1-z_0\phi_0')^{-1}\\&-z_0\phi_0'^2(1-z_0\phi_0')^{-1}-2(z_0+1)^{-1}+4z_0\phi_0'(z_0+1)^{-1}]\\&+(1-2z\phi')A'(z)\}K_2(2\Lambda(\phi_0'-z_0\phi_0'^2)^{1/2}),\end{aligned} \tag{5.25}$$

or, by (5.16) and (5.21),

$$\tfrac{1}{2}(\tau_1/\tau_0)\Lambda^{-1}z_0^{-1/2}K_1(\Lambda z_0^{-1/2})A(z)=(\tfrac{1}{2}z_0^{-1}+z_0\phi_0'')[A(z)+2(z-z_0)A'(z)]K_2(\Lambda z_0^{-1/2}). \tag{5.26}$$

To simplify (5.26) further, we rewrite (5.24) in the form

$$4\Lambda^2K_1(\zeta_0)-(\Lambda^2+\zeta_0^2)\zeta_0K_2(\zeta_0)=0 \tag{5.27}$$

and substitute into (5.26) to get

$$\tfrac{1}{4}(\tau_1/\tau_0)(z_0+1)z_0^{-1}A(z)=(1+2z_0^2\phi_0'')[A(z)+2(z-z_0)A'(z)]. \tag{5.28}$$

This differential equation is of exactly the same form as (3.23), and hence

$$\tfrac{1}{4}(\tau_1/\tau_0)(z_0+1)z_0^{-1}(1+2z_0^2\phi_0'')^{-1}=1+2n \tag{5.29}$$

is the generalization of (3.25). In other words,

$$\tau=\tau_0[\gamma+4(1+2n)z_0(z_0+1)^{-1}(1+2z_0^2\phi_0'')], \tag{5.30}$$

or, by (3.6),

$$16\pi^2\tau_0[\gamma^2+4(1+2n)z_0(z_0+1)^{-1}(1+2z_0^2\phi_0'')\gamma]=g^2, \tag{5.31}$$

or

$$\gamma=\tfrac{1}{4}g\tau_0^{-1/2}/\pi-2(1+2n)z_0(z_0+1)^{-1}(1+2z_0^2\phi_0''). \tag{5.32}$$

In these results, ϕ_0'' is determined by taking the derivative of (5.11) and evaluating the result at the critical point. In order to eliminate ϕ_0'', we expand (5.11) near the critical point z_0:

$$1+2z_0^2\phi_0''=\tfrac{1}{4}z_0^{-1}(z_0+1)^{-1/2}[8z_0^3-\Lambda^2(z_0+1)^2]^{1/2}. \tag{5.33}$$

Therefore,

$$\gamma=\tfrac{1}{4}g\tau_0^{-1/2}/\pi-(n+\tfrac{1}{2})(z_0+1)^{-3/2}[8z_0^3-\Lambda^2(z_0+1)^2]^{1/2}. \tag{5.34}$$

E. Spacing Between Regge Poles

The above result (5.34) is not explicit for the following reason. The desired quantity γ appears not only on the left-hand side but also on the right-hand side through τ_0, which is defined by (5.14) or (5.21). Remember that, due to the definition (5.5), Λ depends on γ. Accordingly, (5.34) really gives g as a function of γ, but not γ explicitly as a function of g. In order to find the distance between the Regge poles corresponding to successive values of n, the dependence of τ_0 on n cannot be neglected.

Let $\delta\gamma$, $\delta\tau_0^{1/2}$, and $\delta\Lambda$ be the changes of γ, τ_0^{-1}, and Λ, respectively, when n is changed by unity. Of course g is held fixed. Since, from (5.34),

$$(4\pi)^{-1}g = \gamma\tau_0^{1/2} + \tau_0^{1/2}(n+\tfrac{1}{2})(z_0+1)^{-3/2} \times[8z_0^3 - \Lambda^2(z_0+1)^2]^{1/2},$$

we get

$$\delta(\gamma\tau_0^{1/2}) = \tau_0^{1/2}(z_0+1)^{-3/2}[8z_0^3 - \Lambda^2(z_0+1)^2]^{1/2}. \quad (5.35)$$

On the other hand, it follows from (5.15) and (5.23) that

$$\delta\tau_0^{-1} = \delta[4\Lambda^2\zeta_0^3(\Lambda^2+\zeta_0^2)^{-2}K_1(\zeta_0)], \quad (5.36)$$

where, on the right-hand side, ζ_0 is held fixed and only Λ is varied. Thus

$$\delta\tau_0^{-1} = \tau_0^{-1}(2\Lambda\delta\Lambda)[\Lambda^{-2} - 2(\Lambda^2+\zeta_0^2)^{-1}] = (2\tau_0^{-1}\delta\Lambda/\Lambda)[1 - 2z_0/(1+z_0)], \quad (5.37)$$

and furthermore

$$\delta(\gamma\tau_0^{1/2}) = 2\tau_0^{1/2}z_0(1+z_0)^{-1}\delta\gamma. \quad (5.38)$$

The substitution of (5.38) into (5.35) then gives the spacing between Regge poles as

$$\delta\gamma = \tfrac{1}{2}z_0^{-1}(1+z_0)^{-1/2}[8z_0^3 - \Lambda^2(z_0+1)^2]^{1/2}. \quad (5.39)$$

In this formula, we may use

$$\Lambda \sim (\tfrac{1}{4}g\tau_0^{1/2}/\pi)\lambda = \tau_0^{1/2}[\lambda g/(4\pi)]. \quad (5.40)$$

We therefore get more explicitly

$$\delta\gamma = \tfrac{1}{2}z_0^{-1}(1+z_0)^{-1/2}\{8z_0^3 - \tau_0(z_0+1)^2[\lambda g/(4\pi)]^2\}^{1/2}. \quad (5.41)$$

In Fig. 2, we plot this spacing $\delta\gamma$ as a function of $\lambda g/(4\pi)$. It is interesting to note that (1) $\delta\gamma$ changes less than $\frac{1}{3}$ for all λ; and (2) $\delta\gamma$ has a small peak for $\gamma g/(4\pi) \sim 0.55$.

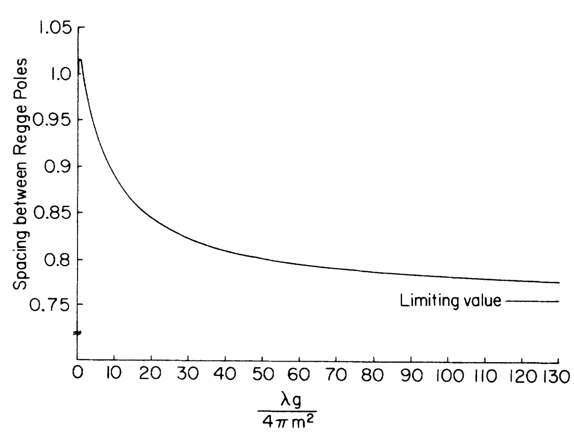

FIG. 2. The distance between successive Regge (or Toller) poles as a function of the mass of the exchanged particle.

6. CASE WHERE $\lambda = O(m)$

In Sec. 3 we consider the Wick-Cutkosky case $\lambda = 0$, while in Sec. 5 we deal with the case $\lambda = O(\gamma^{-1})$. We now try to increase λ further.

Take the results of Sec. 5 and let Λ be very large. First

$$\bar{\zeta}_0 = \lim_{\Lambda\to\infty} \zeta_0 \quad (6.1)$$

exists, and satisfies, from (5.24),

$$2K_1(\bar{\zeta}_0) - \bar{\zeta}_0 K_0(\bar{\zeta}_0) = 0. \quad (6.2)$$

Numerically,

$$\bar{\zeta}_0 = 2.386\,736. \quad (6.3)$$

By (5.15), in this limit of large Λ,

$$\tau_0 = \tfrac{1}{4}\Lambda^2[\bar{\zeta}_0^3 K_1(\bar{\zeta}_0)]^{-1} = 0.215\,991\,5\Lambda^2 \quad (6.4)$$

and of course

$$z_0 = 0.175\,546\,1\Lambda^2. \quad (6.5)$$

In this limit, (5.34) reduces to

$$1 + 2z_0^2\phi_0'' = \tfrac{1}{4}(8 - \bar{\zeta}_0^2)^{1/2} = 0.379\,431\,4. \quad (6.6)$$

It is fortunate that $\bar{\zeta}_0 < 2\sqrt{2}$. Otherwise the situation would become very complicated. Finally, the substitution of (6.4) and (6.6) into (5.31) and (5.32) gives that

$$16\pi^2(0.215\,991\,5)\lambda^2\gamma^2[\gamma^2 + 3.035\,451(n+\tfrac{1}{2})\gamma] = g^2, \quad (6.7)$$

or

$$\gamma = 0.171\,226\,8\,g\lambda^{-1}\gamma^{-1} - 1.517\,725(n+\tfrac{1}{2}). \quad (6.8)$$

We have obtained these results by first assuming a large γ and then let Λ be large. Thus we have implicitly assumed that Λ, although large, is small compared with γ. However, this assumption is unnecessary. This point can be seen in many ways. First, since the result can essentially depend only on λ/m for dimensional reasons, we can let m be small instead of large Λ. In the development of Sec. 5, m appears only in a trivial way, in the denominator $(z'+m^2)^{-2}$. Thus it is not difficult to check that small m does not cause any complication. Alternatively, we can repeat the calculation of Sec. 5 with finite λ, taking care of the fact that both z_0 and τ_0 are of the order of ζ^2. We can thus use (6.7) and (6.8) for finite λ to get

$$\alpha(0) = 0.413\,795\,7(g/\lambda)^{1/2} - 1 - 0.758\,863(n+\tfrac{1}{2}). \quad (6.9)$$

Note that this is dimensionally correct, and hence m does not enter. In particular, even for $m_1 \neq m_2$, (6.9) is still valid in the limit of large g with fixed $\lambda \neq 0$, m_1, m_2, and n.

It is very interesting to compare (6.9) for $\lambda \neq 0$ and (3.29) rewritten in the form

$$\alpha(0) = (4\pi)^{-1} g/m - n - \tfrac{3}{2} \qquad (6.10)$$

for $\lambda = 0$. First, the leading Regge pole in the sense of Sec. 1 is located, for g large, very roughly at a point of order g for $\lambda = 0$, but of order $g^{1/2}$ for $\lambda \neq 0$. Thus, increasing the coupling constant for $\lambda \neq 0$ is much less effective in pushing the Regge pole to the right. Secondly, while the spacing between Regge poles is 1 for $\Lambda = 0$ (although they are not daughters as shown in Sec. 4 and also seen from Sec. 5), the spacing is only somewhat smaller for $\lambda \neq 0$, as seen from (6.9):

$$\delta\gamma = \delta\alpha(0) = 0.758\,863\,. \qquad (6.11)$$

Of course this result (6.11) can also be obtained readily from (5.41). This limiting value is marked on Fig. 2.

7. DISCUSSIONS

(A) As already mentioned, the results of Sec. 6 hold even when $m_1 \neq m_2$. However, the curve shown in Fig. 2 is correct only for an $m_1 = m_2$, although the asymptotic value has a more general validity. Since the special case $m_1 = m_2$ is not a particularly simple one, the appearance of the peak is perhaps not surprising.

(B) A natural question to ask is: Why are the daughter trajectories[17-19] not seen? This question has a very simple answer: the integral equation that we study is really the equation for Toller poles.[21] Since a Toller pole leads to a family of Regge poles, daughter poles are already included. From this point of view, the result of Sec. 4 is immediate. What we have found here is, for large coupling constants, a *family* of approximately equally spaced Toller poles.

(C) As stated in Sec. 2, the procedure here can be extended at least partially to nonforward directions. We believe that the difficulty with the daughter trajectories is only a technical one, not of fundamental nature. However, a great deal of further work remains to be carried out in this direction.

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[25]See Eq. (1) on p. 110 of Ref. 22. This equation has a misprint listed in the errata.

[26]See Eq. (1) on p. 108 of Ref. 22.

[27]See Chap. VI of Ref. 22.

[28]We have used Eq. (17) and Eq. (18) on p. 194 of Vol. II of Ref. 22.