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Regge Poles for Large Coupling Constants. I*

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We study the problem of scattering by an attractive Yukawa potential. In particular, the positions of Regge poles are found approximately when the coupling constant is large.

1. INTRODUCTION

It is the purpose of this and the following paper[1] to study the behavior of Regge poles[2] when the coupling constant is large. Since Regge poles were originally found in connection with the scattering by a superposition of Yukawa potentials, we treat here first this case of potential scattering. Although the procedure of this paper is applicable to a large class of superpositions of Yukawa potentials, we shall restrict ourselves to the case of a single Yukawa potential, where the answer is more explicit in some cases.

In Paper II we shall consider the case of ladder diagrams in ϕ^3 theory, again in the limit where the coupling constant is large.

The differential equation for a partial wave is

$$\left[\frac{d^2}{dr^2}-\kappa^2-l(l+1)r^{-2}+g^2V(r)\right]\psi=0, \tag{1}$$

where $i\kappa$ is the momentum of the incident particle, and $V(r)$ is given by

$$V(r)=r^{-1}e^{-\lambda r}. \tag{2}$$

In (1), l is considered to be a continuous variable. Let

$$r=e^z$$

and (3)

$$\psi=\psi_1 e^{z/2};$$

then it follows from (1) that

$$\left[\frac{d^2}{dz^2}-\kappa^2e^{2z}-(l+\tfrac{1}{2})^2+g^2e^{2z}V(e^z)\right]\psi_1=0. \tag{4}$$

For the Yukawa potential (2), Eq. (4) is more explicitly

$$\left[\frac{d^2}{dz^2}-\kappa^2e^{2z}-(l+\tfrac{1}{2})^2+g^2e^ze^{-\lambda e^z}\right]\psi_1=0. \tag{5}$$

We are thus interested in the behavior of the potential

$$U=\kappa^2e^{2z}-g^2e^ze^{-\lambda e^z}. \tag{6}$$

This is an attractive potential when g is sufficiently large. For large g, its energy levels can be found approximately by replacing U with a harmonic potential located at the minimum of U.

2. CASE OF ZERO MOMENTUM

The case $\kappa = 0$ is particularly simple. The minimum of U is located at

$$z_0 = -\ln\lambda\,, \tag{7}$$

and furthermore, if z is close to z_0,

$$U \sim -(g^2/e\lambda)[1 - \tfrac{1}{2}(z - z_0)^2]\,. \tag{8}$$

When this approximation for U is used in (5), we find that

$$(l + \tfrac{1}{2})^2 \sim g^2/e\lambda - (n + \tfrac{1}{2})(2g^2/e\lambda)^{1/2} \tag{9}$$

or

$$l \sim g(e\lambda)^{-1/2} - \tfrac{1}{2} - (n + \tfrac{1}{2})/\sqrt{2}\,, \tag{10}$$

when $n = 0, 1, 2, \ldots$ is a non-negative integer. Note that the spacing of Regge poles is $1/\sqrt{2}$. (We have taken g to be positive without loss of generality.) Note also that the Regge poles given by (10) are located at $l \gg 1$. They do not include the infinite number of Regge poles at $l = -\frac{1}{2}$, for example.

3. CASE OF SMALL MOMENTA

To find the location of the minimum of U, we differentiate (6):

$$g^2(1 - \lambda e^z)e^{-\lambda e^z} = 2\kappa^2 e^z\,. \tag{11}$$

When κ^2 is sufficiently small, (11) is satisfied at

$$z_0 \sim -\ln\lambda - 2e\kappa^2/(\lambda g^2)\,. \tag{12}$$

Moreover,

$$U \sim -\frac{g^2}{e\lambda}\left[\left(1 - \frac{e\kappa^2}{\lambda g^2}\right) - \tfrac{1}{2}(z - z_0)^2\left(1 + \frac{2e\kappa^2}{\lambda g^2}\right)\right] \tag{13}$$

in the vicinity of this minimum at z_0. Therefore,

$$(l + \tfrac{1}{2})^2 \sim \frac{g^2}{e\lambda}\left(1 - \frac{e\kappa^2}{\lambda g^2}\right) - (n + \tfrac{1}{2})\left(\frac{2g^2}{e\lambda}\right)^{1/2}\left(1 + \frac{e\kappa^2}{\lambda g^2}\right) \tag{14}$$

or

$$l \sim g(e\lambda)^{-1/2}\left(1 - \frac{e\kappa^2}{2\lambda g^2}\right) - \tfrac{1}{2} - (n + \tfrac{1}{2})\frac{1}{\sqrt{2}}\left(1 + \frac{3e\kappa^2}{2\lambda g^2}\right). \tag{15}$$

This is the desired answer. In particular,

$$\frac{dl}{d(-\kappa^2)} \sim \tfrac{1}{2}e^{1/2}g^{-1}\lambda^{-3/2} + 3(n + \tfrac{1}{2})2^{-3/2}e\lambda^{-1}g^{-2}\,. \tag{16}$$

This result, (15), for large g has been obtained here for $\kappa^2 \geq 0$. For all κ^2 the U of (6) has one relative minimum located at z_0 which satisfies (11), provided that g is sufficiently large. Only for $\kappa^2 \geq 0$, this relative minimum is also an absolute minimum; otherwise U is not bounded from below when $z \to \infty$. Therefore, l is real for $\kappa^2 \geq 0$, but complex if $\kappa^2 < 0$. However, when $g \to \infty$ with λ and $-\kappa^2$ fixed at positive values, the imaginary part of l is exponentially small due to barrier penetration. Accordingly, (15) holds for $\kappa^2 < 0$ as well as $\kappa^2 \geq 0$.

4. CASE OF LARGE MOMENTA

The above results, (15) and (16), hold when

$$|\kappa^2| \ll \lambda g^2\,. \tag{17}$$

Otherwise, (12) does not follow from (11).

We now consider the case where $|\kappa^2|$ and λg^2 are comparable. Let

$$u = F(C) \tag{18}$$

be the solution of the equation

$$(1 - u)e^{-u} = 2Cu\,. \tag{19}$$

Equation (19) has more than one real solution for $C < 0$; we choose the branch where $F(0) = 1$. In Fig. 1, we plot this $F(C)$, defined for $C > C_0$, where

$$C_0 = -\tfrac{1}{4}(3 - \sqrt{5})\exp[\tfrac{1}{2}(1 + \sqrt{5})] = -0.037\,869\,67. \tag{20}$$

At this point, the value of $F(C)$ is

$$F(C_0) = \tfrac{1}{2}(1 + \sqrt{5}) = 1.618\,034\,. \tag{21}$$

In terms of this function F, we have from (11)

$$z_0 = -\ln\lambda + \ln F(\kappa^2\lambda^{-1}g^{-2})\,. \tag{22}$$

In the vicinity of this minimum at z_0,

$$U \sim -\kappa^2\lambda^{-2}F^2(1 - F)^{-1} \times [(1 + F) - (1 + F - F^2)(z - z_0)^2]\,, \tag{23}$$

and hence

$$(l + \tfrac{1}{2})^2 \sim \kappa^2\lambda^{-2}F^2(1 - F)^{-1}(1 + F) - (n + \tfrac{1}{2})[4\kappa^2\lambda^{-2}F^2(1 + F - F^2)/(1 - F)]^{1/2}\,, \tag{24}$$

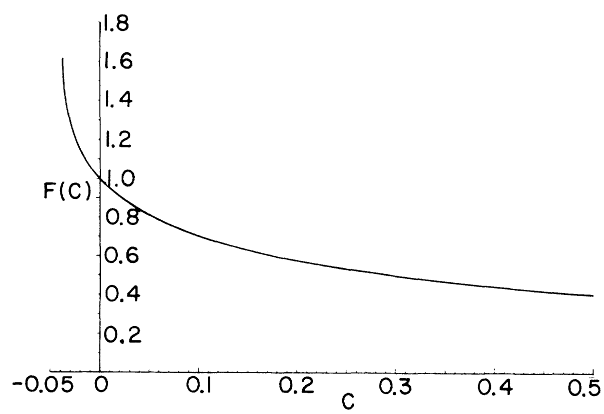

FIG. 1. Plot of the function F that appears in Sec. 4.

or

$$l \sim \kappa\lambda^{-1}F[(1+F)/(1-F)]^{1/2} - \tfrac{1}{2} - (n+\tfrac{1}{2})[(1+F-F^2)/(1+F)]^{1/2}. \qquad (25)$$

In (23)–(25), F of course stands for $F(\kappa^2\lambda^{-1}g^{-2})$.

The approximate formula (25) certainly fails when

$$\kappa^2\lambda^{-1}g^{-2} = C_0. \qquad (26)$$

In other words, for extremely large g, the Regge pole moves rapidly away from the real axis at roughly

$$-\kappa^2 = |C_0|\lambda g^2. \qquad (27)$$

5. SPECIAL CASE OF THE COULOMB FIELD

The limiting case $\lambda \to 0$ is rather singular. This special case can be solved exactly in terms of confluent hypergeometric functions. In particular, as shown by Singh,[3] for $\kappa^2 = 0$ all the Regge poles are at infinity. This is consistent with (9), and also with (27) where the right-hand side is zero.

In the Wick-Cutkosky case of the ladder diagrams,[4,5] the exchanged particle is massless. This case therefore bears some resemblance to the potential case of the Coulomb field. This suggests that for large coupling constants, the Wick-Cutkosky case may be rather different from the ladder diagrams where a massive particle is exchanged. We shall see in Paper II that this is indeed the case.

6. REMARKS

In the analysis of this section, we assume that $g^2 \to \infty$. It is useful to have some idea as to how large g^2 should be. Without loss of generality, let λ be normalized to be 1.

Let us consider first the case $\kappa = 0$ of Sec. 2. It is seen from (10) that we need

$$n + \tfrac{1}{2} \ll g. \qquad (28)$$

For a given large g, (10) is most accurate for the leading Regge pole, but becomes less and less accurate as n increases.

For $\kappa \neq 0$, the approximate results here may be expected to be less accurate. This is most dramatically expressed by (27). With $\lambda = 1$, the validity of (27) requires

$$C_0 g^2 \gg 1$$

or $\qquad (29)$

$$g^2 \gg 26.$$

This is to be compared with $g^2 = 30$, the largest value used by Lovelace and Masson[6] in their numerical calculation. The values used by Ahmadzadeh, Burke, and Tate[7] are smaller.

7. DISCUSSION

Our main interest here is to study the behavior of Regge poles for ladder diagrams in field theory when the coupling constant is large. In Paper II we shall study some simple aspects of this problem. The considerations in this paper on the potential problem serve as a useful introduction. For this purpose, it is sufficient to restrict ourselves to the cases where the Regge pole is real, at least approximately.

The extension of the present considerations to Regge poles not close to the real axis is fairly straightforward. We mention in particular the following special cases:

(i) repulsive Yukawa potential;
(ii) superpositions of Yukawa potentials;
(iii) the case where

$$\kappa^2\lambda^{-1}g^{-2} \leq C_0, \qquad (30)$$

where C_0 is defined by (20). It is interesting to have these cases worked out in detail, although at present they seem to be too complicated to be relevant to the ladder diagrams.

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