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High-Energy Scattering in ϕ^3 Theory and the Breakdown of the Eikonal Approximation. I*

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In order to test the general validity of the impact picture which was extracted from quantum electrodynamics at high energies, we make a systematic study of high-energy scattering amplitudes in ϕ^3 theory. The method we use is the one developed in our treatment of quantum electrodynamics, and it enables us to obtain in closed forms the asymptotic amplitudes for multiple exchanges of Regge poles and scalar particles. Our results are nonperturbative and are valid to all orders of the coupling constant. We show in various examples that (i) the longitudinal momentum of the incident particle is always distributed in positive fractions by the particles created; (ii) an exchanged particle, either composite (Regge pole) or elementary, can carry only transverse momentum; (iii) the scattering amplitude satisfies the impact factor representation; (iv) except in the leading order of the coupling constant, the simple-exponentiation form of the eikonal approximation does not appear. The eikonal approximation fails in ϕ^3 theory because of particle creation: More than one particle may share the incident longitudinal momentum and it is impossible to single out a particular particle to be associated with the eikonal path. The above conclusions are reached by a study of the one-ladder amplitude and the amplitude for the double exchange of a ladder and a scalar particle. An integral equation for the Regge parameters of a ladder was derived and many properties of the Regge parameters were given. In particular, in the Wick-Cutkosky model, the Regge pole at zero invariant mass $\alpha(0)=1$ when the coupling constant g is equal to $4\sqrt{6}\pi m$, where m is the mass of the scalar meson. Thus the physical picture of an expanding black disk in quantum electrodynamics is also realized in the Wick-Cutkosky model as long as $g > 4\sqrt{6}\pi m$. The scattering amplitude for the double exchange of a ladder and a scalar particle is given explicitly in terms of the Regge parameters of the ladder diagrams. Throughout our investigation of ϕ^3 theory at high energies, there is exact correspondence with the results in quantum electrodynamics at every step. This reaffirms our belief of the general validity of the impact picture.

1. INTRODUCTION

Since 1967 we have been pursuing a program of learning about high-energy hadronic scattering by studying relativistic quantum field theories. For this purpose, it is only natural for us to concentrate on quantum electrodynamics, the "best" field theory in the sense of producing theoretical and experimental triumphs in the past. During the last few years, we have obtained a number of concrete results[1] about the high-energy behavior of quantum electrodynamics, and it is very gratifying indeed that each of these results has a simple and natural physical interpretation. In this way, it is possible to apply our results from quantum electrodynamics to hadron physics to obtain a number of definite predictions.[2,3] Whether this extrapolation to hadron physics is justified can only be determined by comparing these predictions with future experiments.

On the theoretical side, it is an interesting question whether our results are peculiar to quantum electrodynamics, i.e., whether they also hold for other relativistic quantum field theories. An affirmative answer to this question would be important evidence supporting our physical interpretations, which depend mostly on very general properties such as time dilation and Lorentz contraction. As a small step in that direction, we have studied scalar electrodynamics,[4] and found that there is no qualitative difference from quantum electrodynamics. However, because of the close similarity between the two cases, this result is to be expected. As a nontrivial test of our physical interpretations, another field theory must be chosen.

Because of the necessity of investing a large amount of time and effort in determining the high-energy behavior of any theory, we must be careful in choosing this field theory for purposes of comparing with quantum electrodynamics. From an experimental point of view, there is no useful field theory aside from quantum electrodynamics. From a theoretical point of view, we must restrict ourselves to *renormalizable* field theories. The simplest renormalizable field theories are, besides quantum electrodynamics and scalar electrodynamics with either a massless photon or a massive vector meson, the following: ϕ^3, ϕ^4, $\bar{\psi}\psi\phi$, and $\bar{\psi}\gamma_5\psi\phi$. All these theories are fundamentally different from the cases of quantum electrodynamics and scalar electrodynamics in the absence of a vector particle. Since the vector particle plays a rather major role in our previous calculations of high-energy behavior, a comparison of any of these four renormalizable field theories with quantum electrodynamics is of great importance.

The choice is basically between ϕ^3 theory on the one hand and ϕ^4, $\bar{\psi}\psi\phi$, and $\bar{\psi}\gamma_5\psi\phi$ theories on the other. In the latter three cases, it is necessary to renormalize a four-scalar-particle vertex. The renormalization program for this vertex has been carried out[5] a number of years ago, but is very complicated by any criterion. We believe that the renormalization of this vertex renders it difficult, without making additional unjustifiable assumptions about this renormalization constant, to determine the high-energy behavior of the elastic scattering amplitude of two scalar particles. By this process of elimination, we study in this paper the high-energy behavior of ϕ^3 theory.

Since 1962, the high-energy behavior of ϕ^3 theory has become a well-explored field.[6,7] Much of the previous work is based on summing leading terms for classes of diagrams, and is valid only in the weak-coupling limit. Unfortunately, ϕ^3 theory, unlike quantum electrodynamics, is not very interesting in such a limit. One of the reasons is that all cross sections vanish rapidly at high energies if the coupling constant is small. This does not agree with the realities in hadron-hadron scattering. A detailed discussion on this point can be found in Sec. 2A.

Since ϕ^3 theory is interesting only if the coupling constant is appreciable, a result in ϕ^3 theory is meaningful only if it holds for all values of the coupling constant. In the context of studying Feynman diagrams, which are perturbative in nature, our attitude must be that of extracting physical principles with general validity. Thus we must distinguish the results which are independent of the size of the coupling constant and the results which are not. In ϕ^3 theory, one of the former is that the one-ladder amplitude is asymptotically of the form $\beta(t)s^{\alpha(t)}$, and one of the latter is that $\alpha(t) = -1$.

The same considerations hold for the multi-ladder amplitude. The observation that the two-ladder amplitude has a Regge cut[8] is independent of the value of the coupling constant, while the recent result that the multi-ladder amplitudes can be summed[9,10] into the eikonal form[11,12] is true only in the weak coupling limit. As a consequence, the conclusions of Chang and Yan[9] on the inelastic differential cross sections, one-particle spectrum, multiplicity, and number distribution have no physical relevance. This is especially true since the eikonal approximation[11,12] indeed breaks down when the coupling constant is not small, as we shall show in Sec. 5.

In this series of papers, we shall study ϕ^3 theory with the goal of gaining general physical insights. We shall first give a discussion on the one-ladder amplitude as a function of the external masses. This discussion is necessary for our later purposes and we have not been able to find it in the literature.[13-15] We shall then concentrate on the amplitude for the exchange of a ladder and a scalar particle and the amplitude for the exchange of two ladders, and calculate these amplitudes to *all* orders of the coupling constant.

A physical picture already emerges from such calculations. It is precisely the impact picture.[1-3] More specifically, we have shown that, in ϕ^3 theory, the following are true:

(i) If the longitudinal momentum of an incident particle is called ω, and if this particle turns into a group of N particles with longitudinal momenta $x_i\omega$, $i=1, 2, \ldots, N$, respectively, then $0 \leq x_i \leq 1$ for all i with $\sum_1^N x_i = 1$.

(ii) A Regge pole which is exchanged in a high-energy collision process can carry only transverse momentum. Specifically, if q_μ is the momentum of a Regge pole in exchange, then q_0 and q_3 are both of the order of ω^{-1}.

(iii) The scattering amplitude satisfies the impact-factor representation.[1] In particular, the amplitude for the process $a+b \to c+d$ with the exchange of two Regge poles is in the form[16] of the impact-factor representation[1]

$$\mathfrak{M}^{(ab\to cd)} \sim i\, s(2\pi)^{-2}\int d\vec{q}_\perp\, \mathcal{g}^{ac}(\vec{r}_1, \vec{q}_\perp)\mathcal{g}^{bd}(\vec{r}_1, \vec{q}_\perp) \times s^{\alpha_1(-(\vec{r}_1+\vec{q}_\perp)^2)+\alpha_2(-(\vec{r}_1-\vec{q}_\perp)^2)-2}. \tag{1.1}$$

In (1.1), α_1 and α_2 denote the two Regge-pole exchanges, $\vec{r}_1+\vec{q}_\perp$ and $\vec{r}_1-\vec{q}_\perp$ denote the momenta carried by the two Regge poles, respectively, and $\mathcal{g}^{ac}$ and $\mathcal{g}^{bd}$ denote the impact factors. The important point is that $\mathcal{g}^{ac}$ is independent of b and d.

Equation (1.1) is satisfied regardless of whether the particles exchanged are composite or elementary. Thus, when vector mesons are exchanged, $\alpha_1=\alpha_2=1$ and (1.1) is the impact-factor representation established in quantum electrodynamics.[1] In the double exchange of a ladder and a scalar meson, $\alpha_1=\alpha$, the Regge pole of the ladder diagrams, and $\alpha_2=0$. Equation (1.1) clearly demonstrates that if one of the particles exchanged is composite (α not a constant), the scattering amplitude has a Regge cut. It further shows that the discontinuity across this branch cut is factorized into a product of the impact factors.

(iv) Just as in quantum electrodynamics,[11] the simple-exponentiation form, which characterizes amplitudes in the eikonal approximation, does *not* occur if the coupling constant is not small. This important point was emphasized in our earlier paper[17] but has apparently been overlooked by other workers in the field. The failure of the eikonal approximation in field theories has a natural interpretation: An eikonal path cannot be defined if the incident particle turns into one or more particles with longitudinal momenta comparable to ω. When the coupling constant is small, the creation and annihilation processes, being of higher order in the coupling constant, can be neglected, and one of the particles takes up all of the longitudinal momentum ω throughout the collision process. An eikonal path can then be defined with respect to this particle, and the eikonal approximation is expected to hold. It is one of the major goals of this series of papers to demonstate explicitly the failure of the eikonal approximation when the coupling constant is not small.

In the first paper of the series, we shall concentrate on the amplitudes for the double exchange of a ladder and a scalar particle. The amplitude for the exchange of two ladders will be studied in the second paper of this series.

2. MAGNITUDE OF COUPLING CONSTANT

A. Case of Small Coupling Constant

When the coupling constant g for the ϕ^3 theory is sufficiently small, the high-energy behavior of scattering amplitudes can be determined completely in a very simple way. The result is merely that of one-particle exchange, and is therefore uninteresting. We discuss this trivial remark in some detail.

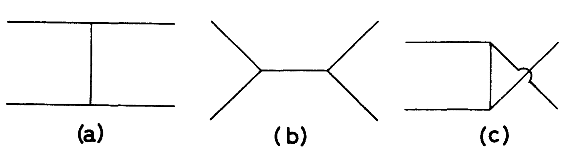

FIG. 1. The lowest-order diagrams for two-body elastic scattering in ϕ^3 theory. The s channel is from left to right and the t channel is from bottom to top.

Consider the two-particle elastic scattering amplitude. In perturbation theory, the lowest-order diagrams are the three shown in Fig. 1, where the s channel is from the left to the right while the t channel is from the bottom to the top. In the limit $s\to\infty$ with fixed t, only the diagram of Fig. 1(a) is important and gives

$$\mathfrak{M}_0=g^2/(|t|+m^2)\,. \tag{2.1}$$

Thus, to this order, the amplitude is of the order s^0. Since for ϕ^3 theory no numerator ever appears in the Feynman rules, no diagram can give an amplitude of a larger order of magnitude. Indeed, the asymptotic behavior for $s\to\infty$ and t fixed can be found to all orders of the coupling constant by considering only diagrams of the form shown in Fig. 2, and the result is simply

$$\mathfrak{M}(s,\,t)=-[\Gamma(t)]^2S_F(t)+o(1)\,. \tag{2.2}$$

In (2.2), $\Gamma(t)$ and $S_F(t)$ are respectively the renormalized vertex function and propagator, where the corresponding bare functions are g and $(t-m^2+i\epsilon)^{-1}$.

Equation (2.2) holds to all orders of the coupling constant g. More precisely, to every finite order of the coupling constant,

$$\mathfrak{M}(s,\,t)+[\Gamma(t)]^2S_F(t)\to 0 \tag{2.3}$$

as $s\to\infty$ with t fixed. In this sense it is a statement about perturbation expansion. Equation (2.2) also holds uniformly in order when the coupling constant g is sufficiently small, as seen from a comparison with the early calculations with the ladder diagrams.[6] The amplitude obtained from the ladder diagrams in the t channel is of the order of

$$s^{-1+O(g^2)} \tag{2.4}$$

for small g, and is therefore negligible. However, when g is not small, the expression (2.4) may be unbounded as $s\to\infty$. In other words, there is a

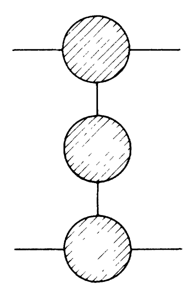

FIG. 2. General form of the diagrams which give amplitudes of the order s^0 in ϕ^3 theory.

critical coupling constant g_c such that the simple result (2.2) holds only for

$$|g| < g_c \tag{2.5}$$

but not for all $|g|$ larger than g_c. (It may or may not hold for $g = \pm g_c$).

In this paper, we shall not be concerned with the uninteresting result (2.2). Therefore, we must consider the coupling constant g to be of the order of m, not a small quantity.

B. Comparison with Quantum Electrodynamics

When (2.2) holds, we have

$$d\sigma/dt = O(s^{-2}) \tag{2.6}$$

as $s \to \infty$ with fixed t. This is very different from the case of quantum electrodynamics, where $d\sigma/dt$ is roughly of the order of 1, not counting factors of $\ln s$. The result of quantum electrodynamics corresponds much more closely to the experimental data of hadronic scattering. In all known cases of elastic scattering, such as pp, $\pi^+ p$, $\pi^- p$, np, etc., the differential cross section $d\sigma/dt$ remains roughly constant, at least when t is not too large. Therefore, both for comparison with the previous results of quantum electrodynamics and for possible extrapolation to hadron physics, the case where (2.2) and hence (2.6) hold is of no interest.

Suppose g is slightly larger than g_c. Then (2.2) and (2.6) fail but $d\sigma/dt$ still approaches zero as $s \to \infty$, although not as fast as s^{-2}. Only for significantly larger values of g, $d\sigma/dt$ does not go to zero as some inverse power of s. We can define a *second critical coupling constant* $\bar{g}_c$ as the greatest lower bound of the positive values of g such that $s^\epsilon d\sigma/dt$, as a function of s, is unbounded for some t and all positive ϵ. Clearly

$$\bar{g}_c > g_c . \tag{2.7}$$

For our purposes, we are interested in the case

$$|g| \geq \bar{g}_c . \tag{2.8}$$

If the mass of the particles in the ladder rungs is equal to zero, it is shown in Appendix A that

$$g_c = 4\sqrt{2}\,\pi m \tag{2.9}$$

and

$$\bar{g}_c = 4\sqrt{6}\,\pi m . \tag{2.10}$$

It is perhaps of interest to recall briefly the physical picture, called the impact picture, that we learned from quantum electrodynamics. According to this picture, in a high-energy collision a particle acts like a Lorentz-contracted disk with a black core whose radius increases logarithmically with energy. Consequently,[3] the total cross section rises indefinitely with energy, the ratio of the total elastic cross section to the total inelastic cross section approaches unity, the product of diffraction width with the total cross section approaches a constant, and, for forward elastic scattering, the imaginary part of the amplitude dominates over the real part. This physical picture is also realized in ϕ^3 theory if $|g| > |\bar{g}_c|$.

C. Potential Theory

Since exponentiation[1,11,12] is of central interest here, we turn briefly to the case of high-energy potential scattering[18,19] where exponentiation is most clearly seen. We shall follow the procedure of Ref. 19, and study the Schrödinger equation

$$(\nabla^2 + k^2 - V)\psi = 0 \tag{2.11}$$

for large k with fixed V/k. In this limit, the matrix element is given by

$$ik\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp[-i(\Delta_1 x + \Delta_2 y)] \times\left[1 - \exp\left(-\tfrac{1}{2} i \int_{-\infty}^{\infty} dz\, U(x, y, z)\right)\right], \tag{2.12}$$

where $\vec{\Delta} = (\Delta_1, \Delta_2)$ is the momentum transfer, and

$$U = V/k . \tag{2.13}$$

The appearance of the last exponential factor is referred to as exponentiation.

The important point is that, for high-energy potential scattering, *exponentiation holds when V/k is fixed.* If instead V is fixed as $k \to \infty$, then it is rather vague to ask whether exponentiation holds or not. Surely (2.12) remains a good approximation because it agrees, to leading order, with the Born approximation

$$-\tfrac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \exp[-i(\Delta_1 x + \Delta_2 y)] V(x, y, z) , \tag{2.14}$$

which does not contain an exponential factor of the potential. Since, in this limit of fixed V, (2.12) is no more accurate than (2.14), there is no compelling reason to include an exponential factor in the matrix element.

The matrix element (2.12) is of the order k for fixed V/k, but of the order 1 for fixed V. A comparison with (2.2) for ϕ^3 theory shows that the ϕ^3 case, with small coupling constant, is closer to the potential case with fixed V. This is a further reason why the case of small coupling constant in ϕ^3 theory is not very interesting.

3. LADDER DIAGRAMS IN THE t CHANNEL

A. Motivation

As we have mentioned in the Introduction, we shall study diagrams in which ladders are inserted. Some examples are illustrated in Fig. 3. Since the external lines of a ladder may represent virtual particles, we need to know, for our purpose, the dependence of the one-ladder amplitude on the external masses. Now the one-ladder amplitude is asymptotically of the Regge form. Thus we must first study the dependence of the Regge parameters on the external masses.

One of the popular methods to study asymptotic amplitudes is summing leading terms. As an example, let us consider the ladder diagrams in Fig. 4. Let us call the amplitude corresponding to the n-rung ladder in Fig. 4 $g^{2n}\mathfrak{M}_L^{(n)}$, where g is the coupling constant, and put

$$\mathfrak{M}_L = \sum_{n=1}^{\infty} g^{2n}\mathfrak{M}_L^{(n)} . \tag{3.1}$$

The method of summing leading terms consists of obtaining the asymptotic form for $\mathfrak{M}_L^{(n)}$ in the high-energy limit, and then summing up these asymptotic terms. In other words, we make the following replacement in the limit $s\to\infty$ with fixed t:

$$\begin{aligned}\text{asymptotic form of } \mathfrak{M}_L &= \text{asymptotic form of } \sum_{n=1}^{\infty} g^{2n}\mathfrak{M}_L^{(n)} \\ &\to \sum_{n=1}^{\infty} g^{2n}\times \text{asymptotic form of } \mathfrak{M}_L^{(n)} .\end{aligned} \tag{3.2}$$

We wish to emphasize that we must be cautious in using this method in the above limit. To see this, we start with the well-known formula

$$\mathfrak{M}_L \sim \beta s^{\alpha}, \tag{3.3}$$

the Regge asymptotic form. The Regge parameters β and α have the perturbation expansion[6]

$$\begin{aligned}\alpha &= -1 + O(g^2) , \\ \beta &= -g^2 + O(g^4) .\end{aligned} \tag{3.4}$$

Thus we make the expansion

$$s^{\alpha} = s^{-1}\sum_{n=0}^{\infty} (\ln s)^n (\alpha+1)^n/n! . \tag{3.5}$$

It is found that the largest terms in the summation above are those with

$$n = O(\ln s) . \tag{3.6}$$

This can be shown easily if we use the Stirling formula for $n!$. Now $(\alpha+1)$ has the perturbation expansion

$$\alpha + 1 = a g^2 + b g^2 + \cdots ; \tag{3.7}$$

thus

$$(\alpha+1)^n/g^{2n} = a^n + n a^{n-1} b g^2 + \cdots . \tag{3.8}$$

The method of summing leading terms is equivalent to replacing the right-hand side of (3.8) by a^n. This is not justified, as the next-order term $na^{n-1}bg^2$ is always larger than the leading term a^n for $n = O(\ln s)$ in the limit $s\to\infty$, no matter what value the coupling constant is.

We may also understand the method of summing leading terms in the following way. Instead of (3.5), let us expand s^{α} in the Taylor series of the coupling constant:

$$s^{\alpha} = s^{-1}\sum_{n=1}^{\infty} g^{2n} A_n . \tag{3.9}$$

Then by (3.7) and (3.9)

$$A_n = (a\ln s)^n/n! + a^{n-2} b (\ln s)^{n-1}/(n-2)! + \cdots . \tag{3.10}$$

The method of summing leading terms is equivalent to replacing the right-hand side of (3.10) by the leading term $(a\ln s)^n/n!$. This is justified only if the next-order term $a^{n-2}b(\ln s)^{n-1}/(n-2)!$ is smaller than the leading term, which is true only

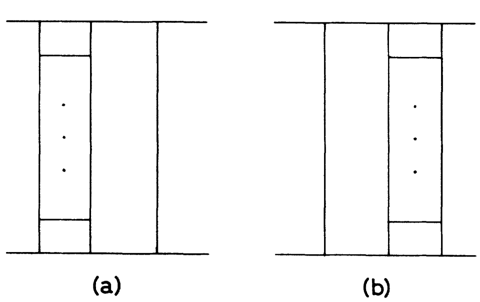

FIG. 3. Two of the diagrams for the double exchange of a ladder and a scalar particle. A ladder with dots inside represents the sum of all ladder diagrams.

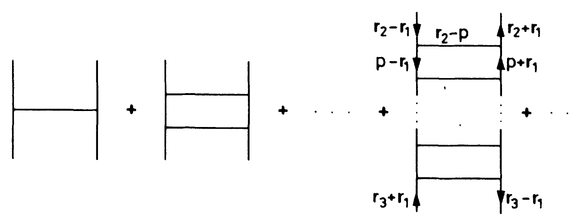

FIG. 4. Ladder diagrams.

for

$$n^2 = O(\ln s)\,. \tag{3.11}$$

Equation (3.11) means that the next-order term is already of the order of the leading term when n is of the order of $(\ln s)^{1/2}$, while by (3.6) the important terms are those with n of the order of $\ln s$. Thus the method of summing leading terms cannot be justified.

If, instead of the limit $s\to\infty$, we consider the following limit,

$$g^2 \to 0, \quad s\to\infty,$$

in such a way that

$$g^2 \ln s \to 0\,, \tag{3.12}$$

then it is justified to replace the right-hand side of (3.8) by a^n. It is in this peculiar weak-coupling limit that the method of summing leading terms is justified.

Since we are interested in ϕ^3 theory with arbitrary coupling constant, we must not resort to the method of summing leading terms. Instead, we shall derive below an integral equation satisfied by α and β to all orders of the coupling constant. This equation turns out to be the same as that derived by Amati, Stanghellini, and Fubini[20] long ago. However, we shall give our derivation here for the purpose of retaining continuity.

B. Equations for the Regge Parameters

We shall adopt our usual notations[1] r_1, r_2, and r_3 for the momenta. The meaning of these momenta can be seen from Fig. 4. We put

$$\begin{aligned} (r_2-r_1)^2 &= M_i^2\,,\\ (r_2+r_1)^2 &= M_f^2\,,\\ (r_3+r_1)^2 &= M_i'^2\,,\\ (r_3-r_1)^2 &= M_f'^2\,. \end{aligned} \tag{3.13}$$

We also have

$$s=(r_2+r_3)^2 \quad\text{and}\quad t=4r_1^2\,.$$

We shall study the Regge parameters as functions of t and the external masses M_i, M_i', M_f, M_f'.

With reference to Fig. 4, the Bethe-Salpeter equation for $\mathfrak{M}_L(r_1, r_2, r_3)$ is

$$\mathfrak{M}_L(r_1, r_2, r_3) = -g^2(s-\lambda^2)^{-1} + ig^2(2\pi)^{-4}\int d^4p\,\frac{\mathfrak{M}_L(r_1, p, r_3)}{[(r_2-p)^2-\lambda^2+i\epsilon][(p+r_1)^2-m^2+i\epsilon][(p-r_1)^2-m^2+i\epsilon]}\,, \tag{3.14}$$

where m is the mass of the particle in a vertical internal line and λ is that in a horizontal internal line. We keep m and λ distinct because, at $t=0$, the case $\lambda=0$ can be solved in closed form, as is done in Appendix A. In Eq. (3.14), p is the independent variable while r_1 and r_3 are regarded as constants. Hence M_i' and M_f' are regarded as constants in (3.14).

In the limit $s\to\infty$ with t and the external masses fixed, we have

$$\mathfrak{M}_L(r_1, r_2, r_3) \sim \beta(t, M_i^2, M_f^2)\, s^{\alpha(t, M_i^2, M_f^2)}\,, \tag{3.15}$$

the Regge asymptotic form. The Regge parameters α and β are functions of M_i' and M_f' also. However, since M_i' and M_f' are held fixed in (3.14), for simplicity we shall not exhibit the dependence on these masses here. Substituting (3.15) into (3.14) and assuming that

$$\mathrm{Re}\,\alpha > -1\,, \tag{3.16}$$

we get

$$\beta(t, M_i^2, M_f^2)\, s^{\alpha(t, M_i^2, M_f^2)} \sim ig^2(2\pi)^{-4}\int d^4p\,\frac{\beta(t, (p-r_1)^2, (p+r_1)^2)[(r_3+p)^2]^{\alpha(t,(p-r_1)^2,(p+r_1)^2)}}{[(r_2-p)^2-\lambda^2+i\epsilon][(p+r_1)^2-m^2+i\epsilon][(p-r_1)^2-m^2+i\epsilon]}\,. \tag{3.17}$$

Let us put[1]

$$p_\pm = p_0 \pm p_3$$

and

$$p_+ = 2\omega x\,. \tag{3.18}$$

Also, we have

$$r_2 \sim [\omega + (4\omega)^{-1}(2\vec{r}_1^{\,2} + M_i^{\,2} + M_f^{\,2}),\ \omega,\ 0]\,, \tag{3.19}$$

where the quantities in the above bracket are the energy, the z component of the momentum, and the transverse components of the momentum, respectively. Then

$$\begin{aligned}(r_2 - p)^2 - \lambda^2 + i\epsilon &\sim (1-x)[\vec{r}_1^{\,2} + \tfrac{1}{2}(M_i^{\,2} + M_f^{\,2}) - 2\omega p_-] - \vec{p}_\perp^{\,2} - \lambda^2 + i\epsilon\,,\\ (p \pm r_1)^2 - m^2 + it &\sim x[\pm \tfrac{1}{2}(M_f^{\,2} - M_i^{\,2}) + 2\omega p_-] - (\vec{p}_\perp \pm \vec{r}_1)^2 - m^2 + i\epsilon\,.\end{aligned} \tag{3.20}$$

We also know that, considered as an analytic function of $M_i^{\,2}$ ($M_f^{\,2}$), $\mathfrak{M}_L$ is regular in the upper half-plane $\mathrm{Im}M_i^{\,2} \geqslant 0$ ($\mathrm{Im}M_f^{\,2} \geqslant 0$). Thus, considered as an analytic function of $M_i^{\,2}$ ($M_f^{\,2}$), the Regge parameters α and β are regular in the upper half-plane $\mathrm{Im}M_i^{\,2} \geqslant 0$ ($\mathrm{Im}M_f^{\,2} \geqslant 0$). Therefore, we may carry out the integration over p_- in (3.17) by closing the contour in the complex plane of p_-. It is easily shown that unless

$$0 < x < 1\,, \tag{3.21}$$

we can always close the contour in such a way that no singularities of the integrand are enclosed. When (3.21) is satisfied, we close the contour to the upper half-plane of p_-, and (3.17) becomes

$$\begin{aligned}&\beta(t,\, M_i^{\,2},\, M_f^{\,2})\, s^{\alpha(t,\, M_i^2,\, M_f^2)}\\ &\quad\sim \tfrac{1}{2} g^2 (2\pi)^{-3} \int d\vec{p}_\perp \int_0^1 dx (1-x)\\ &\qquad\times \frac{\beta(t,\, \mu_i^{\,2},\, \mu_f^{\,2})(xs)^{\alpha(t,\, \mu_i^2,\, \mu_f^2)}}{\{[\vec{p}_\perp + (1-x)\vec{r}_1]^2 + (1-x)m^2 + x\lambda^2 - x(1-x)M_f^{\,2} - i\epsilon\}\{[\vec{p}_\perp - (1-x)\vec{r}_1]^2 + (1-x)m^2 + x\lambda^2 - x(1-x)M_i^{\,2} - i\epsilon\}}\,,\end{aligned} \tag{3.22}$$

where

$$\mu_i^{\,2} = -\{[\vec{p}_\perp - (1-x)\vec{r}_1]^2 - x(1-x)M_i^{\,2} + x\lambda^2\}/(1-x) \tag{3.23}$$

and

$$\mu_f^{\,2} = -\{[\vec{p}_\perp + (1-x)\vec{r}_1]^2 - x(1-x)M_f^{\,2} + x\lambda^2\}/(1-x)\,. \tag{3.24}$$

Since the two sides of (3.22) have different s dependence unless

$$\alpha(t,\, \mu_i^{\,2},\, \mu_f^{\,2}) = \alpha(t)\,, \tag{3.25}$$

we conclude that (3.25) must hold, and thus (3.22) becomes

$$\begin{aligned}&\beta(t,\, M_i^{\,2},\, M_f^{\,2})\\ &\quad= \tfrac{1}{2} g^2 (2\pi)^{-3} \int d\vec{p}_\perp \int_0^1 dx (1-x) x^{\alpha(t)}\\ &\qquad\times \frac{\beta(t,\, \mu_i^{\,2},\, \mu_f^{\,2})}{\{[\vec{p}_\perp + (1-x)\vec{r}_1]^2 + (1-x)m^2 + x\lambda^2 - x(1-x)M_f^{\,2} - i\epsilon\}\{[\vec{p}_\perp - (1-x)\vec{r}_1]^2 + (1-x)m^2 + x\lambda^2 - x(1-x)M_i^{\,2} - i\epsilon\}}\,.\end{aligned} \tag{3.26}$$

Notice that Eq. (3.26) is independent of s, M_i', and M_f'. The factors in the denominator of (3.26) are familiar – similar factors also appear in high-energy amplitudes in quantum electrodynamics.[1] In fact, they are the denominator factors in the impact diagram rules.[21]

Equation (3.26) is an eigenvalue equation, with α the eigenvalue and β the eigenfunction. Since (3.26) is homogeneous, β can be determined only up to a factor f which is independent of $M_i^{\,2}$ and $M_f^{\,2}$. This factor f is a function of t, M_i', and M_f'. Thus

$$\beta(t;\ M_i^{\,2},\, M_f^{\,2};\ M_i'^{\,2},\, M_f'^{\,2}) = f(t,\, M_i'^{\,2},\, M_f'^{\,2})\, b(t,\, M_i^{\,2},\, M_f^{\,2})\,, \tag{3.27}$$

which exhibits explicitly the factorization of the Regge residue function β. In (3.27), we have written out the dependence of β on all four of the external masses. Notice that $b(t, M_i^2, M_f^2)$ is now independent of M_i' and M_f', as these two masses do not appear in (3.26). If all of the external particles are the same scalar meson, we may choose by symmetry

$$f = b$$

and

$$\beta(t;\ M_i^2, M_f^2;\ M_i'^2, M_f'^2) = b(t, M_i'^2, M_f'^2) b(t, M_i^2, M_f^2)\,. \tag{3.28}$$

Substituting (3.28) and (3.25) into (3.15), we get

$$\mathfrak{M}_L(r_1, r_2, r_3) \sim b(t, M_i^2, M_f^2) b(t, M_i'^2, M_f'^2) s^{\alpha(t)}\,. \tag{3.29}$$

Also, Eq. (3.26) can be written as

$$b(t, M_i^2, M_f^2) = \tfrac{1}{2} g^2 (2\pi)^{-3} \int d\vec{p}_\perp \int_0^1 dx (1-x) x^{\alpha(t)} \times \frac{b(t, \mu_i^2, \mu_f^2)}{\{[\vec{p}_\perp + (1-x)\vec{r}_1]^2 + (1-x)m^2 + x\lambda^2 - x(1-x)M_f^2 - i\epsilon\}\{[\vec{p}_\perp - (1-x)\vec{r}_1]^2 + (1-x)m^2 + x\lambda^2 - x(1-x)M_i^2 - i\epsilon\}}\,, \tag{3.30}$$

where μ_i^2 and μ_f^2 are given by (3.23) and (3.24) and b is determined up to a factor which depends on t only.

In summary, we have found that (a) the Regge trajectory α is a function of t only, and is independent of the external masses; (b) the Regge residue β is factorized in the form (3.28); (c) the functions α and b are determined by the eigenvalue equation (3.30). All of these conclusions are of course well known.

It is particularly gratifying that for high-energy scattering, $0<x<1$. This confirms (i) of the physical picture discussed in Sec. 1. It is also interesting that the denominator factors in the impact diagram rules also appear in amplitudes of ϕ^3 theory. In fact, the formulas here are in striking resemblance to those in quantum electrodynamics.

C. Properties of the Regge Parameters

We shall here list the properties of the Regge parameters $\alpha(t)$ and $b(t, M_i^2, M_f^2)$. The derivations of these properties are given in Appendixes A, B, and C.

The perturbation series for the Regge parameters are[14]

$$\alpha(t) = -1 + \frac{g^2}{16\pi^2} \int_0^1 dx [-x(1-x)\,t + m^2 - i\epsilon]^{-1} + \frac{g^4}{16\pi^3} \int d\vec{p}_\perp B(t, -(\vec{p}_\perp - \vec{r}_1)^2, -(\vec{p}_\perp + \vec{r}_1)^2) [(\vec{p}_\perp + \vec{r}_1)^2 + m^2]^{-1} [(\vec{p}_\perp - \vec{r}_1)^2 + m^2]^{-1} + O(g^6) \tag{3.31}$$

and

$$b(t, M_i^2, M_f^2) = f(t)[1 + g^2 B(t, M_i^2, M_f^2) + O(g^4)]\,, \tag{3.32}$$

where

$$B(t, M_i^2, M_f^2) = (16\pi^3)^{-1} \int d\vec{p}_\perp \int_0^1 dx\, x^{-1} \times \left(\frac{1-x}{\{[\vec{p}_\perp + (1-x)\vec{r}_1]^2 + m^2(1-x) + \lambda^2 x - x(1-x)M_f^2 - i\epsilon\}\{[\vec{p}_\perp - (1-x)\vec{r}_1]^2 + m^2(1-x) + \lambda^2 x - x(1-x)M_i^2 - i\epsilon\}} - \frac{1}{[(\vec{p}_\perp + \vec{r}_1)^2 + m^2][(\vec{p}_\perp - \vec{r}_1)^2 + m^2]} \right)\,, \tag{3.33}$$

and $f(t)$ is an over-all factor which cannot be determined from the integral equation (3.30). From previous calculations,[6] we know that

$$[f(t)]^2 = -g^2 + O(g^4)\,. \tag{3.34}$$

In the limit $(|M_i|^2 + |M_f|^2) \to \infty$ with t fixed, we have

$$b(t, M_i^2, M_f^2) \sim c(t)(|M_i^2| + |M_f|^2)^{-\alpha(t)-1} . \quad (3.35)$$

The over-all constant $c(t)$ cannot be determined from (3.30). It has the perturbation expansion[6]

$$c(t) = ig[1 + O(g^2)] . \quad (3.36)$$

The case $t = \lambda^2 = 0$ can be solved in closed form. We have

$$\alpha(0) = -\tfrac{3}{2} + [\tfrac{1}{4} + g^2/(16\pi^2 m^2)]^{1/2} \quad (3.37)$$

and

$$b(0, M^2, M^2) = c(m^2 - M^2 - i\epsilon)^{-\alpha(0)-1} . \quad (3.38)$$

In particular, $\alpha(0) = 0$ if

$$g = \pm 4\sqrt{2}\,\pi m . \quad (3.39)$$

Also, $\alpha(0) = 1$ if

$$g = \pm 4\sqrt{6}\,\pi m . \quad (3.40)$$

Therefore, the Froissart bound[22] is violated by the one-ladder amplitude if $g > 4\sqrt{6}\pi m$ and a particle resembles a black disk with a radius increasing logarithmically with the energy.[3]

4. THE EXCHANGE OF A LADDER AND A PARTICLE

Mandelstam[8] first pointed out that the diagram in Fig. 5 gives the Regge cut arising from the exchange of a ladder and a scalar particle. The corresponding high-energy amplitude has been calculated explicitly by the method of summing leading terms.[23,24]

In this section, we shall calculate the amplitude

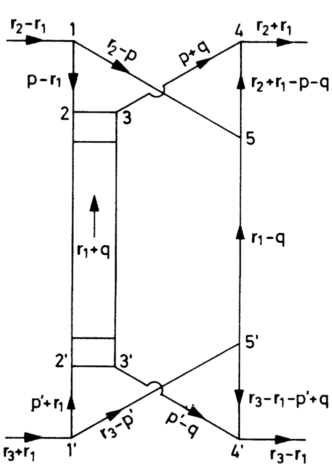

FIG. 5. A diagram for the double exchange of a ladder and a scalar particle. This diagram gives a Regge cut.

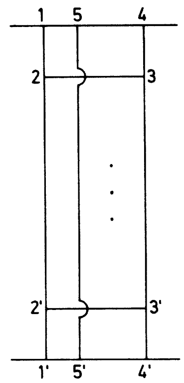

FIG. 6. Another way to draw the diagram in Fig. 5.

corresponding to Fig. 5 in the high-energy limit to *all* orders of the coupling constant. The calculation is actually rather straightforward, as we shall soon demonstrate. Most interestingly, we found that this amplitude is in the form of the impact-factor representation.[1]

A. Preliminaries

Before we plunge into the calculations, let us first decide if there are other diagrams which con-

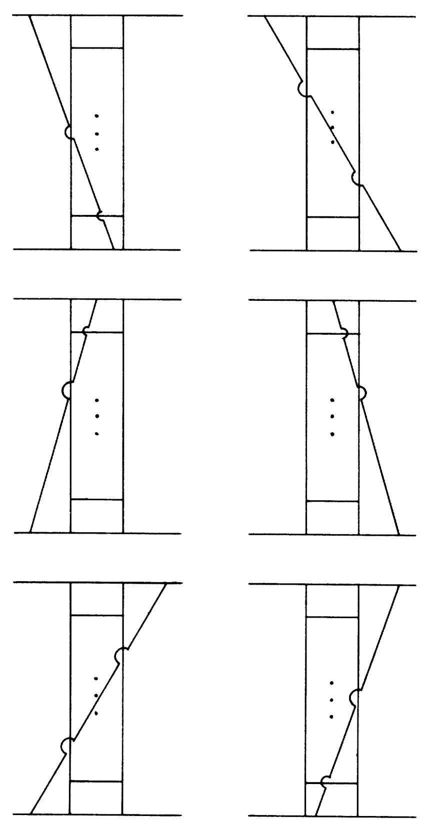

FIG. 7. More diagrams for the double exchange of a ladder and a scalar particle.

tribute to the exchange in consideration. For this purpose, it is appropriate to take hints from quantum electrodynamics.

In quantum electrodynamics, we have gone considerably further in the study of exchanges of composite or elementary systems.[11] For simplicity, let us restrict ourselves to electron-electron scattering. Then the counterparts of ladder diagrams are tower diagrams.[3,25] We have shown in this connection that, to the leading order, an eikonal form is obtained if we include all multi-tower exchange diagrams with different permutations of the tower legs. To go back to ϕ^3 theory, this means that, in addition to the diagrams in Fig. 3, we must consider the diagrams in Fig. 6 and Fig. 7.

We first note that the diagram in Fig. 6 is precisely the Mandelstam diagram in Fig. 5. This can be seen by comparing the arabic numerals in the two figures. Next we observe that the scattering amplitudes corresponding to the diagrams in Fig. 3 and Fig. 7 are negligible in the high-energy limit.[26] Although this observation is well known, we shall give an explanation of it from our viewpoint. In the process of doing so, we shall also justify (ii) of the physical picture discussed in Sec. 1.

All of the diagrams in Fig. 3 and Fig. 7 share one feature: Either their top or bottom halves, or both, are of one of the forms illustrated in Fig. 8. Let us denote the momenta carried by the Regge pole and the scalar particle exchanged as r_1+q and r_1-q, respectively. We shall assume that q_+q_- is small and shall justify this assumption in a moment. Then it is fairly easy to show that if we carry out the integration over q_- $(=q_0-q_3)$, we get zero. Consider first the diagram 8(a). The q_--dependent factors in the scattering amplitude are

$$[(r_2+q)^2-m^2+i\epsilon]^{-1}\sim(2\omega q_-+\vec{r}_1^{\,2}+i\epsilon)^{-1}, \tag{4.1}$$

$$[(r_1-q)^2-m^2+i\epsilon]^{-1}\sim[-(\vec{r}_1-\vec{q}_\perp)^2-m^2]^{-1}, \tag{4.2}$$

and

$$b((r_1+q)^2,\, m^2,\, (r_2+q)^2) \sim b(-(\vec{r}_1+\vec{q}_\perp)^2,\, m^2,\, 2\omega q_-+\vec{r}_1^{\,2}+m^2)\,, \tag{4.3}$$

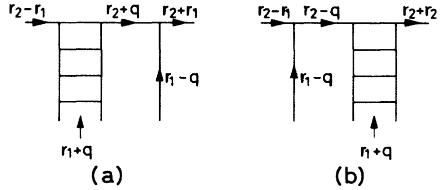

FIG. 8. The top halves of the diagrams in Fig. 3.

where (3.29) has been used. All of the above factors are regular in the upper half-plane of q_-. Also, by the asymptotic form (3.35), the Regge parameter b vanishes when one or both of the external masses approach infinity. Thus we may close the contour in the upper half-plane of q_- and the integral over q_- vanishes. The same considerations apply to diagram 8(b).

It is probably worth pointing out that if we replace the Regge parameter b by its lowest-order term ig, the above argument would not go through, as the integrand does not vanish fast enough at infinity. This is one example of the pitfalls in using leading terms.

We shall now justify the assumption that q_+q_- is small. In brief, if we keep q_+q_-, there are, in the q_- plane, additional singularities which are, in the high-energy limit, so far away that they do not contribute. For the sake of clarity, let us treat the diagram 3(a) in detail. The corresponding scattering amplitude is proportional to

$$\begin{aligned}\int d^4q\, &b((r_1+q)^2,\, m^2,\, (r_2+q)^2)\\ &\times b((r_1+q)^2,\, m^2,\, (r_3-q)^2)\, s^{\alpha((r_1+q)^2)}\\ &\times[(r_2+q)^2-m^2+i\epsilon]^{-1}[(r_3-q)^2-m^2+i\epsilon]^{-1}\\ &\times[(r_1-q)^2-m^2+i\epsilon]^{-1}.\end{aligned} \tag{4.4}$$

Now

$$(r_1\pm q)^2\sim q_+q_- -(\vec{r}_1\pm\vec{q}_\perp)^2, \tag{4.5}$$

$$(r_2+q)^2\sim(2\omega+q_+)[q_-+(m^2+\vec{r}_1^{\,2})/(2\omega)]-\vec{q}_\perp^{\,2}, \tag{4.6}$$

$$(r_3-q)^2\sim(2\omega-q_-)[-q_++(m^2+\vec{r}_1^{\,2})/(2\omega)]-\vec{q}_\perp^{\,2}. \tag{4.7}$$

Substituting (4.5)–(4.7) into (4.4), we find that the integrand in (4.4) is an analytic function in the complex q_- plane and, unless

$$-2\omega\leqslant q_+\leqslant 0\,, \tag{4.8}$$

the singularities are always located on the same side of the real axis. Thus we may set

$$q_+=-2\omega x\,, \tag{4.9}$$

and restrict ourselves to $0\leqslant x\leqslant 1$. The singularities from

$$(r_1\pm q)^2-m^2=0 \tag{4.10}$$

are located at

$$\omega q_-=O(x^{-1})\,. \tag{4.11}$$

Thus, evaluated at these singularities,

$$(r_2+q)^2=O(x^{-1})\,. \tag{4.12}$$

Hence, if x is very small, $b((r_1+q)^2,\, m^2,\, (r_2+q)^2)$ vanishes like $x^{\alpha+1}$ as a result of the asymptotic form (3.35). On the other hand, if x is nonzero,

$$(r_3-q)^2\sim 4\omega^2 x\,, \tag{4.13}$$

and $b((r_1+q)^2,\, m^2,\, (r_3-q)^2)$ vanishes like $(\omega^2)^{-\alpha-1}$. In either case, the contributions by the singularities deterimined from (4.10) are negligible.

In the same way, we may see that the singularity in the q_- plane from

$$(r_3-\mathrm{q})^2-m^2=0$$

is also of negligible contribution. Thus all additional singularities arising from retaining q_+q_- have negligible contribution.

In general, the longitudinal momentum carried by a Regge pole is always small compared to those of the incident particles. Otherwise the Regge parameter b would become very small as a result of (3.35). Thus we have verified (ii) of the physical picture discussed in Sec. 1.

We wish to emphasize that the composite nature of a Regge pole is crucial to the validity of our arguments above. For example, if the Regge pole in Fig. 3 is replaced by a scalar particle, the amplitude corresponding to the diagrams in Fig. 3 is no longer negligible. For an individual diagram with exchange of elementary particles such as the one in Fig. 9(a), q_+q_- cannot be neglected. Only when we consider the *sum* of all diagrams with different permutations in the ordering of the exchange can we neglect q_+q_-. For example, it is justified to neglect q_+q_- if we consider the sum of diagrams in Fig. 9, but we should not neglect q_+q_- if we consider any particular one of the diagrams in Fig. 9.

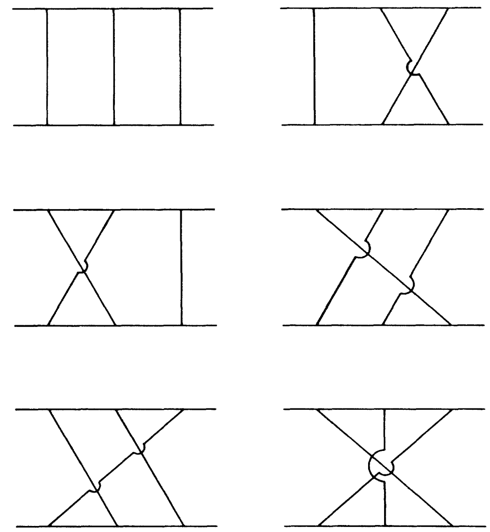

FIG. 9. Some sixth-order diagrams.

Applying the above arguments, we see that, for the exchange of two composite systems, the legs of the two systems must be intertwined, *both* for the top *and* the bottom parts. More precisely, if, in either the top or the bottom parts of a diagram, the legs of one system can be completely separated from the legs of the other by cutting one internal line, then this diagram is negligible.

B. The Impact-Factor Representation

Let us calculate the amplitude $\mathfrak{M}'_{LS}$ corresponding to the diagram of Fig. 5. We have

$$\begin{aligned}\mathfrak{M}'_{LS}=-ig^6(2\pi)^{-12}\int d^4p\,d^4p'\,d^4q\,&[(p-r_1)^2-m^2+i\epsilon]^{-1}[(p+q)^2-m^2+i\epsilon]^{-1}\\ &\times[(r_2+r_1-p-q)^2-m^2+i\epsilon]^{-1}[(r_2-p)^2-m^2+i\epsilon]^{-1}[(p'+r_1)^2-m^2+i\epsilon]^{-1}\\ &\times[(p'-q)^2-m^2+i\epsilon]^{-1}[(r_3-p')^2-m^2+i\epsilon]^{-1}[(r_3-r_1-p'+q)^2-m^2+i\epsilon]^{-1}\\ &\times[(r_1-q)^2-m^2+i\epsilon]^{-1}\mathfrak{M}_L(p+q,p'-q;\;p-r_1,p'+r_1)\,.\end{aligned} \tag{4.14}$$

In (4.14), $\mathfrak{M}_L(p_f,p_f';\;p_i,p_i')$ is the amplitude for the sum of ladder diagrams discussed in Sec. 3, and (r_1+q) is the momentum carried by the ladder. The external particles will be taken to be on the mass shell and all particles except those in the ladder have mass m.

In the c.m. system, we have

$$r_{2+}\sim 2\omega\,,\qquad r_{3-}\sim 2\omega\,, \tag{4.15}$$

where

$$s\sim 4\omega^2\,. \tag{4.16}$$

We put

$$p_+ = 2x\omega, \tag{4.17}$$

$$p'_- = 2x'\omega; \tag{4.18}$$

then by (3.29),

$$\mathfrak{M}_L(p+q,\ p'-q;\ p-r_1,\ p'+r_1) \sim b((q+r_1)^2,\ (p-r_1)^2,\ (p+q)^2)\,b((q+r_1)^2,\ (p'+r_1)^2,\ (p'-q)^2)(xx's)^{\alpha((q+r_1)^2)}. \tag{4.19}$$

Also,

$$(p-r_1)^2 - m^2 + i\epsilon \sim 2x\omega p_- - (\vec{p}_\perp - \vec{r}_1)^2 - m^2 + i\epsilon, \tag{4.20}$$

$$(r_2+r_1-p-q)^2 - m^2 + i\epsilon \sim (1-x)(\vec{r}_1{}^2 + m^2 - 2\omega p_- - 2\omega q_-) - (\vec{r}_1 - \vec{p}_\perp - \vec{q}_\perp)^2 - m^2 + i\epsilon, \tag{4.21}$$

$$(p+q)^2 - m^2 + i\epsilon \sim 2x\omega(p_- + q_-) - (\vec{p}_\perp + \vec{q}_\perp)^2 - m^2 + i\epsilon, \tag{4.22}$$

$$(r_2-p)^2 - m^2 + i\epsilon \sim (1-x)(\vec{r}_1{}^2 + m^2 - 2\omega p_-) - \vec{p}_\perp{}^2 - m^2 + i\epsilon, \tag{4.23}$$

and similarly for the other factors dependent on p'. Let us call

$$\begin{aligned} g^{LS}(\vec{r}_1, \vec{q}) = -\omega g^3(2\pi)^{-5} \int d^4p\,dq_-[(p-r_1)^2 - m^2 + i\epsilon]^{-1}[(p+q)^2 - m^2 + i\epsilon]^{-1} \\ \times[(r_2+r_1-p-q)^2 - m^2 + i\epsilon]^{-1}[(r_2-p)^2 - m^2 + i\epsilon]^{-1} \\ \times b((q+r_1)^2,\ (p-r_1)^2,\ (p+q)^2)x^{\alpha((q+r_1)^2)}. \end{aligned} \tag{4.24}$$

Then by substituting (4.20)–(4.23) into (4.24) and carrying out the integration over p_- and q_-, we get

$$\begin{aligned} g^{LS}(\vec{r}_1, \vec{q}_\perp) = g^3(32\pi^3)^{-1}\int d\vec{p}_\perp \int_0^1 dx[(\vec{p}_\perp + \vec{Q})^2 + (1-x+x^2)m^2]^{-1} \\ \times[(\vec{p}_\perp - \vec{Q})^2 + (1-x+x^2)m^2]^{-1} b(-(\vec{q}_\perp + \vec{r}_1)^2;\ \lambda_i{}^2,\ \lambda_f{}^2)x^{\alpha(-(\vec{q}_\perp+\vec{r}_1)^2)}, \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} \lambda_i{}^2 &= -[(\vec{p}_\perp - \vec{Q})^2 + x^2m^2]/(1-x), \\ \lambda_f{}^2 &= -[(\vec{p}_\perp + \vec{Q})^2 + x^2m^2]/(1-x), \end{aligned} \tag{4.26}$$

with

$$\vec{Q} = \tfrac{1}{2}\vec{q}_\perp + (\tfrac{1}{2} - x)\vec{r}_1. \tag{4.27}$$

In deriving (4.25), a translation in the integration variable $\vec{p}_\perp$ has been made.

From (4.14), (4.19), and (4.24), we finally have

$$\mathfrak{M}'_{LS} = 2i(2\pi)^{-2}\int d\vec{q}_\perp s^{\alpha(-(\vec{q}_\perp+\vec{r}_1)^2)-1}[g^{LS}(\vec{r}_1, \vec{q}_\perp)]^2 \times [(\vec{q}_\perp - \vec{r}_1)^2 + m^2]^{-1}, \tag{4.28}$$

which is the impact-factor representation.

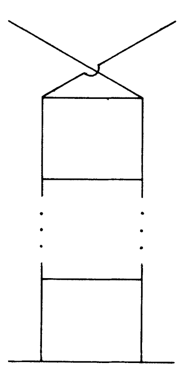

FIG. 10. The crossed ladder.

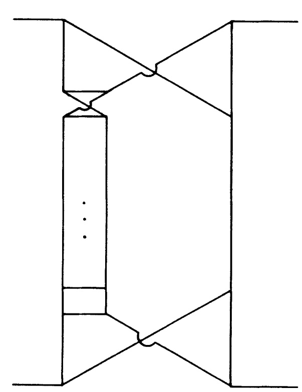

FIG. 11. A diagram for the double exchange of a crossed ladder and a scalar particle.

The integral in (4.28) does not converge at the infinity of $\vec{q}_\perp$. This divergence can be cured by putting in the signature factor for the Regge pole. The amplitude corresponding to the sum of ladder diagrams in Fig. 10 is asymptotically

$$e^{i\pi\alpha(t)}\beta(t)s^{\alpha(t)}. \tag{4.29}$$

Thus the scattering amplitude corresponding to the diagram in Fig. 11 is asymptotically given by the impact-factor representation

$$2i(2\pi)^{-2}\int d\vec{q}_\perp s^{\alpha(-(\vec{q}_\perp+\vec{r}_1)^2)-1}e^{i\pi\alpha(-(\vec{q}_\perp+\vec{r}_1)^2)}[g^{LS}(\vec{r}_1,\vec{q}_\perp)]^2[(\vec{q}_\perp-\vec{r}_1)^2+m^2]^{-1}. \tag{4.30}$$

Adding up the amplitudes (4.28) and (4.30), we get

$$\mathfrak{M}_{LS}\sim 2i(2\pi)^{-2}\int d\vec{q}_\perp s^{\alpha(-(\vec{q}_\perp+\vec{r}_1)^2)-1}(1+e^{i\pi\alpha(-(\vec{q}_\perp+\vec{r}_1)^2)})[g^{LS}(\vec{r}_1,\vec{q}_\perp)]^2[(\vec{q}_\perp-\vec{r}_1)^2+m^2]^{-1}. \tag{4.31}$$

The integral in (4.31) converges as the signature factor vanishes at $|\vec{q}_\perp|\to\infty$.

We once again emphasize that no assumption on the size of the coupling constant has been made. Thus (4.31) gives the asymptotic form of the high-energy amplitude for the exchange of a ladder plus a scalar particle, with the neglected terms smaller than (4.31) by a power of s.

If $\alpha(t)$ is a monotonically increasing function of t, a steepest-descent calculation easily reduces (4.31) to

$$\mathfrak{M}_{LS}\sim i(2\pi)^{-1}[\alpha'(0)\ln s]^{-1}s^{\alpha(0)-1}(-t+m^2)^{-1}[1+e^{i\pi\alpha(0)}][g^{LS}(\vec{r}_1,\vec{r}_1)]^2[1+O((\ln s)^{-1})], \tag{4.32}$$

where

$$\alpha'(t)=d\alpha(t)/dt\,.$$

5. BREAKDOWN OF EIKONAL APPROXIMATION

Let us first give a precise definition of simple exponentiation. By simple exponentiation of a Regge exchange and a scalar-meson exchange, we mean that the scattering amplitude can be written in the form

$$2is\int d\vec{x}_\perp e^{-i\vec{\Delta}\cdot\vec{x}_\perp}(1-\exp\{\tfrac{1}{2}i[A(s,\vec{x}_\perp)+B(s,\vec{x}_\perp)]\}), \tag{5.1}$$

where the terms in (5.1) proportional to $A, B, AB, \ldots$ are equal to the amplitudes for the exchange of a Regge pole, the exchange of a scalar meson, the exchange of a Regge pole plus a scalar meson, ..., respectively. Thus

$$[1+e^{i\pi\alpha(t)}]\mathfrak{M}_L(s,t)\sim s\int dx\, e^{-i\vec{\Delta}\cdot\vec{x}_\perp}A(s,\vec{x}_\perp), \tag{5.2}$$

$$g^2(-t+m^2)^{-1}\sim s\int d\vec{x}\, e^{-i\vec{\Delta}\cdot\vec{x}_\perp}B(s,\vec{x}_\perp), \tag{5.3}$$

and

$$\mathfrak{M}_{LS}(s,t)\sim\tfrac{1}{2}is\int d\vec{x}_\perp e^{-i\vec{\Delta}\cdot\vec{x}_\perp}A(s,\vec{x}_\perp)B(s,\vec{x}_\perp), \tag{5.4}$$

etc. By a Fourier transform (5.2) and (5.4) become

$$A(s,\vec{x}_\perp)\sim s^{-1}(2\pi)^{-2}\int d\vec{\Delta}\, e^{i\vec{\Delta}\cdot\vec{x}_\perp}\mathfrak{M}_L(s,t)(1+e^{i\pi\alpha(t)}) \tag{5.5}$$

and

$$B(s,\vec{x}_\perp)\sim s^{-1}(2\pi)^{-2}\int d\vec{\Delta}\, e^{i\vec{\Delta}\cdot\vec{x}_\perp}g^2(\vec{\Delta}^2+m^2)^{-1}. \tag{5.6}$$

Now (3.29) reads

$$\mathfrak{M}_L(s,t)\sim b^2(t,m^2,m^2)s^{\alpha(t)} \tag{5.7}$$

when the external particles are on the mass shell. Substituting (5.7) into (5.5), we get

$$A(s,\vec{x}_\perp)\sim s^{-1}(2\pi)^{-2}\int d\vec{\Delta}\, e^{i\vec{\Delta}\cdot\vec{x}_\perp}[b(t,m^2,m^2)]^2 s^{\alpha(t)}(1+e^{i\pi\alpha(t)}). \tag{5.8}$$

Substituting (5.6) and (5.8) into (5.4), we get

$$\mathfrak{M}_{LS}(s,t)\sim\tfrac{1}{2}is^{-1}(2\pi)^{-2}\int d\vec{q}_\perp[b(-(\vec{r}_1+\vec{q}_\perp)^2,m^2,m^2)]^2 s^{\alpha(-(\vec{r}_1+\vec{q}_\perp)^2)} \times g^2[(\vec{r}_1-\vec{q}_\perp)^2+m^2]^{-1}(1+e^{i\pi\alpha(-(\vec{r}_1+\vec{q}_\perp)^2)}), \tag{5.9}$$

if simple exponentiation holds.

Since $\mathfrak{M}_{LS}$ is given by (4.31), the test for the validity of simple exponentiation is

$$2\mathcal{g}^{LS}(\vec{r}_1,\vec{q}_\perp)=gb(-(\vec{r}_1+\vec{q}_\perp)^2,m^2,m^2). \tag{5.10}$$

A. The Leading Order

In the limit $g^2\to 0$, we have by (3.32) and (3.34) that

$$b(t)\sim ig, \tag{5.11}$$

and by (3.31) that

$$\alpha(t)\sim -1+g^2\alpha_0(t), \tag{5.12}$$

where $\alpha_0(t)$ is given by (B4). Thus (4.25) becomes

$$\mathcal{g}^{LS}(\vec{r}_1,\vec{q}_\perp)\sim\frac{ig^4}{32\pi^3}\int d\vec{p}_\perp\int_0^1 dx[(\vec{p}_\perp+\vec{Q})^2+(1-x+x^2)m^2]^{-1} \times[(\vec{p}_\perp-\vec{Q})^2+(1-x+x^2)m^2]^{-1}x^{-1+g^2\alpha_0(-(\vec{q}_\perp+\vec{r}_1)^2)}. \tag{5.13}$$

Now $\int_0^1 x^{-1+g^2\alpha_0}dx$ is of the order of g^{-2}, which diverges as $g^2\to 0$. The divergence occurs in the neighborhood of $x=0$. Thus, in the leading order, we may set $x=0$ in the other factors of the integrand in (5.13) and obtain

$$\mathcal{g}^{LS}(\vec{r}_1,\vec{q}_\perp)\sim(32\pi^3)^{-1}ig^2[\alpha_0(-(\vec{q}_\perp+\vec{r}_1)^2)]^{-1}\int d\vec{p}_\perp[(\vec{p}_\perp+\tfrac{1}{2}\vec{q}_\perp+\tfrac{1}{2}\vec{r}_1)^2+m^2]^{-1}[(\vec{p}_\perp-\tfrac{1}{2}\vec{q}_\perp-\tfrac{1}{2}\vec{r}_\perp)^2+m^2]^{-1}=\tfrac{1}{2}ig^2, \tag{5.14}$$

where (4.27) and (B4) have been used. Equations (5.11) and (5.14) show that (5.10) is satisfied in the leading order. Thus simple exponentiation holds in the weak-coupling limit.[23]

Substituting (5.12) and (5.14) into (4.32), we also get

$$\mathfrak{M}_{LS}^{(0)}(s,t)\sim-\tfrac{1}{8}g^4\alpha_0(0)[\alpha_0'(0)\ln s]^{-1}s^{g^2\alpha_0(0)-2}(-t+m^2)^{-1}. \tag{5.15}$$

From (B4), we have

$$\alpha_0(0)=(16\pi^2m^2)^{-1}$$

and

$$\alpha_0'(0)=(16\pi^2)^{-1}m^{-4}\int_0^1 x(1-x)dx=(96\pi^2m^4)^{-1}. \tag{5.16}$$

Thus (5.15) becomes

$$\mathfrak{M}_{LS}^{(0)}\sim-\tfrac{3}{4}g^4m^2s^{-2+g^2/(16\pi^2m^2)}(\ln s)^{-1}(-t+m^2)^{-1}. \tag{5.17}$$

B. General Order

It is clear that (5.10) cannot hold in general. This is because the left-hand side of (5.10) is a function of two variables, $\vec{r}_1$ and $\vec{q}_\perp$, while the right-hand side of (5.10) is a function of $\vec{r}_1+\vec{q}_\perp$ only.

To verify the failure of (5.10) explicitly, let us consider the case $\vec{r}_1=\vec{q}_\perp=\lambda=0$. Then we have from (4.25) and (A21) that

$$\mathcal{g}^{LS}(0,0)=g^2(32\pi^3)^{-1}\int d\vec{p}_\perp\int_0^1 dxc[\vec{p}_\perp^{\,2}+(1-x+x^2)m^2]^{-1-\gamma}(1-x)^{\gamma-1} =cg^2(32\pi^2\gamma)^{-1}m^{-2\gamma}\int_0^1 dx(1-x+x^2)^{-\gamma}(1-x)^{\gamma-1}x^{\gamma-2}, \tag{5.18}$$

where γ is given by (A14). Since

$$\gamma \geq 1, \tag{5.19}$$

where the equality sign holds only if $g^2=0$, we see that (5.18) is finite as long as $g^2 \neq 0$. However, by (A21) we see that

$$b(0, m^2, m^2) = b(-m^2) = \infty . \tag{5.20}$$

Thus (5.10) does not hold and the eikonal approximation is not valid.

Just as in quantum electrodynamics,[11] the failure of the eikonal approximation here is due to the fact that there are two particles sharing the incident momentum, and it is not justified to favor any particular one by associating the eikonal path with it.[27] More precisely, referring to Fig. 5, we see that the incident particle with momentum $r_2 - r_1$ splits into two particles with momentum $p - r_1$ and $r_2 - p$, respectively. According to (4.17), the ratio of the longitudinal momenta of these particles is $x/(1-x)$. Since the full range $0 \leq x \leq 1$ contributes to $\mathcal{g}^{LS}$ defined by (4.25), we cannot associate the eikonal path to either of these particles. Thus the eikonal approximation in the most straightforward form fails. In fact, if we substitute (4.25) into (5.10), the resulting integral equation has a kernel which is equal to that of (3.30) only when $x \to 0$, the limit in which we can associate the eikonal path with the particle with momentum $r_2 - p$.

Throughout this paper, we have used the ladder diagrams to represent the exchanged Regge pole. Although this is an oversimplification, the above discussion makes it clear that the violation of simple exponentiation is by no means limited to the ladder diagrams.

In closing, we emphasize that our experimental predictions[3] are independent of the validity of the eikonal approximation. As long as $|g| > \bar{g}_c$ so that the one-ladder amplitude violates the Froissart bound, the scattering at large transverse distances must increase with the energy.[3] A particle then acts like a black disk with a radius increasing with the energy. Thus the impact picture is realized in ϕ^3 theory when the coupling constant is sufficiently strong.

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APPENDIX A

The integral equation (3.29) for the Regge parameters is considerably simpler if the momentum transfer is zero, i.e.,

$$r_1 = 0 . \tag{A1}$$

As a result of (A1),

$$M_i = M_f . \tag{A2}$$

Let us introduce the notations

$$z = -M_i^2 = -M_f^2$$

and

$$b(z) = b(0, M_i^2, M_i^2) ; \tag{A3}$$

then (3.29) becomes

$$b(z) = \frac{g^2}{16\pi^2} \int_0^\infty dy \int_0^1 dx (1-x) x^\alpha \times b\left(xz + \frac{y + x\lambda^2}{1-x}\right) \times [y + (1-x)m^2 + x\lambda^2 + x(1-x)z]^{-2} , \tag{A4}$$

where

$$\alpha = \alpha(0) . \tag{A5}$$

Let us put

$$xz + \frac{y + x\lambda^2}{1-x} = z' ; \tag{A6}$$

then (A4) becomes

$$b(z) = \frac{g^2}{16\pi^2} \int_0^1 dx\, x^{\alpha(0)} \int_{xz + x\lambda^2/(1-x)}^\infty dz' b(z')/(z' + m^2)^2 . \tag{A7}$$

If $\lambda \neq 0$, (A7) is reduced to

$$b(z) = \frac{g^2}{16\pi^2} \int_0^\infty dz' b(z')(z' + m^2)^{-2} \int_0^{x_0} dx\, x^{\alpha(0)} = \frac{g^2[1+\alpha(0)]^{-1}}{16\pi^2} \int_0^\infty dz' b(z')(z' + m^2)^{-2} x_0^{\alpha(0)+1} , \tag{A8}$$

where

$$x_0 = \{z + z' + \lambda^2 - [(z + z' + \lambda^2)^2 - 4zz']^{1/2}\}/2z . \tag{A9}$$

In the rest of this appendix we shall concentrate on the Wick-Cutkosky case, i.e., the case $\lambda = 0$.[28-30]

When $\lambda=0$, (A8) becomes

$$b(z)=\frac{g^2[1+\alpha(0)]^{-1}}{16\pi^2}\left(\int_0^z dz'b(z')(z'+m^2)^{-2}(z'/z)^{\alpha(0)+1}+\int_z^\infty dz'b(z')(z'+m^2)^{-2}\right). \tag{A10}$$

Differentiating (A10), we get

$$\frac{d}{dz}\left[z^{\alpha(0)+2}\frac{d}{dz}b(z)\right]=-\frac{g^2}{16\pi^2}b(z)(z+m^2)^{-2}z^{\alpha(0)+1} \tag{A11}$$

or

$$\frac{d^2}{dz^2}b(z)+\frac{[\alpha(0)+2]}{z}\frac{d}{dz}b(z)+g^2(16\pi^2)^{-1}z^{-1}(z+m^2)^{-2}b(z)=0. \tag{A12}$$

The solution of (A12) is

$$b(z)=(z+m^2)^\gamma[aF(\gamma+\alpha+1,\gamma;\alpha+2;-z/m^2)+a'z^{-\alpha-1}F(\gamma,\gamma-\alpha-1;-\alpha;-z/m^2)], \tag{A13}$$

where a and a' are constants, F is the hypergeometric function, and

$$\gamma=\frac{1}{2}+\left(\frac{1}{4}+\frac{g^2}{16\pi^2m^2}\right)^{1/2}. \tag{A14}$$

Since the solution of (A10) cannot diverge like $z^{-\alpha-1}$ at $z=0$, we require that

$$a'=0 \tag{A15}$$

or

$$b(z)=a(z+m^2)^\gamma F(\gamma+\alpha+1,\gamma;\alpha+2;-z/m^2). \tag{A16}$$

Also, the solution of (A10) cannot approach a constant as $z\to\infty$. Thus from (A16) and the asymptotic form of the hypergeometric function, we have

$$\alpha+2-\gamma=-n,\quad n=0,1,2,\ldots . \tag{A17}$$

Since, by (3.16), α must be greater than -1, (A17) should be modified as

$$\alpha=\gamma-n-2,\quad n=0,1,2,\ldots,[\gamma]-1 \tag{A18}$$

where $[\gamma]$ is the largest integer smaller than γ. Equation (A18) shows that more and more Regge poles satisfying $\alpha>-1$ appear as the coupling constant increases. From (A16) and (A18), we get

$$b(z)=a(z+m^2)^\gamma F(2\gamma-n-1,\gamma;\gamma-n;-z/m^2)=am^{4\gamma+2}(z+m^2)^{1-\gamma}F(1-\gamma,-n;\gamma-n;-z/m^2). \tag{A19}$$

The largest Regge pole corresponds to $n=0$:

$$\alpha=\gamma-2=-\frac{3}{2}+\left(\frac{1}{4}+\frac{g^2}{16\pi^2m^2}\right)^{1/2}, \tag{A20}$$

which satisfies (3.16) for all values of the coupling constant, and

$$b(z)=c(z+m^2)^{1-\gamma}=c(z+m^2)^{-\alpha-1}, \tag{A21}$$

where c is a constant. It may appear peculiar that when the external particles are on the mass shell ($z=-m^2$), the residue function $b(z)$ diverges. However, we remember that, since $\lambda=0$, the scattering amplitude $\mathfrak{M}_L$ itself has an infrared divergence if the external particles are on the mass shell. Thus $b(z)$ has a divergent branch point at $z=-m^2$. The function α is independent of the external masses, and therefore has no infrared divergence. Equations (A18)–(A21) were first obtained by Nakanishi with a different method.[30,31]

In the strong-coupling limit $g\to\infty$, we have

$$\alpha=\frac{g}{4\pi m}-\tfrac{3}{2}+O(g^{-1}). \tag{A22}$$

Thus α can be much larger than 1 at $t=0$, if the coupling constant is sufficiently large.

In the weak-coupling limit $g\to0$, we have

$$\alpha=-1+\frac{g^2}{16\pi^2m^2}-\frac{g^4}{(16\pi^2m^2)^2}+O(g^6) \tag{A23}$$

and

$$b(z)=c[1+O(g^2)]. \tag{A24}$$

Notice that $b(z)$ has no infrared divergence in the lowest order of perturbation.

APPENDIX B

In this appendix we give the first two terms in the perturbation series for $b(t, M_i^2, M_f^2)$ and $[1+\alpha(t)]$.

Consider $g^2\to0$ in (3.30). Then the right-hand side of (3.30) appears to be smaller than the left-hand side by a factor g^2. Thus (3.30) can hold only if the integral diverges in the limit $g^2\to0$. This can happen if

$$\lim_{g^2\to0}\alpha(t)=-1.$$

Let us therefore express α and b in the perturbation series

$$\alpha(t)=-1+g^2\alpha_0(t)+g^4\alpha_1(t)+\cdots,$$

$$b(t, M_i^2, M_f^2)=f(t)[b_0(t, M_i^2, M_f^2)+g^2b_1(t, M_i^2, M_f^2)+\cdots]. \tag{B1}$$

We shall calculate α_0, α_1, b_0, and b_1.

A. Leading Order

To obtain α_0 and b_0, we note that $\int_0^1 x^\alpha dx$ diverges at $x=0$ if $\alpha=-1$. This means that, in the integration over x, the important region is the neighborhood of $x=0$. Thus we set $x=0$ in the other factors in (3.30), obtaining

$$b_0(t, M_i^2, M_f^2)=\tfrac{1}{2}(2\pi)^{-3}\alpha_0^{-1}\int d\vec{p}_\perp \frac{b_0(t, -(\vec{p}_\perp-\vec{r}_1)^2, -(\vec{p}_\perp+\vec{r}_1)^2)}{[(\vec{p}_\perp+\vec{r}_1)^2+m^2-i\epsilon][(\vec{p}_\perp-\vec{r}_1)^2+m^2-i\epsilon]}\,. \tag{B2}$$

Since the right-hand side of (B2) is independent of M_i and M_f, b_0 is independent of M_i and M_f. Thus we may choose

$$b_0(t, M_i^2, M_f^2)=1 \tag{B3}$$

as the t-dependent factor can be absorbed by $f(t)$ defined in (B1). Then (B2) becomes

$$\begin{aligned}\alpha_0(t)&=(16\pi^3)^{-1}\int d\vec{p}_\perp[(\vec{p}_\perp+\vec{r}_1)^2+m^2-i\epsilon]^{-1}[(\vec{p}_\perp-\vec{r}_1)^2+m^2-i\epsilon]^{-1}\\ &=(16\pi^2)^{-1}\int_0^1 dx[-x(1-x)t+m^2]^{-1}\,.\end{aligned} \tag{B4}$$

Equation (B4) is well known.[6]

B. Next Order

We shall now calculate α_1 and b_1 as defined by (B1).

Substituting (B1) and (B3) into (3.30), we get

$$1+g^2b_1(t, M_i^2, M_f^2)\sim I_1+g^2I_2, \tag{B5}$$

where

$$\begin{aligned}\begin{bmatrix}I_1\\ I_2\end{bmatrix}&=\tfrac{1}{2}g^2(2\pi)^{-3}\int d\vec{p}_\perp\int_0^1 dx\,(1-x)x^{-1+g^2\alpha_0(t)+g^4\alpha_1(t)}\{[\vec{p}_\perp+(1-x)\vec{r}_1]^2+(1-x)m^2+x\lambda^2-x(1-x)M_f^2-i\epsilon\}^{-1}\\ &\quad\times\{[\vec{p}_\perp-(1-x)\vec{r}_1]^2+(1-x)m^2+x\lambda^2-x(1-x)M_i^2-i\epsilon\}^{-1}\begin{bmatrix}1\\ b_1(t, \mu_i^2, \mu_f^2)\end{bmatrix},\end{aligned} \tag{B6}$$

where μ_i and μ_f are defined by (3.23) and (3.24).

For I_2, we need to calculate only the leading term (of the order of 1). This calculation is fairly easy: We simply replace $x^{-1+g^2\alpha_0+g^4\alpha_1}$ by $x^{-1+g^2\alpha_0}$ and set $x=0$ in the other factors of the integrand above, obtaining

$$\begin{aligned}I_2&\sim\tfrac{1}{2}g^2(2\pi)^{-3}\int d\vec{p}_\perp\int_0^1 dx\,x^{-1+g^2\alpha_0(t)}[(\vec{p}_\perp+\vec{r}_1)^2+m^2-i\epsilon]^{-1}[(\vec{p}_\perp-\vec{r}_1)^2+m^2-i\epsilon]^{-1}b_1(t, -(\vec{p}_\perp-\vec{r}_1)^2, -(\vec{p}_\perp+\vec{r}_1)^2)\\ &=\frac{1}{2}\,\frac{(2\pi)^{-3}}{\alpha_0(t)}\int d\vec{p}_\perp[(\vec{p}_\perp+\vec{r}_1)^2+m^2-i\epsilon]^{-1}[(\vec{p}_\perp-\vec{r}_1)^2+m^2-i\epsilon]^{-1}b_1(t, -(\vec{p}_\perp-\vec{r}_1)^2, -(\vec{p}_\perp+\vec{r}_1)^2),\end{aligned} \tag{B7}$$

which is independent of M_i and M_f.

The function I_1 can be written as

$$I_1\sim A+g^2B, \tag{B8}$$

where

$$A\sim\tfrac{1}{2}g^2(2\pi)^{-3}\int d\vec{p}_\perp\int_0^1 dx\,x^{-1+g^2\alpha_0(t)+g^4\alpha_1(t)}[(\vec{p}_\perp+\vec{r}_1)^2+m^2-i\epsilon]^{-1}[(\vec{p}_\perp-\vec{r}_1)^2+m^2-i\epsilon]^{-1} \tag{B9}$$

and

$$\begin{aligned}B&\sim\tfrac{1}{2}(2\pi)^{-3}\int d\vec{p}_\perp\int_0^1 dx\,x^{-1+g^2\alpha_0(t)+g^4\alpha_1(t)}\big((1-x)\{[\vec{p}_\perp+(1-x)\vec{r}_1]^2+(1-x)m^2+x\lambda^2-x(1-x)M_f^2-i\epsilon\}^{-1}\\ &\quad\times\{[\vec{p}_\perp-(1-x)\vec{r}_1]^2+(1-x)m^2+x\lambda^2-x(1-x)M_i^2-i\epsilon\}^{-1}\\ &\quad-[(\vec{p}_\perp+\vec{r}_1)^2+m^2-i\epsilon]^{-1}[(\vec{p}_\perp-\vec{r}_1)^2+m^2-i\epsilon]^{-1}\big)\,.\end{aligned} \tag{B10}$$

By carrying out the integration over x and making use of (B4), we easily get

$$A \sim \alpha_0(t)/[\alpha_0(t)+g^2\alpha_1(t)] . \tag{B11}$$

For B, we keep the leading term only. Since the quantity in the bold parentheses of (B10) vanishes at $x=0$, we may replace $x^{-1+g^2\alpha_0+g^4\alpha_1}$ in (B10) by x^{-1} and obtain

$$\begin{aligned} B=\tfrac{1}{2}(2\pi)^{-3}\int d\vec{p}_\perp\int_0^1 dx\,x^{-1}\Big((1-x)&\{[\vec{p}_\perp+(1-x)\vec{r}_1]^2+(1-x)m^2+x\lambda^2-x(1-x)M_f^2-i\epsilon\}^{-1}\\ &\times\{[\vec{p}_\perp-(1-x)\vec{r}_1]^2+(1-x)m^2+x\lambda^2-x(1-x)M_i^2-i\epsilon\}^{-1}\\ &-[(\vec{p}_\perp+\vec{r}_1)^2+m^2-i\epsilon]^{-1}[(\vec{p}_\perp-\vec{r}_1)^2+m^2-i\epsilon]^{-1}\Big). \end{aligned} \tag{B12}$$

Substituting (B8) and (B11) into (B5), we get

$$b_1(t, M_i^2, M_f^2)=F(t)+B(t, M_i^2, M_f^2), \tag{B13}$$

where

$$F(t)=-\alpha_1(t)/\alpha_0(t)+I_2(t). \tag{B14}$$

The function $B(t, M_i^2, M_f^2)$ is completely known and is explicitly given by (B12). The function $F(t)$ is arbitrary and cannot be determined from the integral equation. This is related to the fact that $b(t, M_i^2, M_f^2)$ is determined up to a factor which depends on t. The function $F(t)$ can be absorbed into $f(t)$ and we obtain (3.32).

The function $\alpha_1(t)$ can be determined by substituting (B13) and (B14) into (B7). The result is

$$\alpha_1(t)=(16\pi^3)^{-1}\int d\vec{p}_\perp B(t, -(\vec{p}_\perp-\vec{r}_1)^2, -(\vec{p}_\perp+\vec{r}_1)^2)[(\vec{p}_\perp+\vec{r}_1)^2+m^2-i\epsilon]^{-1}[(\vec{p}_\perp-\vec{r}_1)^2+m^2-i\epsilon]^{-1}. \tag{B15}$$

The results in this appendix may be checked by comparing with the results in Appendix A. Setting $r_1=\lambda=0$ in (B1), (B4), and (B15), we easily obtain (A23).

APPENDIX C

In this appendix we calculate the asymptotic form of $b(t, M_i^2, M_f^2)$ when one or both of $-M_i^2$ and $-M_f^2$ approach infinity.

Case 1: $r_1=0$

Let us first consider the case $r_1=0$, which implies that $M_i=M_f$ and that b satisfies (A8):

$$b(z)=\frac{a}{\alpha(0)+1}\int_0^\infty dz'\,b(z')(z'+m^2)^{-2}\{z+z'+\lambda^2-[(z+z'+\lambda^2)^2-4zz']^{1/2}\}^{1+\alpha(0)}(2z)^{-1-\alpha(0)}, \tag{C1}$$

where

$$a=g^2/16\pi^2 \tag{C2}$$

and

$$z=-M_i^2=-M_f^2 .$$

In the limit $z\to\infty$, we have

$$z+z'+\lambda^2-[(z+z'+\lambda^2)^2-4zz']^{1/2}\to 2z' . \tag{C3}$$

Thus we get, from (C1) and (C3),

$$b(z)\sim cz^{-\alpha(0)-1}, \quad z\to\infty \tag{C4}$$

where

$$c=\frac{a}{\alpha(0)+1}\int_0^\infty dz'\,b(z')(z'+m^2)^{-2}z'^{1+\alpha(0)}. \tag{C5}$$

We notice that, with the asymptotic form (C4), c is a finite number because the integral in (C5) is convergent. Indeed, (C4) is in agreement with the closed-form solution (A21).

Since $b(z)$ can be determined from (C1) only up to a factor, c is undetermined. In the weak-coupling limit $g\to 0$, we know that

$$c \sim ig,$$

$$\alpha(0)+1 \sim g^2/16\pi^2 m^2;$$

thus

$$b(z) \sim igz^{-g^2/16\pi^2 m^2}, \quad z\to\infty, \quad g\to 0. \tag{C6}$$

Case 2: r_1 Nonzero and Fixed

Consider the limit $|M_i|^2+|M_f|^2\to\infty$ with r_1 nonzero and fixed. Let us put

$$M_i^2/(|M_i|^2+|M_f|^2)=\rho_i,$$

$$M_f^2/(|M_i|^2+|M_f|^2)=\rho_f.$$

Let us study the integrand of (3.30). We observe that if μ_i^2 and μ_f^2 are kept fixed, by (3.23) and (3.24) x must be very small. Thus, putting

$$y=x(|M_i|^2+|M_f|^2), \tag{C7}$$

we reduce (3.30) into

$$b(t, M_i^2, M_f^2) \sim c(t)(|M_i|^2+|M_f|^2)^{-\alpha(t)-1}, \tag{C8}$$

where

$$c(t)=\frac{g^2}{16\pi^3}\int d\vec{p}_\perp \int^\infty dy\, y^{\alpha(t)} b(t, -y\rho_i-(\vec{p}_\perp-\vec{r}_1)^2, -y\rho_f-(\vec{p}_\perp+\vec{r}_1)^2)$$
$$\times[(\vec{p}_\perp+\vec{r}_1)^2+m^2+y]^{-1}[(\vec{p}_\perp-\vec{r}_1)^2+m^2+y]^{-1}, \tag{C9}$$

which is dependent on ρ_i and ρ_f. We also know that in the weak-coupling limit

$$c(t)\sim ig. \tag{C10}$$

Thus in the limit (C7) and for small coupling constant

$$b(t, M_i^2, M_f^2) \sim ig(|M_i|^2+|M_f|^2)^{-g^2\alpha_0(t)}, \tag{C11}$$

where $\alpha_0(t)$ is given by (B4).

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Regge Poles for Large Coupling Constants. I*

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We study the problem of scattering by an attractive Yukawa potential. In particular, the positions of Regge poles are found approximately when the coupling constant is large.

1. INTRODUCTION

It is the purpose of this and the following paper[1] to study the behavior of Regge poles[2] when the coupling constant is large. Since Regge poles were originally found in connection with the scattering by a superposition of Yukawa potentials, we treat here first this case of potential scattering. Although the procedure of this paper is applicable to a large class of superpositions of Yukawa potentials, we shall restrict ourselves to the case of a single Yukawa potential, where the answer is more explicit in some cases.

In Paper II we shall consider the case of ladder diagrams in ϕ^3 theory, again in the limit where the coupling constant is large.

The differential equation for a partial wave is

$$\left[\frac{d^2}{dr^2}-\kappa^2-l(l+1)r^{-2}+g^2V(r)\right]\psi=0, \tag{1}$$

where $i\kappa$ is the momentum of the incident particle, and $V(r)$ is given by

$$V(r)=r^{-1}e^{-\lambda r}. \tag{2}$$

In (1), l is considered to be a continuous variable. Let

$$r=e^z$$

and (3)

$$\psi=\psi_1e^{z/2};$$

then it follows from (1) that

$$\left[\frac{d^2}{dz^2}-\kappa^2e^{2z}-(l+\tfrac{1}{2})^2+g^2e^{2z}V(e^z)\right]\psi_1=0. \tag{4}$$

For the Yukawa potential (2), Eq. (4) is more explicitly

$$\left[\frac{d^2}{dz^2}-\kappa^2e^{2z}-(l+\tfrac{1}{2})^2+g^2e^ze^{-\lambda e^z}\right]\psi_1=0. \tag{5}$$

We are thus interested in the behavior of the potential

$$U=\kappa^2e^{2z}-g^2e^ze^{-\lambda e^z}. \tag{6}$$

This is an attractive potential when g is sufficiently large. For large g, its energy levels can be found approximately by replacing U with a harmonic potential located at the minimum of U.