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PHYSICAL REVIEW D VOLUME 5, NUMBER 12 15 JUNE 1972

Spontaneously Broken Gauge Symmetries. II. Perturbation Theory and Renormalization

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The second paper in this series is devoted to the formulation of a renormalizable perturbation theory of Higgs phenomena (spontaneously broken gauge theories). In Sec. II, we reformulate the renormalization prescription for massless Yang-Mills theories in terms of gauge-invariant renormalization counterterms in the action. Section III gives a group-theoretic discussion of Higgs phenomena. We discuss the possibility that an asymmetric vacuum is stable, and show how the symmetry of the physical vacuum determines the mass spectrum of the gauge bosons. We show further that in a special gauge (U gauge), all unphysical fields can be eliminated. Section IV discusses the quantization of a spontaneously broken gauge theory in the R gauge, where, as we show in Sec. V, Green's functions are made finite by the renormalization counterterms of the symmetric theory (in which the gauge invariance is not spontaneously broken). The R-gauge formulation makes use of redundant fields for the sake of renormalizability. Section VI is a discussion of the low-energy limits of propagators in the R-gauge formulation. In Sec. VII we show that the particles associated with redundant fields peculiar to the R-gauge formulation are unphysical, i.e., they do not contribute to the sum over intermediate states.

I. INTRODUCTION

In this paper we give a renormalization method and a proof of finiteness of renormalized Green's functions of spontaneously broken gauge theories. For definiteness we consider a very simple model in which SU(2) gauge bosons are coupled to a triplet of scalar mesons. There is an extra complication when chiral fermions are included in the model, as pointed out by Veltman,[1] and more recently by Gross and Jackiw.[2] This difficulty can be circumvented in a realistic model of electromagnetic and weak interactions. We shall not discuss this problem further in this paper, but postpone the discussion until we deal with the renormalizability of a realistic theory in a sequel to this paper.

We give in this paper a method of renormalization which is based on the observation that, in a spontaneously broken symmetry theory, divergences in Feynman integrals can be classified according to, and identified with, those of a comparison theory in which the symmetry is not broken.[3] This method is successfully used for the σ model and we borrow many concepts and techniques from that study.

Let us summarize the contents of this paper. In Sec. II, we give a brief recapitulation of the results of the first paper on the renormalization of a massless Yang-Mills theory. We write down explicitly the effective action in terms of renormalized fields and gauge-invariant counterterms. The renormalized version of the Ward-Takahashi identity for the generating functional of renormalized Green's functions is recorded. The reader who is not particularly interested in the details may be able to gather enough background for the subsequent discussions by studying Secs. II and V of the previous paper concurrently with this section.

Section III is a discussion of group theory of Higgs phenomena. To a large extent, this section is a review of Kibble's work.[4] The discussion here is carried out in the context of classical field theory. We show how the instability of the symmetric vacuum arises, and how the symmetry of the physical vacuum determines the mass spectrum of gauge bosons. The study culminates in a theorem which shows which gauge bosons become massive in a spontaneously broken gauge theory. The theorem is an analog of that due to Bludman and Klein,[5] which shows in what quantum channels Goldstone bosons appear in a spontaneously broken symmetry theory.

There exists a special choice of gauge in which Goldstone boson fields combine with gauge boson fields to form massive vector fields with three degrees of polarizations. This is the gauge used by Kibble[4] in his discussion of Higgs phenomena. In this gauge, there are no redundant fields and the physical interpretation of the theory is straightforward. We shall call this gauge the U gauge (unitary gauge). Unfortunately the renormalizability of the U-gauge formulation is not obvious, even though indications are that the T matrix in this formulation is renormalizable.[6,7]

In Sec. IV, we quantize the simple model mentioned at the beginning in a class of gauges, which includes, in quantum electrodynamics, the transverse Landau gauge and the Feynman gauge. We shall call these gauges collectively as R gauge (renormalizable gauge). The R-gauge formulation contains redundant field components so that the unitarity of the S matrix is not manifest. As we show in Sec. V, Green's functions in the R-gauge formulation are rendered finite by the renormalization counterterms of the corresponding symmetric theory. Here lies the advantage of this formulation. Since the renormalization counterterms which make the theory finite are gauge invariant, the renormalized Green's functions of a spontaneously broken gauge theory satisfy appropriate Ward-Takahashi identities.

In Sec. VI we discuss the low-energy limits of propagators in the R-gauge formulation.

In Sec. VII we show that renormalized T-matrix elements are independent of the parameter which characterizes a particular R gauge chosen, and the redundant massless scalar fields peculiar to the R-gauge formulation are unphysical, i.e., they do not contribute to the sum over intermediate states when one computes the absorptive part of T-matrix elements by the Landau-Cutkosky rule.[8,9] These discussions are based on the Ward-Takahashi identities. For the proof of unitarity of the R-gauge formulation, we have identified the set of relations that are needed. The proof is worked out in detail for intermediate states containing one, two, and three such unphysical quanta.

In the sequel we wish to consider the renormalizability aspect of Weinberg's theory of weak and electromagnetic interactions in detail and the equivalence of the S matrix in the U and R gauges.

II. GAUGE-INVARIANT COUNTERTERMS

As Bogoliubov and Shirkov[10] have shown, the R operation can be formally implemented by the inclusion of counterterms in the Lagrangian. The discussion in the previous paper implies that these counterterms are themselves gauge-invariant. We can in fact reexpress the effective action (I2. 10) (where the prefix I refers to the equations of paper I) in terms of the renormalized field $\vec{A}_r^\mu$ and the renormalized coupling constant g_r,

$$\vec{A}^\mu = Z_3^{1/2}\vec{A}_r^\mu ,$$

$$g = g_r Z_1/Z_3^{3/2}$$

and making explicit the renormalization counterterms. With

$$\alpha = \alpha_r Z_3 ,$$

we write

$$\int d^4x\{-\tfrac{1}{4}(\partial^\mu\vec{A}_r^\nu-\partial^\nu\vec{A}_r^\mu-g_r\vec{A}_r^\mu\times\vec{A}_r^\nu)^2-(1/2\alpha_r)(\partial_\mu\vec{A}_r^\mu)^2-\tfrac{1}{4}(Z_3-1)(\partial^\mu\vec{A}_r^\nu-\partial^\nu\vec{A}_r^\mu)^2$$
$$+\tfrac{1}{2}g_r(Z_1-1)\vec{A}_{r\mu}\times\vec{A}_{r\nu}\cdot(\partial^\mu\vec{A}_r^\nu-\partial^\nu\vec{A}_r^\mu)-\tfrac{1}{4}g_r{}^2(Z_1{}^2/Z_3-1)(\vec{A}_r^\mu\times\vec{A}_r^\nu)^2\}$$
$$-i\,\mathrm{Tr}\ln(1-g_r\vec{t}\cdot\vec{A}_r^\mu\partial_\mu/\partial^2)-i\,\mathrm{Tr}\ln\left\{1+\frac{1}{\partial^2-g_r\vec{t}\cdot\vec{A}_r^\mu\partial_\mu}[(\tilde{Z}_3-1)\partial^2-g_r(\tilde{Z}_1-1)\vec{t}\cdot\vec{A}^\mu\partial_\mu]\right\}, \tag{2.1}$$

with

$$Z_1/Z_3=\tilde{Z}_1/\tilde{Z}_3 \tag{2.2}$$

which is a restatement of Eq. (I 6.4). We may choose Z_3 and $\tilde{Z}_3$ such that

$$\lim_{k^2\to-a^2}[\Delta_{\mu\nu}(k)]_r=\left(g_{\mu\nu}+\frac{k_\mu k_\nu}{a^2}\right)\left(\frac{1}{-a^2}\right)+\text{gauge-dependent terms}, \tag{2.3}$$

$$[\mathcal{G}(-a^2)]_r=-1/a^2 \tag{2.4}$$

and Z_1, so that

$$\lim_{p^2=q^2=r^2=a^2} i\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)=\epsilon^{abc}\{[(p-q)_\nu g_{\lambda\mu}+(q-r)_\lambda g_{\mu\nu}+(r-p)_\mu g_{\nu\lambda}]+\cdots\}, \tag{2.5}$$

as we described in Eq. (I 6.7).

Clearly the construction of Eq. (2.1) can be extended when there are matter fields present in the Lagrangian. The part that has to do with the gauge invariance, for the triplet of scalar fields discussed in paper I, is

$$\tfrac{1}{2}(\partial_\mu\vec{\phi}_r-g_r\vec{A}_{r\mu}\times\vec{\phi}_r)^2+\tfrac{1}{2}(Z_2-1)(\partial_\mu\vec{\phi}_r)^2+g_r\left[Z_1\left(\frac{Z_2}{Z_3}\right)-1\right]\vec{A}_r^\mu\cdot(\vec{\phi}_r\times\partial_\mu\vec{\phi}_r)+\tfrac{1}{2}g_r{}^2\left[\frac{Z_1{}^2}{Z_3}\left(\frac{Z_2}{Z_3}\right)-1\right](\vec{A}_r^\mu\times\vec{\phi}_r)^2, \tag{2.6}$$

where Z_2 may be chosen to ensure the normalization condition for the scalar propagator, Eq. (I 7.15),

$$\lim_{k^2\to-a^2}[\Delta^{-1}(k^2)]_r\sim k^2+a^2-M^2. \tag{2.7}$$

It is perhaps useful to rephrase the Bogoliubov-Parasiuk-Hepp (BPH) renormalization procedure in terms of the Lagrangian of (2.1) and (2.6). First we include the regulator term (I 5.7) and other regulator terms in the Lagrangian. Feynman integrals are now finite and we can choose the renormalization constants, Z's, which depend on the cutoff Λ^2, in such a way that the renormalization conditions (2.2)–(2.5) and (2.7) are satisfied. As $\Lambda^2\to\infty$, the renormalized Feynman amplitudes are well defined and finite.

If we make the scale change

$$\vec{J}^\mu=Z_3{}^{-1/2}\vec{J}_r^\mu,$$
$$\vec{K}=Z_2{}^{-1/2}\vec{K}_r$$

in the definition of the generating functional of Green's functions, then functional derivatives of Z with respect to the renormalized sources are the renormalized Green's functions. The Ward-Takahashi identity (3.13) may be written in terms of renormalized quantities:

$$\frac{i}{\alpha_r}\partial_\mu\frac{\epsilon W}{\delta J^a_{r\mu}(x)}+\partial^\mu J^a_{r\mu}(x)W-ig_r\tilde{Z}_1\int d^4y\,d^4z\left[J_r^{c\mu}(y)g^{\mathrm{tr}}_{\mu\nu}(y-z)t^{cbd}\frac{\delta}{\delta J^b_{r\nu}(z)}\right]G_r^{da}(z,x;i\delta/\delta\vec{J}_r)W=0, \tag{2.8}$$

where

$$g^{\mathrm{tr}}_{\mu\nu}(x-y)=g_{\mu\nu}\delta^4(x-y)+\partial_\mu\partial_\nu\tilde{D}_F(x-y)$$

and

$$G(x,y;i\delta/\delta\vec{J})=\tilde{Z}_3G_r(x,y;i\delta/\delta\vec{J}_r),$$
$$G_r(x,y;i\delta/\delta\vec{J}_r)=\left\langle x\left|\left[\partial^2-ig_r\vec{t}\cdot\partial_\mu\frac{\delta}{\delta\vec{J}_{r\mu}}+\left((\tilde{Z}_3-1)\partial^2-ig_r(\tilde{Z}_1-1)\vec{t}\cdot\partial_\mu\frac{\delta}{\delta\vec{J}_{r\mu}}\right)\right]^{-1}\right|y\right\rangle. \tag{2.9}$$

III. GROUP THEORY OF HIGGS PHENOMENA

We will describe here the Higgs phenomenon[11,4] in the context of classical (nonquantized) field theory. Alternatively, one may interpret the following discussion as applying to the tree approximation to quantum field theory. The following discussion is essentially a review of Kibble's work.[4] We include it here, mainly to make this paper self-contained and to establish notations, terminology and concepts. For simplicity, we shall consider the system of gauge bosons interacting with scalar mesons.

Let G be the local gauge symmetry (compact, but not necessarily semisimple) of the Lagrangian. We denote by $\{L_a\}$ the set of generators of the group G. The Yang-Mills gauge bosons $\{A_a^\mu\}$ belong to the adjoint representation of the group G, so they can be put in one-to-one correspondence with the generators $\{L_a\}$. We assume that there are scalar multiplets $\phi^{(\alpha)}$ of dimensionalities n_α,

$$\phi^{(\alpha)}=\begin{pmatrix}\phi_1^{(\alpha)}\\ \cdot\\ \cdot\\ \cdot\\ \phi_{n_\alpha}^{(\alpha)}\end{pmatrix}. \tag{3.1}$$

The multiplet $\phi^{(\alpha)}$ transforms like an irreducible representation of the group G. We denote by $\{L^{(\alpha)}\}$ the matrix representation of the generators. The renormalizable Lagrangian in which the gauge bosons are coupled in the minimal way is of the form

$$\mathcal{L}=\sum_\alpha (D_\mu\phi^{(\alpha)})^\dagger\cdot(D^\mu\phi^{(\alpha)}) - \tfrac{1}{4}\sum_a(\partial^\mu A_a^\nu-\partial^\nu A_a^\mu-gf_{abc}A_b^\mu A_c^\nu)^2-V(\phi), \tag{3.2}$$

where D_μ is the covariant derivative

$$D_\mu=\partial_\mu+ig\vec{L}^{(\alpha)}\cdot\vec{A}_\mu\,, \tag{3.3}$$

f_{abc} is the structure constant,

$$[L_a,L_b]=if_{abc}L_c\,, \tag{3.4}$$

and $V(\phi)$ is an invariant polynomial in the $\phi^{(\alpha)}$, which is at most quartic in the scalar fields. The Lagrangian (3.2) is invariant under the local gauge transformation

$$\begin{aligned}\phi^{(\alpha)}&\to e^{i\vec{L}^{(\alpha)}\cdot\vec{\omega}}\phi^{(\alpha)}\,,\\ \vec{A}_\mu\cdot\vec{L}&\to e^{i\vec{L}\cdot\vec{\omega}}\vec{A}_\mu\cdot\vec{L}\,e^{-i\vec{L}\cdot\vec{\omega}}\\ &\quad-\frac{i}{g}(\partial_\mu e^{i\vec{L}\cdot\vec{\omega}})e^{-i\vec{L}\cdot\vec{\omega}}\,,\end{aligned} \tag{3.5}$$

where ω_a is a function of space-time.

The vacuum expectation values of the scalar fields $\phi^{(\alpha)}\equiv v^{(\alpha)}$ are determined by the conditions

$$\left.\frac{\delta V(\phi)}{\delta\phi_i^{(\alpha)}}\right|_{\phi=v}=0\,, \tag{3.6}$$

$$\left.\frac{\delta^2 V(\phi)}{\delta\phi_i^{(\alpha)}\delta\phi_j^{(\beta)}}\right|_{\phi=v}\geqslant 0\,. \tag{3.7}$$

The second condition (3.8) is necessary in order that the physical masses of the scalar particles be non-negative. The solutions of Eqs. (3.6) and (3.7),

$$\phi^{(\alpha)}=v^{(\alpha)}\,, \tag{3.8}$$

may be null vectors, in which case the vacuum is invariant under G. It may be that the minimum of V occurs at some finite $v^{(\alpha)}$. Let $\{l\}$ be the subset of $\{L\}$ which map all of $v^{(\alpha)}$'s to null vectors:

$$l_i^{(\alpha)}v^{(\alpha)}=0\,. \tag{3.9}$$

Then the set $\{l\}$ generates a subgroup S of G. We call S the little group of the vacuum.

The nature of the little group S depends on the polynomial $V(\phi)$. We give some examples below.

Example 1. Let

$$V(\phi)=\tfrac{1}{2}\mu^2\phi^2+\tfrac{1}{4}\lambda(\phi^2)^2,$$

where ϕ is an n-dimensional real vector. The group G of invariance is $O(n)$. The parameter λ has to be $\geqslant 0$ in order that $|\phi|$ is bounded, or the Hamiltonian is positive definite. If $\mu^2\geqslant 0$, the minimum of $V(\phi)$ occurs at $\phi=0$ and the little group S is equal to $O(n)$. If $\mu^2<0$, the minimum lies in the orbit $|\phi|^2=-\mu^2/\lambda$. Because of the invariance of $V(\phi)$ under $O(n)$ we can always put v in the standard form

$$v=\begin{pmatrix}(-\mu^2/\lambda)^{1/2}\\ 0\\ \cdot\\ \cdot\\ 0\end{pmatrix}.$$

The little group of the vacuum is $O(n-1)$.

Example 2. Let

$$M_\alpha^\beta=\sum_{i=0}^{8}(\lambda_i)_{\alpha\beta}(s^i+ip^i),$$

where λ_i, $i=0,\ldots,8$ are Gell-Mann's 3×3 matrices with $\lambda_0=(2/3)^{1/2}\,\underline{1}$ and $\alpha,\beta=1,2,3$. s and p are nonets of scalar and pseudoscalar fields. We consider

$$V(s,p)=\alpha\,\mathrm{Tr}(MM^\dagger)^2+\beta[\mathrm{Tr}(MM^\dagger)]^2+\gamma(\det M+\det M^\dagger)+\delta\,\mathrm{Tr}(MM^\dagger),$$

which is SU(3)×SU(3) invariant. Let us concentrate on the case in which parity is conserved, so that the minimum V lies on the hyperplane $p_i = 0$, $i = 0, \ldots, 8$. Let us assume that the minimum occurs at

$$M = M^\dagger = v\,,$$

where v is a 3×3 Hermitian matrix. We can diagonalize v by an SU(3) transformation so v takes the form

$$v = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}.$$

Equation (3.6) then demands that

$$4\alpha a^3 + 4\beta a(a^2 + b^2 + c^2) + 2\gamma bc + 2\delta a = 0$$

and two more equations obtained from the above by cyclic permutations of a, b, and c. The three equations imply that the three eigenvalues a, b, c cannot be all unequal. Therefore, the little group S cannot be smaller than SU(2).

When $\phi^{(\alpha)}$'s have nonvanishing vacuum expectation values we can perform nonlinear canonical transformations on $\phi^{(\alpha)}$'s and eliminate a certain number of field components from $V(\phi)$. Let the dimensionalities of G and S be N and M, respectively. There are, then, $m = N - M$ generators, $\{t\}$ of G, which span the cosets $S^{-1}G$:

$$\{l\} + \{t\} = \{L\}\,. \tag{3.10}$$

We may choose the generators to be orthonormal with respect to the Cartan inner product. Let us write

$$\phi^{(\alpha)} = D^{(\alpha)}[e^{i\vec{\xi}\cdot\vec{t}}](v^{(\alpha)} + \rho^{(\alpha)})\,, \tag{3.11}$$

where $\vec{\xi}$ has m components and choose $\rho^{(\alpha)}$'s, such that the mapping

$$\phi^{(\alpha)} \to (\vec{\xi}, \rho^{(\alpha)})$$

is canonical. [A nonlinear mapping $\phi_i \to \rho_j(\{\phi_i\})$ is called canonical if $(\delta\phi_i/\delta\rho_j)|_{\rho=0}$ is a nonsingular matrix.] Both ξ and $\rho^{(\alpha)}$'s have null vacuum expectation values. The collection of $\rho^{(\alpha)}$'s will have $[(\sum_\alpha n_\alpha) - m]$ components. Clearly, $V(\phi)$ is independent of the fields $\vec{\xi}$ since the invariance of V under G implies $V(\phi) = V(v + \rho)$. If there were no gauge bosons, the Lagrangian would depend on $\vec{\xi}$ only through $\partial_\mu\vec{\xi}$, arising from the terms $(\partial_\mu\phi^{(\alpha)})^\dagger \times (\partial^\mu\phi^{(\alpha)})$ in the Lagrangian. Consequently, the fields $\vec{\xi}$ would represent massless scalar particles, coupled to other particles gradiently. They would be the Goldstone fields.

When the theory is invariant under local gauge transformations, the $\vec{\xi}$ fields can be eliminated from the Lagrangian completely. We define the vector fields B_μ^a by

$$\vec{L}\cdot\vec{A}_\mu = e^{i\vec{\xi}\cdot\vec{t}}\,\vec{L}\cdot\vec{B}_\mu e^{-i\vec{\xi}\cdot\vec{t}} - \frac{i}{g}(\partial_\mu e^{i\vec{\xi}\cdot\vec{t}})e^{-i\vec{\xi}\cdot\vec{t}}\,. \tag{3.12}$$

The mapping $(A_\mu, \phi^{(\alpha)}) \to (B_\mu, \rho^{(\alpha)})$ expressed in Eqs. (3.11) and (3.12) is a gauge transformation (3.5) which leaves the Lagrangian (3.2) invariant. We have

$$\begin{aligned}\mathcal{L} = {}& \sum_\alpha [\Delta_\mu(v^{(\alpha)} + \rho^{(\alpha)})]^\dagger \cdot [\Delta^\mu(v^{(\alpha)} + \rho^{(\alpha)})] \\ & - \tfrac{1}{4}\sum_a (\partial^\mu B_a^\nu - \partial^\nu B_a^\mu - g f_{abc} B_b^\mu B_c^\nu)^2 - V(v + \rho)\,.\end{aligned} \tag{3.13}$$

Here Δ_μ stands for

$$\Delta_\mu = \partial_\mu + gi\,\vec{L}^{(\alpha)}\cdot\vec{B}_\mu\,. \tag{3.14}$$

Some of the gauge bosons are no longer massless. As the vector-meson mass term, we have

$$g^2\sum_\alpha (v^{(\alpha)}, L_a^\dagger L_b v^{(\alpha)}) B_\mu^a B_\nu^b g^{\mu\nu}$$

so that the vector-meson mass matrix is given by

$$(M^2)_{ab} = 2g^2\sum_\alpha (v^{(\alpha)}, L_a^\dagger L_b v^{(\alpha)})\,. \tag{3.15}$$

It is convenient to adopt the following convention: We order L_a's so that L_a, $a = 1, 2, \ldots, M$ form the set $\{l\}$. We see from Eq. (3.15) that M^2 is block diagonal, the upper $M\times M$ diagonal matrix being zero. The lower $m\times m$ matrix is positive definite (the lower matrix cannot have a null eigenvalue, for if it did, the little group would have a dimension larger than M).

Let us summarize the result of this section in a theorem [Kibble's theorem[4]]: Let G be the gauge symmetry of the Lagrangian and S, $G \supset S$, be the little group of the vacuum. The generators $\{L\}$ of G can be divided into two sets, the generators $\{l\}$ of S and the rest $\{t\}$. The gauge bosons corresponding to $\{l\}$ are massless. The gauge bosons corresponding to $\{t\}$ are massive. This theorem is an analog of that of Bludman and Klein[5] to spontaneously broken gauge theories.

If the symmetry is not spontaneously broken, i.e., $G = S$, the gauge bosons are endowed with the two transverse polarizations. If the symmetry is broken, some gauge bosons become massive and have three polarizations. How do the longitudinal components of massive vector bosons come about? We see from Eq. (3.12) that

$$\vec{L}\cdot\vec{B}_\mu = \vec{L}\cdot\vec{A}_\mu - \frac{1}{g}\vec{t}\cdot\partial_\mu\vec{\xi} + O(\vec{\xi}^2),$$

i.e., the would-be Goldstone fields serve as the longitudinal components of the massive vector bosons.

The discussions given above can be generalized to quantum field theory, if we use the generating functional of proper vertices instead of $\mathcal{L}$ in Eqs. (3.2), (3.6), and (3.7). This was done for the σ model in the last paper of Ref. 3.

IV. QUANTIZATION OF HIGGS PHENOMENA

In the preceding section we disposed of the general group-theoretical problem associated with the Higgs mechanism in the context of classical field theory. We shall now proceed to the quantization problem. To be specific we consider a simple model: SU(2) gauge bosons coupled to an isotriplet of scalar fields. The inclusion of fermions will be discussed in a sequel, when we discuss a more realistic model.

The Lagrangian of this model is, with $\mu^2 < 0$,

$$\mathcal{L} = -\tfrac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} + \tfrac{1}{2}(D_\mu\vec{\phi})^2 - \tfrac{1}{2}\mu^2\vec{\phi}^2 - \tfrac{1}{4}\lambda(\vec{\phi}^2)^2 - \tfrac{1}{2}\delta\mu^2\vec{\phi}^2\,, \tag{4.1}$$

where

$$D_\mu = \partial_\mu - g\vec{A}_\mu\times\,,$$
$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu - g\vec{A}_\mu\times\vec{A}_\nu\,, \tag{4.2}$$

and $\delta\mu^2$ is the scalar mass counterterm. If μ^2 is positive, we can quantize the theory in the manner described in Sec. II, and choose, for example, $M^2 - a^2 = \mu^2$, where M^2 and a^2 are defined in Eqs. (2.6) and (2.7).

Irrespective of the sign of μ^2, we can write the generating functional of the Green's functions as

$$W = \exp\{iZ[\vec{J}_\mu,\vec{K}]\} = \int[d\vec{A}][d\vec{\phi}]\exp\left[i\left(S_\alpha[\vec{J}_\mu,\vec{K}] + \int d^4x\,[\vec{K}\cdot\vec{\phi} - \vec{J}_\mu\cdot\vec{A}^\mu](x)\right)\right], \tag{4.3}$$

where S_α is the effective action:

$$S_\alpha = \int d^4x\left[\mathcal{L}(x) - \frac{1}{2\alpha}[\partial_\mu\vec{A}^\mu(x)]^2\right] - i\,\mathrm{Tr}\ln\left(1 - g\vec{t}\cdot\vec{A}_\mu\partial^\mu\frac{1}{\partial^2}\right). \tag{4.4}$$

The important fact one should bear in mind is that Eq. (4.3) applies equally well to the broken-symmetry case as it does to the symmetric case, and therefore *the same functional Ward-Takahashi identity* (I 7.1),

$$\frac{i}{\alpha}\partial_\mu\frac{\delta W}{\delta J^a_\mu(x)} - \int d^4y\,J^c_\lambda(y)D^\lambda_y[i\delta/\delta\vec{J}]^{cb}G^{ba}(y,x;i\delta/\delta\vec{J})W + ig\int d^4y\,K^c(y)t^{cdb}\frac{\delta}{\delta K^b(y)}G^{da}(y,x;i\delta/\delta\vec{J})W = 0\,, \tag{4.5}$$

holds in the broken-symmetry case also.

If we were to write down the Feynman rules for the Lagrangian (4.1) as in the symmetric case, then we would get imaginary masses for scalar bosons. The correct way of generating the perturbation expansion for the generating functional (4.3) is to expand the $V(\phi)$ about its minimum

$$V(\phi) = \tfrac{1}{4}\lambda(\vec{\phi}^2)^2 + \tfrac{1}{2}\mu^2\vec{\phi}^2 \tag{4.6}$$

and define the free Lagrangian as the quadratic part in the new expansion parameters of the Lagrangian. Let

$$\left.\frac{\partial V}{\partial\vec{\phi}}\right|_{\vec{\phi}=\vec{v}} = 0\,,\quad \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\vec{\phi}=\vec{v}} \geqslant 0\,. \tag{4.7}$$

We shall write

$$\vec{v} = v\vec{\eta}\,, \tag{4.8}$$

where

$$v^2 = -\mu^2/\lambda \tag{4.9}$$

and $\vec{\eta}$ is a unit vector in the isospin space, pointing in the 3 axis, say. We shall denote the components of an isovector transverse to $\vec{\eta}$ by the subscript t: thus

$$(\phi_t)_i = (\delta_{ij} - \eta_i\eta_j)\phi_j\,. \tag{4.10}$$

We shall further define

$$\vec{\eta}\cdot\vec{\phi}=v+\psi\,, \tag{4.11}$$

$$\vec{\eta}\cdot\vec{A}^{\mu}=A^{\mu}\,. \tag{4.12}$$

We shall insist that v is the vacuum expectation value of the field $\vec{\eta}\cdot\vec{\phi}=\phi_3$, so that

$$\left.\frac{\delta Z}{\delta K_3(x)}\right|_{\vec{J}_\mu=\vec{K}=0}=v\,. \tag{4.13}$$

Equation (4.9) should really be thought as defining the part μ^2 of $\mu_0{}^2=\mu^2+\delta\mu^2$.

The generating functional (4.3) may be written as

$$W=\int[d\vec{A}_t]\,[dA]\,[d\vec{\phi}_t]\,[d\psi]\exp i\Big\{S^0_\alpha[\vec{A}^\mu_t, A^\mu, \vec{\phi}_t, \psi]+S^I[\vec{A}^\mu_t, A^\mu, \vec{\phi}_t, \psi] + \int d^4x[\vec{K}_t\cdot\vec{\phi}_t+K(v+\psi)-\vec{J}^\mu_t\cdot\vec{A}_{t\mu}-J^\mu A_\mu]\,(x)\Big\}\,, \tag{4.14}$$

where $K=K_3$ and $J^\mu=J^\mu_3$. In Eq. (4.14), S^0_α and S^I are respectively

$$S^0_\alpha=\int d^4x\{\tfrac12(\partial_\mu\vec{\phi}_t)^2+\tfrac12(\partial_\mu\psi)^2-\tfrac12(2\lambda v^2)\,\psi^2 - \tfrac14(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu)^2+\tfrac12(gv)^2(\vec{\eta}\times\vec{A}_\mu)^2+gv\vec{\eta}\cdot(\vec{A}^\mu\times\partial_\mu\vec{\phi})-(1/2\alpha)(\partial_\mu\vec{A}^\mu)^2\} \tag{4.15}$$

and

$$S^I=\int d^4x\{\tfrac12 g\vec{A}_\mu\times\vec{A}_\nu\cdot(\partial^\mu\vec{A}^\nu-\partial^\nu\vec{A}^\mu)-\tfrac14 g^2(\vec{A}_\mu\times\vec{A}_\nu)^2+g\vec{A}_\mu\cdot(\vec{\phi}\times\partial^\mu\vec{\phi})+\tfrac12 g^2(\vec{\phi}\times\vec{A}_\mu)^2 - \tfrac14\lambda(\vec{\phi}_t{}^2+\psi^2)^2-\lambda v\psi(\psi^2+\vec{\phi}_t{}^2)-\tfrac12\delta\mu^2(\vec{\phi}_t{}^2+\vec{\psi}^2)-v\delta\mu^2\psi\}-i\,\mathrm{Tr}\ln(1-g\vec{t}\cdot\vec{A}_\mu\partial^\mu/\partial^2)\,. \tag{4.16}$$

The perturbation expansion for the generating functional (4.14) is obtained from the formula

$$W[\vec{J}_\mu,\vec{K}]=\left[\exp iv\int d^4x\,K(x)\right]\left\{\exp iS^I\left[\frac{i\delta}{\delta\vec{J}_{t\mu}},\frac{i\delta}{\delta J_\mu},-\frac{i\delta}{\delta\vec{K}_t},-\frac{i\delta}{\delta K}\right]\right\}W_0[\vec{J}_\mu,\vec{K}]\,, \tag{4.17}$$

where

$$W_0[\vec{J}_\mu,\vec{K}]=\int[d\vec{A}]\,[d\vec{\phi}_t]\,[d\psi]\exp i\Big\{S^0_\alpha+\int d^4x[\vec{K}_t\cdot\vec{\phi}_t+K\psi+\vec{J}_\mu\cdot A^\mu]\,(x)\Big\}\,. \tag{4.18}$$

The right-hand side of Eq. (4.18) may be evaluated by the elementary method[12] and yields

$$W_0[\vec{J}_\mu,\vec{K}]=\exp\frac{i}{2}\int d^4x\int d^4y\,\{\vec{K}_t(z)\cdot\tilde{D}_F(x-y)\vec{K}_t(y)+K(x)\tilde{\Delta}(x-y;\,2\lambda v^2)K(y)-J^\mu(x)\tilde{D}^{(\alpha)}_{\mu\nu}(x-y)J^\nu(y) - \vec{J}^\mu_t(x)\cdot\tilde{\Delta}^{(\alpha)}_{\mu\nu}(x-y;\,g^2v^2)\vec{J}^\nu_t(y)+2\vec{\eta}\cdot\vec{J}^\mu_t\times\tilde{\Delta}^{(\alpha)}_\mu(x-y)\vec{K}_t(y)\}\,, \tag{4.19}$$

where

$$\begin{Bmatrix}\tilde{D}_F(x-y)\\ \tilde{\Delta}(x-y;\,\mu^2)\\ \tilde{D}^{(\alpha)}_{\mu\nu}(x-y)\\ \tilde{\Delta}^{(\alpha)}_{\mu\nu}(x-y;\,m^2)\\ \tilde{\Delta}^{(\alpha)}_\mu(x-y)\end{Bmatrix}=\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-y)}\begin{Bmatrix}\dfrac{1}{k^2+i\epsilon}\\ \dfrac{1}{k^2-\mu^2+i\epsilon}\\ \left[\dfrac{1}{k^2+i\epsilon}\left(g_{\mu\nu}-\dfrac{k_\mu k_\nu}{k^2}\right)+\alpha\dfrac{k_\mu k_\nu}{(k^2)^2}\right]\\ \left[\dfrac{1}{k^2-m^2+i\epsilon}\left(g_{\mu\nu}-\dfrac{k_\mu k_\nu}{k^2}\right)+\alpha\dfrac{k_\mu k_\nu}{(k^2)^2}\right]\\ gvk_\mu(k^2+i\epsilon)^{-2}\end{Bmatrix}. \tag{4.20}$$

Equation (4.17), together with Eq. (4.19), gives the Feynman rules and the Dyson-Wick expansion theorem.

It is convenient to expand the Green's functions in powers of g with g^2v^2 and λv^2 fixed (this implies $\lambda\sim g^2$).

It was shown[3] that such an expansion coincides with the expansion in the number of loops in Feynman diagrams.

The interaction Lagrangian in Eq. (4.16) contains the term linear in ψ

$$-v\delta\mu^2\psi\,. \tag{4.21}$$

Since ψ is supposed to have no vacuum expectation value,

$$\begin{aligned}\frac{\delta Z}{\delta K_3}\bigg|_{\vec{J}_\mu=\vec{K}=0} -v&=0\\ &=\int [d\vec{A}]\,[d\vec{\phi}_t]\,[d\psi]\,\psi(x)e^{i[S_\alpha^0+S^I]}\,,\end{aligned} \tag{4.22}$$

the role of the term (4.21) is to cancel the ψ-to-vacuum diagrams with one or more loops (the so-called tadpole diagrams). Let $ivS(v,\lambda)$ be the sum of the contributions from such diagrams. Then

$$v[S(v,\lambda)-\delta\mu^2]=0\,, \tag{4.23}$$

which determines $\delta\mu^2\equiv\mu_0^{\,2}+\lambda v^2$. As we shall see, we can express Eq. (4.23) more elegantly:

$$v\Delta_{\phi_t}^{\;-1}(0)=0\,, \tag{4.24}$$

where $\Delta_{\phi_t}(k^2)$ is the full propagator for the $\vec{\phi}_t$ field. Equation (4.24) is the mathematical expression for the Goldstone theorem. A detailed consideration shows that $\Delta_{\phi_t}(0)$ does not suffer from infrared divergence. Contributions from intermediate states of two massless particles to the self energy of $\vec{\phi}_t$ are explicitly proportional to k^2 to within logarithm in the Landau gauge for example, so that $\Delta_{\phi_t}(0)$ is finite (see Appendix D).

In the next section, we shall show that Green's functions are finite if we choose $\delta\mu^2$ to satisfy Eq. (4.23) or (4.24), and renormalize fields and sources according to

$$\begin{aligned}(v,\psi,\vec{\phi}_t)&=Z_2^{\,1/2}(v,\psi,\vec{\phi}_t)_r\,,\\ A_\mu&=Z_3^{\,1/2}(A_\mu)_r\,,\\ \vec{J}_\mu&=Z_3^{\,-1/2}(\vec{J}_\mu)_r\,,\\ \vec{K}&=Z_2^{\,-1/2}(\vec{K})_r\,,\end{aligned} \tag{4.25}$$

and coupling constants according to

$$\begin{aligned}g&=g_r\,Z_1/Z_3^{\,3/2}=g_r\,\tilde{Z}_1/\tilde{Z}_3Z_3^{\,1/2}\,,\\ \lambda&=\lambda_r\,Z_4/Z_2^{\,2}\,,\end{aligned} \tag{4.26}$$

where Z_1, Z_2, Z_3, Z_4, $\tilde{Z}_1$, and $\tilde{Z}_3$ are to be chosen to make the symmetric theory (the theory with the same λ and g but with $\mu^2>0$) finite. These renormalizations can be implemented in Eq. (4.14) if we write S_α^0 and S^I in terms of renormalized quantities and add to S^I counterterms. We shall omit the subscript r. The expression S_α^0 remains the same as Eq. (4.15) and

$$\begin{aligned}S^I=\;&\text{right-hand side of Eq. (4.16)}\\ &+\int d^4x\bigg\{(Z_2-1)[\tfrac{1}{2}(\partial_\mu\vec{\phi}_t)^2+\tfrac{1}{2}(\partial_\mu\psi)^2]+(Z_1-1)\lambda v^2\psi^2-(Z_3-1)\tfrac{1}{4}(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu)^2\\ &\qquad+\tfrac{1}{2}(gv)^2\left(\frac{Z_1^{\,2}}{Z_3}\frac{Z_2}{Z_3}-1\right)(\vec{\eta}\times\vec{A}_\mu)^2+gv\left(Z_1\frac{Z_2}{Z_3}-1\right)\vec{\eta}\cdot\vec{A}^\mu\times\partial_\mu\vec{\phi}\\ &\qquad+\tfrac{1}{2}g(Z_1-1)\vec{A}_\mu\times\vec{A}_\nu\cdot(\partial^\mu\vec{A}^\nu-\partial^\nu\vec{A}^\mu)-\tfrac{1}{4}g^2\left(\frac{Z_1^{\,2}}{Z_3}-1\right)(\vec{A}_\mu\times\vec{A}_\nu)^2+g\left(Z_1\frac{Z_2}{Z_3}-1\right)\vec{A}_\mu\cdot(\vec{\phi}\times\partial^\mu\vec{\phi})\\ &\qquad+\tfrac{1}{2}g^2\left(\frac{Z_1^{\,2}}{Z_3}\frac{Z_2}{Z_3}-1\right)(\vec{\phi}\times\vec{A}_\mu)^2-\tfrac{1}{4}\lambda(Z_4-1)(\vec{\phi}_t^{\,2}+\psi^2)-\lambda v(Z_4-1)\psi(\vec{\phi}_t^{\,2}+\psi^2)\\ &\qquad-\tfrac{1}{2}\delta\mu^2(Z_2-1)(\vec{\phi}_t^{\,2}+\psi^2)-v\delta\mu^2(Z_2-1)\psi\bigg\}\\ &-i\,\mathrm{Tr}\ln\bigg\{1+\frac{1}{\partial^2-g\vec{t}\cdot\vec{A}^\mu\partial_\mu}[(\tilde{Z}_3-1)\partial^2-g(\tilde{Z}_1-1)\vec{t}\cdot\vec{A}^\mu\partial_\mu/\partial^2]\bigg\}\,.\end{aligned} \tag{4.27}$$

V. PROOF OF FINITENESS

The discussion in the previous section may suggest to the alert reader that all we have to do to renormalize the Green's functions of the spontaneously broken gauge theory is to construct the generating functional (4.3) of the *renormalized* Green's functions for $\mu^2>0$, and then continue the resulting functional analytically to $\mu^2<0$. Unfortunately, the Green's functions are not analytic in μ^2 at $\mu^2=0$,[3] so that we need a little bit of machinery to implement the above idea.

Let us set up this machinery. We consider the generating functional of Eq. (4.3) for $\mu^2>0$ and expand the generating functional about $\vec{J}_\mu=0$ and $\vec{K}=\vec{\gamma}$, where $\vec{\gamma}$ is a constant vector in the isospin space. The expansion coefficients are the Green's functions of the theory whose formal action is given by

$$S_\alpha(\vec{\gamma})=\int d^4x\{\mathcal{L}(x)-(1/2\alpha)[\partial^\mu\vec{A}_\mu(x)]^2+\vec{\gamma}\cdot\vec{\phi}(x)\}-i\,\mathrm{Tr}\ln(1-g\vec{t}\cdot\vec{A}_\mu\partial^\mu/\partial^2)\,. \tag{5.1}$$

Of course, the action of Eq. (5.1) does not follow from any local Lagrangian which makes sense. The action (5.1) is just our device of connecting the $\mu^2>0$ and $\mu^2<0$ cases, as we shall see.

The term $\int d^4x\gamma\cdot\phi(x)$ induces a vacuum expectation value of $\phi(x)$. Let

$$v_i(\vec{\gamma})=\delta Z/\delta K_i(x)|_{\vec{J}_\mu=0,\vec{K}=\vec{\gamma}}\;. \tag{5.2}$$

As we shall see $\vec{v}$ and $\vec{\gamma}$ are necessarily parallel, and we write

$$\vec{\gamma}=c\vec{\eta}\;, \tag{5.3}$$

$$\vec{v}(\vec{\gamma})=v_c\vec{\eta}\,. \tag{5.4}$$

We shall now decompose the fields ϕ and A_μ as

$$\vec{\phi}=\vec{\phi}_t+\vec{\eta}(v_c+\psi)\,, \tag{5.5}$$

$$\vec{A}^\mu=\vec{A}_t^\mu+\vec{\eta}A^\mu\,, \tag{5.6}$$

with $\vec{\eta}\cdot\vec{\phi}_t=\vec{\eta}\cdot\vec{A}_t^\mu=0$. The action (5.1) can be written as

$$S_\alpha(\vec{\gamma})=S_\alpha^0(\vec{\gamma})+S^I(\vec{\gamma})\,, \tag{5.7}$$

where

$$S_\alpha^0(\vec{\gamma})=\int d^4x\{\tfrac{1}{2}(\partial_\mu\vec{\phi}_t)^2-\tfrac{1}{2}m^2\vec{\phi}_t^{\,2}+\tfrac{1}{2}(\partial_\mu\psi)^2-\tfrac{1}{2}(m^2+2\lambda v_c^{\,2})\psi^2-\tfrac{1}{4}(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu)^2+\tfrac{1}{2}g^2v_c^{\,2}(\vec{A}_t^\mu)^2$$
$$+gv_c\vec{\eta}\cdot(\vec{A}^\mu\times\partial_\mu\vec{\phi})-(1/2\alpha)(\partial^\mu\vec{A}_\mu)^2\}\;, \tag{5.7$'$}$$

with

$$m^2=\mu^2+\lambda v_c^{\,2} \tag{5.8}$$

and S^I is given by Eq. (4.16) with $v=v_c$ except that the linear term in ψ should now be written as

$$[c-v_c(m^2+\delta\mu^2)]\psi\,. \tag{5.9}$$

Note that Eqs. (4.15) and (4.16) are recovered from Eqs. (5.7)–(5.9) as we let $c=0$ and $m^2=0$. Again, the role of the term (5.9) is to cancel the ψ-to-vacuum diagrams with one or more loops. Let iv_cS_c be the sum of the contributions from such diagrams. Then

$$v_c(m^2+\delta\mu^2-S_c)=c\,. \tag{5.10}$$

We will now give a brief summary of the ensuing argument. We will first show that the Green's functions for the action (5.1) are renormalized by the counterterms of the symmetric theory ($\mu^2>0$, $c=0$). We shall then show that the renormalized Green's functions of the spontaneously broken gauge theory ($\mu^2<0$, $c=0$) are obtained from those of the action (5.1) in the limit $c=0$, $m^2=0$. We shall make precise the meaning of this limit in due course. In the course of our discussion, it is important to note whether μ^2 or m^2 is kept fixed.

Following Jona-Lasinio, we will introduce the generating functional Γ of the proper (i.e., single-particle irreducible) vertices. First define

$$\Phi_i(x)=\delta Z/\delta K_i(x) \tag{5.11}$$

and

$$-g^{\mu\nu}\mathfrak{A}_\nu^a(x)=\delta Z/\delta J_\mu^a(x)\,. \tag{5.12}$$

The generating functional Γ is obtained from Z by a Legendre transformation:

$$\Gamma[\vec{\mathfrak{A}}_\mu,\vec{\Phi}]=Z[\vec{J}_\mu,\vec{K}]-\int d^4x[\vec{K}\cdot\vec{\Phi}-\vec{J}^\mu\cdot\vec{\mathfrak{A}}_\mu](x)\,. \tag{5.13}$$

We have the Maxwell equations dual to Eqs. (5.11) and (5.12):

$$-K_i(x)=\delta\Gamma/\delta\Phi_i(x) \tag{5.14}$$

and

$$g^{\mu\nu}J_\nu^a(x)=\delta\Gamma/\delta\mathfrak{A}_\mu^a(x)\,. \tag{5.15}$$

In particular, we obtain

$$\vec{\Phi}(x)|_{\vec{J}_\mu=0,\,\vec{K}=\vec{\gamma}}=\vec{v}(\vec{\gamma})=v_c\vec{\eta} \tag{5.16}$$

and its dual

$$-\gamma_i=\delta\Gamma/\delta\Phi_i|_{\vec{\mathfrak{A}}_\mu=0,\,\vec{\Phi}=\vec{v}(\vec{\gamma})}\,. \tag{5.17}$$

According to the analysis of Jona-Lasinio, the expansion coefficients of Γ about $\mathfrak{A}_\mu=0$, $\vec{\Phi}=\vec{v}(\vec{\gamma})$ are the proper vertices for the action (5.1):

$$\tilde{\Pi}(x_1\cdots x_n;y_1\cdots y_m;z_1\cdots z_l|v)=\left.\frac{\delta^{n+m+l}\Gamma[\vec{\mathfrak{A}}_\mu,\vec{\Phi}]}{\delta\mathfrak{A}(x_1)\cdots\delta\Phi_t(y_1)\cdots\delta\Phi_l(z_1)\cdots}\right|_{\mathfrak{A}_\mu=0,\,\vec{\Phi}=\vec{v}}, \tag{5.18}$$

where we have written

$$\vec{\Phi}=\vec{\Phi}_t+\vec{\eta}\,\Phi_l$$

and suppressed all isospin and tensor indices. We define the Fourier transform Π by

$$(2\pi)^4\delta(\Sigma p+\Sigma q+\Sigma r)\Pi(p_1\cdots p_n;q_1\cdots q_m;r_1\cdots r_l|v)$$
$$=\int\prod_{i=1}^n d^4x_ie^{ip_i\cdot x_i}\prod_{j=1}^m d^4y_je^{iq_j\cdot y_j}\prod_{k=1}^l d^4z_ke^{ir_k\cdot z_k}\tilde{\Pi}(x_1\cdots x_n;y_1\cdots y_m;z_1\cdots z_l|v)\,. \tag{5.19}$$

The expansion coefficients of Γ about $\vec{\mathfrak{A}}_\mu=0$, $\vec{\Phi}=0$ are the proper vertices $\Pi(\cdots|v=0)$ of the symmetric theory. Therefore, we have for $\mu^2>0$, and μ^2 held fixed,

$$\Pi(p_1\cdots p_n;q_1\cdots q_m;r_1\cdots r_l|v,g,\lambda)=\sum_{s=0}^\infty\frac{(v)^s}{s!}\Pi(p_1\cdots p_n,q_1\cdots q_m,r_1\cdots r_l00\cdots0|0,g,\lambda)\,. \tag{5.20}$$

Where $00\cdots0$ consists of s factors. Equation (5.20) expresses a proper vertex for the action (5.1) in terms of those of the symmetric theory which we know how to renormalize. The proper vertices appearing in the right-hand side contain $(l+s)\phi_3$ lines of which s lines disappear into the vacuum. We recall from Sec. II that the renormalized vertex $\Pi_r(\cdots|0,g_r,\lambda_r)$,

$$\Pi_r(p_1\cdots p_n;q_1\cdots q_m;r_1\cdots r_l\cdots r_{l+s}|0,g_r,\lambda_r)$$
$$\equiv(Z_3)^{n/2}(Z_2)^{(m+l+s)/2}\Pi(p_1\cdots p_n;q_1\cdots q_m;r_1\cdots r_{l+s}|0,g_rZ_1Z_3^{-3/2},\lambda_rZ_4Z_2^{-2})\,, \tag{5.21}$$

is finite with an appropriate choice of Z_1, Z_2, Z_3, Z_4, $\tilde{Z}_1$, and $\tilde{Z}_3$ with $Z_1Z_3^{-1}=\tilde{Z}_1\tilde{Z}_3^{-1}$. We define the renormalizations of the left-hand side of Eq. (5.17) by

$$\Pi_r(p_1\cdots p_n;q_1\cdots q_m;r_1\cdots r_l|v_r,g_r,\lambda_r)$$
$$\equiv(Z_3)^{n/2}(Z_2)^{(m+l)/2}\Pi(p_1\cdots p_n;q_1\cdots q_m;r_1\cdots r_l|Z_2^{1/2}v_r,g_rZ_1Z_3^{-3/2},\lambda_rZ_4Z_2^{-2})\,. \tag{5.22}$$

Then we see that

$$\Pi_r(\cdots|v_r,g_r,\lambda_r)=\sum_{s=0}^\infty\frac{(v_r)^s}{s!}\Pi_r(\cdots00\cdots0|0,g_r,\lambda_r) \tag{5.23}$$

where $00\cdots0$ consists of s factors. It shows that if we renormalize the wave functions and coupling constants, and choose the mass counterterm $\delta\mu^2$ as in the symmetric theory, then the proper vertex $\Pi_r(\cdots|v_r)$ is finite if $v_r = Z_2^{-1/2}v$ is. [*Note added in proof.* The expansion of Eq. (5.17) entails in part developing the vector-boson propagators in powers of v^2:

$$\frac{1}{k^2-(gv)^2}=\frac{1}{k^2}+\sum_{n=1}^{\infty}\frac{1}{k^2}\left(g^2v^2\frac{1}{k^2}\right)^n .$$

The terms with $n\geq 1$ will cause infrared divergence of the integral. This is a reflection of the fact that Π is not analytic in v near $v=0$. To circumvent this difficulty we may replace the nth term ($n\geq 1$) by

$$\frac{1}{k^2+i\epsilon}\left(g^2v^2\frac{1}{k^2+\lambda^2+i\epsilon}\right)^n .$$

Since primitively divergent parts which include this term are at most logarithmically divergent, renormalization constants of the symmetric theory render them finite. *After* the summation indicated on the right-hand side of Eq. (5.20) is carried out, λ^2 may be let go to zero. This process gives the desired $\Pi_r(\cdots|v_r, g_r, \lambda_r)$, which is independent of λ^2 and finite. Alternatively, we may define a modification of the series in Eq. (5.17). We substitute for massive vector-boson propagators the expression

$$\frac{1}{k^2+i\epsilon}+\frac{1}{k^2+i\epsilon}(gv)^2\frac{1}{k^2+\lambda^2+i\epsilon}$$

and for scalar-boson propagators similar expressions. A Feynman integral for Π becomes a sum of terms. In these terms, subdiagrams consisting entirely of the propagators of the symmetric theory are made finite by the counterterms of the symmetric theory. Subdiagrams in which one of the propagators is replaced by the second term above are at most logarithmically divergent. Divergence in such a subdiagram is also removed by a symmetric counterterm. See reference for a similar discussion for the σ model.] In the symmetric theory, $\delta\mu^2$ and Z_2 may be chosen to satisfy Eq. (2.7) with $\mu^2 = M^2 - a^2$, for example. For the purpose of making $\Pi_r(\cdots|0)$ finite, however, we need not choose the finite parts of $\delta\mu^2$ and Z_2 in this manner. For our purpose, it is more convenient to choose $\delta\mu^2$ so that $\Delta_{\phi_t}(0)$ has the value

$$\Delta_{\phi_t}(0) = -m^{-2}, \tag{5.24}$$

where m^2 is the quantity appearing in Eq. (5.8) (see Appendix D). Obviously the vertices $\Pi_r(\cdots|v_r)$ may be regarded as functions of m^2 rather than of μ^2. Henceforth we shall treat m^2 defined by Eq. (5.24) as an independent variable.

How does one determine v_r in Eq. (5.23)? It must be determined from Eq. (5.10) which is the condition that ψ have a null vacuum expectation value. To determine the structure of S_c, we turn to our sheep, the Ward-Takahashi identity (4.5). We show in Appendix A that Eq. (4.5) implies

$$\epsilon^{abc}\int d^4x\left[J_\mu^b(x)\frac{\delta}{\delta J_\mu^c(x)}+K^b(x)\frac{\delta}{\delta K^c(x)}\right]W=0 . \tag{5.25}$$

Differentiating Eq. (5.25) with respect to K and taking the limit $\vec{J}_\mu = 0$ and $\vec{K}=\vec{\gamma}$, we obtain

$$\vec{\gamma}\Delta_{\phi_t}(0) = -\vec{v}(\vec{\gamma}), \tag{5.26}$$

which shows $\vec{\gamma}$ and $\vec{v}$ are parallel [see Eqs. (5.3) and (5.4)] and

$$c_r = v_r m^2, \tag{5.27}$$

where

$$c_r = cZ_2^{1/2} . \tag{5.28}$$

Equation (5.27) is the renormalized version of Eq. (5.10). Thus if c_r is finite, so is v_r.

Let us summarize the results so far. We have shown that the Green's functions for the action (5.1) become finite if we renormalize the fields, sources and coupling constants according to

$$\begin{aligned}
&(\vec{\phi}, \vec{v}) = Z_2^{1/2}(\vec{\phi}, \vec{v})_r ,\\
&(\vec{K}, \vec{\gamma}) = Z_2^{-1/2}(\vec{K}, \vec{\gamma})_r ,\\
&\vec{A}_\mu = Z_3^{1/2}(\vec{A}_\mu)_r ,\\
&\vec{J}_\mu = Z_3^{-1/2}(\vec{J}_\mu)_r ,\\
&g = g_r(Z_1/Z_3^{3/2}) = g_r(\tilde{Z}_1/\tilde{Z}_3 Z_3^{1/2}),\\
&\lambda = \lambda_r(Z_4/Z_2^2),
\end{aligned} \tag{5.29}$$

and choose $\delta\mu^2$ and Z_2 to satisfy Eq. (5.24), and other Z's to be those of the symmetric theory. By the regularization method developed in paper I, the renormalizations implied in Eqs. (5.21) and (5.23) are made unambiguous and to preserve gauge invariance. That is to say, the Green's functions constructed from $\Pi_r(\cdots|v)$ of Eq. (5.23) satisfy the Ward-Takahashi identities generated from Eq. (4.5) by expanding Z about $(\vec{J}_\mu)_r=0$ and $\vec{K}_r=\vec{\gamma}_r$.

Equation (5.27) is the Goldstone theorem. In the spontaneously broken case, $c_r = c = 0$, so that $m^2 = 0$, which is Eq. (4.24). In this case, the renormalization conditions given in this section reduce to those of the last section. The finiteness proof of Eqs. (5.23) and (5.27) applies to the spontaneous breaking case (i.e., $c_r=0$, v_r finite) as well.

VI. LOW-ENERGY BEHAVIORS OF PROPAGATORS

In this and the following sections we will deal exclusively with renormalized quantities. We shall therefore drop the subscripts r consistently.

From Eq. (4.5) we learn that the longitudinal part of the vector meson propagator is unrenormalized. The derivation of this fact is completely analogous to that given in Sec. IV of the previous paper. Therefore, the full vector propagator has the form

$$\Delta_{\mu\nu}(k) = (g_{\mu\nu} - k_\mu k_\nu/k^2)f(k^2) + \alpha k_\mu k_\nu/(k^2)^2 . \quad (6.1)$$

For the $a=1$ and 2 components of the vector propagator, Eq. (6.1) leads to useful relations. Let $\Gamma_{\mu\nu}$, Γ_μ, and Γ be defined by

$$\tilde{\Gamma}^{\mu\nu}(x-y) = \delta^2\Gamma[\vec{\mathfrak{a}}_\mu, \vec{\Phi}]/\delta\mathfrak{a}^1_\mu(x)\delta\mathfrak{a}^1_\nu(y)|_{\vec{a}_\mu=0,\vec{\Phi}=\vec{v}} ,$$

$$\tilde{\Gamma}^{\mu}(x-y) = \delta^2\Gamma[\vec{\mathfrak{a}}_\mu, \vec{\Phi}]/\delta\mathfrak{a}^1_\mu(x)\delta\Phi^2(y)|_{\vec{a}_\mu=0,\vec{\Phi}=\vec{v}} , \quad (6.2)$$

$$\Gamma(x-y) = \delta^2\Gamma[\vec{\mathfrak{a}}_\mu, \vec{\Phi}]/\delta\Phi^2(x)\delta\Phi^2(y)|_{\vec{a}_\mu=0,\vec{\Phi}=\vec{v}} ,$$

and

$$\begin{bmatrix}\tilde{\Gamma}^{\mu\nu}\\ \tilde{\Gamma}^{\mu}\\ \tilde{\Gamma}\end{bmatrix}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \begin{bmatrix}\Gamma^{\mu\nu}\\ \Gamma^{\mu}\\ \Gamma\end{bmatrix}(k) . \quad (6.3)$$

In Eq. (6.2) $\vec{v}$ is chosen to be along the third axis of the isospin space, and is determined by the condition

$$\frac{\delta\Gamma}{\delta\Phi_i}\bigg|_{\vec{a}_\mu=0,\vec{\Phi}=\vec{v}} = 0 . \quad (6.4)$$

The propagators defined as

$$\delta^2 Z/\delta K^2(x)\delta K^2(y)|_{\vec{J}_\mu=\vec{K}=0} = -\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)}\Delta(k^2) ,$$

$$\delta^2 Z/\delta J^1_\mu(x)\delta K^2(y)|_{\vec{J}_\mu=\vec{K}=0} = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)}\Delta^\mu(k) , \quad (6.5)$$

$$\delta^2 Z/\delta J^1_\mu(x)\delta J^1_\nu(y)|_{\vec{J}_\mu=\vec{K}=0} = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)}\Delta^{\mu\nu}(k) ,$$

and the proper vertices of Eq. (6.3) are the inverses of each other, in the sense that

$$\begin{pmatrix}\Gamma(k^2) & \Gamma_\lambda(-k)\\ \Gamma_\mu(k) & \Gamma_{\mu\lambda}(k)\end{pmatrix}\begin{pmatrix}1 & \\ & -g^{\lambda\rho}\end{pmatrix}\begin{pmatrix}\Delta(k^2) & \Delta_\nu(-k)\\ \Delta_\rho(k) & -\Delta_{\rho\nu}(k)\end{pmatrix} = \begin{pmatrix}1 & \\ & -g_{\mu\nu}\end{pmatrix} . \quad (6.6)$$

We shall parametrize the proper vertices of Eq. (6.3) as

$$\begin{aligned}\Gamma_{\mu\nu}(k) &= -g_{\mu\nu}A(k^2) + k_\mu k_\nu B(k^2) ,\\ \Gamma_\mu(k) &= ik_\mu C(k^2) , \quad (6.7)\\ \Gamma(k^2) &= k^2 D(k^2) .\end{aligned}$$

When the above expressions are substituted into Eq. (6.6), we obtain for $\Delta_{\mu\nu}$

$$\Delta_{\mu\nu}(k) = (g_{\mu\nu} - k_\mu k_\nu/k^2)A^{-1} + \frac{k_\mu k_\nu}{k^2}\frac{D}{D(A-k^2B)+C} . \quad (6.8)$$

Comparing the longitudinal parts of Eqs. (6.1) and (6.8), we obtain

$$\alpha(AD - BDk^2 + C^2) = k^2 D , \quad (6.9)$$

which is the desired relation.

The propagators in Eq. (6.5) can be written as

$$\Delta_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)A^{-1} + \alpha\frac{k_\mu k_\nu}{(k^2)^2} ,$$

$$\Delta_\mu(k) = i\alpha\frac{k_\mu}{(k^2)^2}\left(\frac{C}{D}\right), \quad (6.10)$$

and

$$\Delta(k^2) = \frac{1}{k^2 D(k^2)} - \frac{\alpha}{(k^2)^2}\left(\frac{C}{D}\right)^2 .$$

Let us consider the low-energy limits of propagators in Eq. (6.10). Now taking the limit $k^2 \to 0$ in Eq. (6.9), we find that

$$\lim_{k^2\to 0}(AD + C^2) = 0. \quad (6.11)$$

It is instructive to see what happens in the $a=3$ channel. The invariances of the Lagrangian and the vacuum expectation value under $\phi_{1,3} \to +\phi_{1,3}$, $\phi_2 \to -\phi_2$ and $A_\mu{}^2 \to +A_\mu{}^2$, $A^{1,3}_\mu \to -A^{1,3}_\mu$ imply that $C(k^2)=0$ in Eq. (6.7). Equation (6.9) becomes

$$\alpha(A - k^2 B) = k^2 .$$

Writing $A = k^2 J$ we see that

$$\Gamma_{\mu\nu}(k) = -(k^2 g_{\mu\nu} - k_\mu k_\nu)J - (1/\alpha)k_\mu k_\nu . \quad (6.12)$$

VII. GAUGE INDEPENDENCE AND THE UNITARITY OF THE S MATRIX

By using Eq. (2.8) repeatedly, we obtain, for $k \leq l$,

$$\left(\frac{i}{\alpha}\right)^k \frac{\partial}{\partial x_1^{\mu_1}} \cdots \frac{\partial}{\partial x_k^{\mu_k}} \frac{\delta^{k+l} i Z}{\delta J_{\mu_1}(x_1)\cdots\delta J_{\mu_k}(x_k)\delta J_{\nu_1}(y_1)\cdots\delta J_{\nu_l}(y_l)}\bigg|_{J_\mu=K=0}$$

$$=\sum_{\text{part}}\sum_{\text{perm}(k)} W^{-1}\left\{\prod_{i=1}^{k}\int d^4z_i\, g^{\text{tr}}_{\nu_{j_i},\lambda_i}(y_{j_i}-z_i)\left[-ig\tilde{Z}_1\frac{\delta}{\delta J_{\lambda_i}(z_i)}\right]G(z_i,x_i;i\delta/\delta J)\right\}\left[\prod_{m=k+1}^{l}\frac{\delta}{\delta J_{\nu_{j_m}}(y_{j_m})}\right]W\bigg|_{J_\mu=K=0}, \quad (7.1)$$

where $\sum_{\text{part}}$ is the summation over all possible partitions of $(1,2,\ldots,l)$ into two subsets, $\{j_i\}$, $i=1,\ldots,k$ and $\{j_m\}$, $m=k+1,\ldots,l$, and $\sum_{\text{perm}(k)}$ is the summation over all permutations of k elements of $\{j_i\}$. We have suppressed all references to the isospin which is not crucial in our discussion. We used the symbol $g^{\text{tr}}_{\mu\nu}$ for

$$g^{\text{tr}}_{\mu\nu}(x-y)=g_{\mu\nu}\delta^4(x-y)+\partial_\mu\partial_\nu\tilde{D}_F(x-y).$$

For $k=l+1$, we have

left-hand side of (7.1)

$$=\sum_{\text{perm}(l)} W^{-1}\left\{\prod_{i=1}^{l}\int d^4z_i\, g^{\text{tr}}_{\nu_{j_i},\lambda_i}(y_{j_i}-z_i)\left[-ig\tilde{Z}_1\frac{\delta}{\delta J_{\lambda_i}(z_i)}\right]G(z_i,x_i;i\delta/\delta J)\right\}\frac{i}{\alpha}\frac{\partial}{\partial x_k^{\mu_k}}\frac{\delta}{\delta J_{\mu_k}(x_k)}W\bigg|_{J_\mu=K=0}. \quad (7.2)$$

There are $(k-1)$ more equations of this kind in which the privileged role of (x_k, μ_k) on the right-hand side is taken up by $(x_1,\mu_1),\ldots,(x_{k-1},\mu_{k-1})$. For $k>l+1$, we have simply

$$\text{left-hand side of (7.1)}=0. \quad (7.3)$$

The three equations above are the bases of our discussion on the gauge independence and the unitarity of the S matrix. By the gauge independence of the S matrix, we mean that the on-shell S matrix is independent of α in the gauge defining term in the action (2.1). The proof given in Ref. 13 can be carried over to our case. First note that

$$\left(\prod_{i=1}^{m}\frac{i}{\alpha}\frac{\partial}{\partial x_i^{\mu_i}}\right)\frac{\delta^m Z}{\delta J_{\mu_1}(x_1)\cdots\delta J_{\mu_m}(x_m)}\bigg|_{J_\mu=K=0}=0, \quad (7.4)$$

which is a special case of (7.3). Equation (7.4) corresponds exactly to Eq. (6.11) of Ref. 13, and by the argument given there we conclude that the T matrix is independent of the parameter α.

We wish, next, to show that the massless scalar particles we encounter in the construction of Green's functions are unphysical, i.e., do not contribute to the sum over intermediate states when we compute the absorptive part of a physical (i.e., on-shell) amplitude by the Landau-Cutkosky rule.[8,9] Recall that there are in general three different massless scalars: the negative metric scalar excitation (the first kind) associated with the transverse vector propagator

$$\sim\frac{k_\mu k_\nu}{k^2}\frac{1}{A}$$

the Goldstone boson (the second kind), with the propagator

$$1/k^2D$$

and the fermion scalars associated with the gauge field quantization (the third kind).

Let us begin with the simplest example. Let $T_\mu(k\cdots)$ be the amputated Green's function with one vector boson off the mass shell and all other lines on the mass shell. We have shown explicitly the momentum k and the tensor index μ for the vector boson, but suppressed all other variables. Let $T(k\cdots)$ be the amputated Green's function with one Goldstone boson off shell (with momentum k) and all other external lines on shell. Consider now the combination

$$T^{(2)}_\mu\left(\frac{k^\mu k^\nu}{k^2+i\epsilon}\frac{1}{A}\right)T^{(1)}_\nu+T^{(2)}\frac{1}{(k^2+i\epsilon)D}T^{(1)} \quad (7.5)$$

and compute the absorptive part of this amplitude arising form the two kinds of scalars being on the mass shell. By the Cutkosky rule it is given by

$$- T_1^{(2)*} T_1^{(1)} + T_2^{(2)*} T_2^{(1)} , \tag{7.6}$$

where

$$T_2 = \lim_{k^2 \to 0} T(k^2)/[D(k^2)]^{1/2}$$

is the amplitude for the Goldstone boson (massless particle of the second kind), and

$$T_1 = i \lim_{k^2 \to 0} k_\mu T^\mu(k)/[A(k^2)]^{1/2}$$

is the normalized amplitude for the massless scalar particle of the first kind. (The infrared divergences in D and A *always* cancel the similar ones in the vertices to which the propagators are attached, so that T_1 and T_2 are free of divergences as $k^2 \to 0$). Since

$$\frac{i}{\alpha} \partial_\mu \frac{\delta Z}{\delta J_\mu(x)}\bigg|_{J_\mu = K = 0} = 0$$

we have the relation

$$\frac{1}{\alpha} k_\mu \left\{ \left[\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) A^{-1} + \alpha k^\mu k^\nu (k^2)^{-2} \right] T_\nu + i\alpha \frac{k^\mu}{(k^2)^2} \frac{C}{D} T \right\} = -i \frac{\sqrt{A}}{k^2} \left(\frac{ik^\mu}{\sqrt{A}} T_\mu - \frac{1}{\sqrt{D}} T \right) = 0 ,$$

which gives, in the limit $k^2 = 0$,

$$T_1 = T_2 . \tag{7.7}$$

Therefore the expression (7.6) is identically zero, and neither of the scalars contributes to the sum over states.

To proceed further, it is necessary to extract more information from Eqs. (7.1)–(7.3). Let

$$T_{i_1 i_2 \cdots i_s}(1, 2, \ldots, s)$$

be the amplitude for s massless scalar excitations of the first and second kinds, the subscripts $i_1, \ldots, i_s$, which take the value 1 or 2 indicating which kinds are involved. We suppress, as before, all references to other particles which are on their mass shells. Let

$$G_{i_1 \cdots i_s}(1, 2, \ldots, s \,|\, (j_1, k_1), (j_2, k_2), \ldots, (j_t, k_t))$$

be the amplitude for $s + 2t$ massless scalar excitations, s being either of the first or second kind, and $2t$ being of the third kind. The ghost "particles" of the third kind appear in pairs, and their pairings are unambiguous, because the ghost lines are continuous. In the pair (j_n, k_n), the ordering is important, because the ghost line is orientable (say, from the dotted end j_n to the undotted end k_n). Equation (7.1) tells us that for $k \leq l$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_k=1}^{2} T_{i_1 \cdots i_k 1 1 \cdots 1}(1, \ldots, k, k+1, \ldots, k+l) = \sum_{\text{part}} \sum_{\text{perm}(k)} G_{\{i_{jm}\}}(\{jm\} \,|\, (j_1, 1), \ldots, (j_k, k)) , \tag{7.8}$$

where $11 \cdots 1$ consists of l factors and, as before, $\sum_{\text{part}}$ means the summation over all possible partitions of $(k+1, k+2, \ldots, k+l)$ into two subsets, $\{j_i\}$ $i = 1, 2, \ldots, k$ and $\{j_m\}$ $m = k+1, \ldots, k+l$, and $\sum_{\text{perm}(k)}$ means the summation over all permutations of $\{j_i\}$, $i = 1, 2, \ldots, k$. For $k = l+1$, Eq. (7.2) tells us that

$$\text{left-hand side of Eq. (7.8)} = \sum_{i_k=1}^{2} \sum_{\text{perm}(l)} G_{i_k}(k \,|\, (j_1, 1), (j_2, 2), \ldots, (j_l, l)) \tag{7.9}$$

For $k > l+1$, we have from Eq. (7.3)

$$\text{left-hand side of Eq. (7.8)} = 0 . \tag{7.10}$$

We claim that Eqs. (7.8)–(7.10) are sufficient to prove that the contributions from three kinds of zero-mass excitations always cancel in the sum over intermediate states, no matter how many massless excitations there are in a given intermediate state. To see how it works, let us consider two cases in detail.

Suppose there are two massless scalars in the intermediate states. The unitarity sum is

$$U = \sum_{i_1, i_2} e^{i\pi(i_1 + i_2)} T_{i_1 i_2}^{(2)*}(1, 2) T_{i_1 i_2}^{(1)}(1, 2) - G^{(2)*}(\,|(1, 2)) G^{(1)}(\,|(2, 1)) - G^{(2)*}(\,|(2, 1)) G^{(1)}(\,|(1, 2)) . \tag{7.11}$$

The last two terms have negative signs because the scalars of the third kind are fermions. Equation (7.8) gives

$$\sum_{i_1} T_{i_1 1}(1,2) = G(\,|(21)) \equiv G(21)\,,$$
$$\sum_{i_2} T_{1 i_2}(1,2) = G(\,|(12)) \equiv G(12)\,, \tag{7.12}$$

and Eq. (7.10) gives

$$\sum_{i_1,i_2} T_{i_1 i_2}(1,2) = 0\,. \tag{7.13}$$

Equations (7.12) and (7.13) allow us to express T_{11}, T_{12}, and T_{21} in terms of others:

$$T_{11} = T_{22} + G(12) + G(21)\,,$$
$$T_{21} = -T_{22} - G(21)\,,$$
$$T_{12} = -T_{22} - G(12)\,.$$

When the above expressions are substituted in Eq. (7.11), we find

$$U = 0\,.$$

Now consider the case of three massless scalars. We will use the abbreviations $T_{i_1 i_2 i_3} = T_{i_1 i_2 i_3}(1,2,3)$, $G_{i_1}(1|23) = G_{i_1}(1|(2,3))$. The unitarity sum is

$$U = \sum_{i_1 i_2 i_3} e^{i\pi(i_1+i_2+i_3)} T^{(2)*}_{i_1 i_2 i_3} T^{(1)}_{i_1 i_2 i_3}$$
$$- \sum_{k=1}^{2} e^{i\pi k}[G_k^{(2)*}(1|23)G_k^{(1)}(1|32) + G_k^{(2)*}(1|32)G_k^{(1)}(1|23) + G_k^{(2)*}(2|31)G_k^{(1)}(2|13)$$
$$+ G_k^{(2)*}(2|13)G_k^{(1)}(2|31) + G_k^{(2)*}(3|12)G_k^{(1)}(3|21) + G_k^{(2)*}(3|21)G_k^{(1)}(3|12)]\,. \tag{7.14}$$

The relations among various amplitudes we can get from Eqs. (7.8)–(7.10) are

$$\sum_i T_{i11} = G_1(3|12) + G_1(2|13)\,,$$
$$\sum_i T_{1i1} = G_1(3|21) + G_1(1|23)\,,$$
$$\sum_i T_{11i} = G_1(2|31) + G_1(1|32)\,,$$
$$\sum_{i,j} T_{ij1} = \sum_j G_j(2|13) = \sum_i G_i(1|23)\,, \tag{7.15}$$
$$\sum_{i,j} T_{i1j} = \sum_j G_j(3|12) = \sum_i G_i(1|32)\,,$$
$$\sum_{i,j} T_{1ij} = \sum_j G_j(3|21) = \sum_i G_i(2|31)\,,$$
$$\sum_{i,j,k} T_{ijk} = 0\,.$$

The relations (7.15) are enough to show that U of (7.14) is identically zero.

This process can be pushed *ad infinitum*. We have not found a sufficiently convenient and compact notation to carry out the calculation efficiently for N massless particles. In verifying the cancellation for $N=4$, for example, it is important to bear in mind the fermion nature of the particles of the third kind, so that in the unitarity sum we have

$$+G^{(2)*}(\,|(12),(34))\,G^{(1)}(\,|(21),(43)) - G^{(2)*}(\,|(12),(34))\,G^{(1)}(\,|(41),(23))\,.$$

Note the relative signs.

APPENDIX A: THE σ-MODEL-LIKE IDENTITY

The simplest way of deriving Eq. (5.25) is to consider a constant gauge transformation on the variables of integration in the functional integral (4.3). We give here an alternative derivation of Eq. (5.25) from Eq. (4.5).

From Eq. (5.25) we obtain

$$\left(\partial^2 - i\vec{t}\cdot\frac{\delta}{\delta\vec{J}_\mu}\partial_\mu\right)^{ab}\frac{i}{\alpha}\partial_\lambda\frac{\delta W}{\delta J^b_\lambda} + \partial^\mu J^a_\mu W + ig\left(J_\mu\vec{t}\cdot\frac{\delta}{\delta\vec{J}_\mu} + K\vec{t}\cdot\frac{\delta}{\delta\vec{K}}\right)^a W$$
$$- i\int d^4y\,\delta(y-x)t^{acd}D^\lambda_y[i\delta/\delta J]^{cb}\frac{\partial}{\partial x^\lambda}G^{bd}(y,x;i\delta/\delta\vec{J})W = 0\,. \tag{A1}$$

Since

$$\epsilon^{abc}\partial_\mu\left[\frac{\delta}{\delta J^b_\mu}\partial_\lambda\frac{\delta W}{\delta J^c_\nu}\right] = \epsilon^{abc}\frac{\delta}{\delta J^b_\mu}\partial_\mu\partial_\lambda\frac{\delta W}{\delta J^c_\nu}$$

and

$$\epsilon^{acd}\left\{D^\lambda_y[i\delta/\delta\vec{J}]^{cb}\frac{\partial}{\partial x^\lambda}G^{bd}(y,x;i\delta/\delta\vec{J})\right\}_{x=y} = \epsilon^{acd}\frac{\partial}{\partial x^\lambda}\{D^\lambda_y[i\delta/\delta\vec{J}]^{cb}G^{bd}(y,x;i\delta/\delta\vec{J})\}_{x=y},$$

we can write all but the third term on the left-hand side of Eq. (A1) as divergences of vectors. Equation (5.25) follows upon integration over x.

APPENDIX B: CONSTRUCTION OF RENORMALIZABLE MASSIVE VECTOR-MESON THEORIES

In this appendix, we pose and discuss the following problem: How does one construct a theory in which all of the gauge bosons associated with the gauge group G become massive while the vacuum is invariant under the little group S, which is not a local gauge group? The construction here may be of interest in providing models of strong interactions.

We shall now consider the following set of groups:

$$G^{(L)}\times S^{(R)}\supset S^{(L)}\times S^{(R)}\supset S^{(D)}\,.$$

$S^{(L)}, S^{(R)}, S^{(D)}$ are isomorphic to S, and $S^{(D)}$ is the diagonal subgroup of $S^{(L)}\times S^{(R)}$.

We construct a theory with the following properties:

(1) The Lagrangian is invariant under local gauge transformations of the group $G^{(L)}$ and constant gauge transformations of the group $S^{(R)}$. A^a_μ are the gauge fields associated with the group $G^{(L)}$.

(2) $\phi^{(\alpha)}$ is a set of scalar fields, with nonzero vacuum expectation value $v^{(\alpha)}$. The little group of the vacuum is $S^{(D)}$.

(3) All other fields present in the Lagrangian are invariant under transformations of the group $S^{(R)}$.

In the notation of Sec. III, $\{L\}$ are the generators of $G^{(L)}\times S^{(R)}$, $\{l\}$ are the generators of $S^{(D)}$. The generators of $G^{(L)}$ complete the set of generators $\{L\}$. $\{t\}$ will be this set:

$$\{t\}+\{l\}=\{L\}\,.$$

Now one can choose fields $\rho^{(\alpha)}$ and $\vec{\xi}$ such that

$$\phi^{(\alpha)} = D[e^{i\vec{\xi}\cdot\vec{t}}](v^{(\alpha)}+\rho^{(\alpha)})\,.$$

Using the local gauge invariance, one can eliminate the fields $\vec{\xi}$ and all the gauge fields A^a_μ become massive. In this way one has constructed a theory in which there appears a set of *massive* Yang-Mills fields associated with a given spontaneously broken symmetry G, the theory remaining symmetric under a subgroup S of G. (S is not a local gauge group).

In order to illustrate this mechanism, we will give some examples:

(1) Let G and S be isomorphic to SU(2). ϕ belongs to the $(\frac{1}{2},\frac{1}{2})$ representation of SU(2)×SU(2). In this model massive Yang-Mills are associated with an exact SU(2) symmetry. This is one of the models proposed by 't Hooft.[14]

(2) G is SU(2)×SU(2), S is isomorphic to SU(2). We let $\phi^{(L)}$ belong to a $(\frac{1}{2},0,\frac{1}{2})$ representation; $\phi^{(R)}$ to a $(0,\frac{1}{2},\frac{1}{2})$; $(\sigma,\vec{\pi})$ to a $(\frac{1}{2},\frac{1}{2},0)$. In this way one can construct a model in which a set of massive Yang-Mills fields is associated with the broken chiral symmetry SU(2)×SU(2).

(3) G is isomorphic to SU(3), S is isomorphic to SU(3) or SU(2). ϕ belongs to the $(3,\overline{3})+(\overline{3},3)$ representation of SU(3)×SU(3) or the $(3,\frac{1}{2})+(\overline{3},\frac{1}{2})$ representation of SU(3)×SU(2).

(4) SU(3)×SU(3) can be treated by a combination of the two preceding methods.

APPENDIX C: MASSIVE YANG-MILLS THEORY AS A LIMIT OF SPONTANEOUSLY BROKEN GAUGE THEORIES

In all the models discussed previously, it is easy to see that the masses corresponding to the fields having nonzero vacuum expectation values are free parameters. When these masses become infinite, one finds as a limit ordinary massive Yang-Mills field models. This can be most easily seen in the U gauge, in which the would-be Goldstone bosons have been eliminated.

In this sense these theories, when the masses are finite, can be considered as regularization of the ordinary massive Yang-Mills theories, in the same way as the linear σ model can be understood as a regularization of the nonlinear σ model.[15] It is possible that this limit, as in the case of the σ model, is less singular than the direct power counting of the limiting theory suggests.

We shall study a particular model here, but all the arguments will be completely general.

1. The Lagrangian. We set

$$\mathcal{L} = -\tfrac{1}{4}\mathrm{Tr}\{\partial_\mu V_\nu - \partial_\nu V_\mu + ig[V_\mu, V_\nu]\}^2 + \tfrac{1}{2}\mathrm{Tr}\{\partial_\mu M + ig[V_\mu, M]\}\{\partial^\mu M^\dagger - ig[M^\dagger, V^\mu]\}$$
$$+ V(M) + \text{other matter fields}\,.$$

The gauge group is SU(n), V_μ^b is a Hermitian traceless $n\times n$ matrix and M is an $n\times n$ complex matrix. $V(M)$ is a polynomial in M and $M^\dagger$ which can be chosen such that the vacuum expectation value of M is of the form

$$\langle M\rangle_0 = F\,,$$

where F is a real diagonal matrix. Furthermore, when the masses of the M fields become infinite $V(M)$ gives in the limit in the Feynman path integral a δ function of the form $\delta(MM^\dagger - F^2)$. In order to quantize the theory we add to the Lagrangian:

$$\delta\mathcal{L} = -\frac{1}{2\alpha}\mathrm{Tr}[\partial_\mu V^\mu - i\lambda(FM^\dagger - MF)]^2 + \mathrm{Tr}\{\bar{c}\partial^2 c + ig\bar{c}\partial_\mu[V^\mu, c]\} + \lambda g(\bar{c}FM^\dagger c + cMF\bar{c})\,,$$

where c and $\bar{c}$ are $n\times n$ matrices representing the usual scalar fermions ghosts, and λ can be chosen such that the term $-i(\lambda/\alpha)\mathrm{Tr}\partial^\mu V_\mu(FM^\dagger - MF)$ cancels the corresponding term in the Lagrangian which is obtained when one replaces M by $M' + F$. This is the gauge introduced by 't Hooft.[14]

Now in the limit of the infinite mass of scalar fields M the generating functional becomes

$$\exp iZ = \int [dV_\mu][dM]\cdots\prod_x \delta(MM^\dagger - F)\exp i\int d^4x[\mathcal{L} + \delta\mathcal{L} + \text{source terms}]\,.$$

We can make the following change of variable:

$$M = (e^{iH})\Omega\,,$$

where $H = H^\dagger$ and $\Omega = \Omega^\dagger$. The generating functional can now be written

$$\exp iZ = \int [dV_\mu][dH]J(H)d\cdots\exp i\int d^4x[\mathcal{L} + \delta\mathcal{L} + \text{source term}]\,,$$

where we have used the δ function, and $J(H)$ is the Jacobian.

2. Power counting. It is well known that in the unitary gauge, the most divergent graphs have a superficial degree of divergence δ of the form:

$$\delta = 6L\,,$$

where L is the number of loops. But it has been shown[16] that on the mass shell, cancellations occur which reduce the degree of divergence.

We will give here a new derivation of this result, using another gauge. We shall use the following identities in order to calculate the superficial degree of divergence δ of a graph:

$$\delta = 4 - E_B - \tfrac{3}{2}E_F + \textstyle\sum n_i(\delta_i - 4)\,,$$
$$\delta_i = m_i + v_i^B + \tfrac{3}{2}v_i^F\,,$$
$$E_B + E_F + 2I = \textstyle\sum n_i(v_i^B + v_i^F)\,,$$
$$L = I + 1 - \textstyle\sum n_i\,,$$

where E_B and E_F are the number of external bosons (or ghosts) and fermions, respectively, n_i is the number of vertices of type i, m_i is the number of derivatives at the vertex, v_i^B is the number of bosons, and v_i^F is the number of fermions at the vertex i. I is the number of internal lines. A straightforward calculation gives

$$\delta = 2L + 2 - \tfrac{1}{2}E_F + \sum n_i(m_i + \tfrac{1}{2}v_i^F - 2) .$$

Not returning to the Lagrangian one sees that either

$$v_i^F = 2 \quad \text{and} \quad m_i = 0$$

or

$$v_i^F = 0 \quad \text{and} \quad m_i \leq 2 .$$

The most divergent contributions are given by $\mathrm{Tr}\,\partial_\mu M \partial_\mu M^\dagger$ with $m_i = 2$. So,

$$\delta \leq 2L + 2 .$$

A closer examination, using the fact that we are not interested in Green functions with external particles associated to the fields H, c and $\bar{c}$, shows actually

$$\delta \leq 2L .$$

3. One-loop approximation. When the current is conserved, we have

$$F = f\underline{1} .$$

In the one-loop approximation the Lagrangian can be replaced by the following effective Lagrangian:

$$\mathfrak{L} = -\tfrac{1}{4}\mathrm{Tr}\{\partial_\mu V_\nu - \partial_\nu V_\mu + ig[V_\mu, V_\nu]\}^2 - (1/2\alpha)\mathrm{Tr}(\partial^\mu V_\mu)^2 + \tfrac{1}{2}(fg)^2\,\mathrm{Tr}\,V_\mu{}^2 + \tfrac{1}{2}f^2\mathrm{Tr}\{(\partial_\mu H)^2 - 2g^2f^2H^2 + igV^\mu[H, \partial_\mu H]\}$$
$$+ \mathrm{Tr}\{\bar{c}\partial^2 c + ig\bar{c}\partial_\mu[V^\mu, c]\} .$$

With this effective Lagrangian it is clear that the massless Yang-Mills theory is not the limit of the massive Yang-Mills theory. The massless Yang-Mills theory is obtained for $f = 0$. If f is different from zero, one can integrate over H, c, and $\bar{c}$, and obtains

$$\exp iZ = \int [dV^\mu] \exp i[S + \text{source terms}]\, \Delta_1(V^\mu)\Delta_2(V^\mu) ,$$

$$\Delta_1(V^\mu) = \exp[\mathrm{Tr}\ln(\partial^2 + ig\overleftarrow{\partial}^\mu[V^\mu, \;])] ,$$

$$\Delta_2(V^\mu) = \exp\{-\tfrac{1}{2}\mathrm{Tr}\ln(\partial^2 + \tfrac{1}{2}ig(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)[V^\mu, \;])\} .$$

In the Landau gauge ($\alpha = 0$) we have the relation

$$\Delta_1\Delta_2 = (\Delta_1)^{1/2}$$

because the two expressions inside Tr ln differ only by a term proportional to $\partial_\mu V^\mu$.

We, therefore, see in this way the origin of the difference of a factor 2 in front of the ghost loops, between the massless and the massive Yang-Mills cases.

APPENDIX D: INFRARED PROBLEM

Let us consider the contributions of intermediate states of two or more massless particles to the inverse of the propagator $\Delta_{\phi_t}(k^2)$. Since the phase space for N mass particles, ρ_N, goes as $(k^2)^{N-2}$, the integral

$$\int_0^\infty \frac{ds'}{s' - k^2}\rho_N(s')\,|T_N(s')|^2$$

is not infrared divergent for $N \geq 3$.

It suffices, therefore, to consider only the intermediate states of two massless particles. There are two such states: $\phi_1(p) \to \phi_2(p-q) + A_\mu{}^3(q)$ and $A_\mu{}^3(q)$ + the massless scalar associated with the longitudinal part of the A_μ' propagator. Since in the Landau gauge the $A_\mu{}^3$ propagator is purely transverse,

$$(g^{\mu\nu} - q^\mu q^\nu/q^2)[q^2 J(q^2)]^{-1} ,$$

and since any vector to be contracted with μ or ν of the above propagator may be expressed as a linear combination of $p_\mu(p_\nu)$ and $q_\mu(q_\nu)$ we see that the contributions of two massless particles to the self-energy are necessarily of order $p_\mu p_\nu$, disregarding logarithmic factors.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 12 15 JUNE 1972

Spontaneously Broken Gauge Symmetries. III. Equivalence

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We discuss the equivalence of the S matrix in the R- and U-gauge formulations of spontaneously broken gauge theories. We give definitions of the U-gauge Green's functions in terms of the R-gauge ones, for both Abelian and non-Abelian cases. Based on the equivalence theorem, we give a renormalization prescription of the U-gauge formulation.

I. INTRODUCTION

In this paper, we wish to demonstrate the equivalence of the S matrix in the R- and U-gauge formulations of spontaneously broken gauge theories. We have discussed the advantages and disadvantages of the two formulations in a previous paper (paper II).

We shall carry out this demonstration by expressing Green's functions in the U gauge in terms of those in the R gauge. What we shall show in this paper is a concrete realization of the remarks made previously by Weinberg[1] and by Salam and Strathdee[2] about the equivalence of the two formulations. But more importantly, the present work gives *definitions* of the U-gauge Green's functions in terms of the well-defined R-gauge ones.

This paper is organized as follows. In Sec. II we consider the equivalence of the two formulations for the Abelian model considered previously. In Sec. III, we give some illustrations of the equivalence and formulate the renormalization prescription in the U gauge. In Sec. IV, we deal with the generalization to non-Abelian cases.

It is empirically known that the T matrix for the Abelian case computed in the U gauge is finite.[1,3]