[7]Y. Nambu, Phys. Rev. D 4, 1193 (1971); I. T. Drummond, Nucl. Phys. B35, 269 (1971); F. Csikor, ITP-Budapest Report No. 292, 1971 (unpublished).

[8]As an example, we consider the case $l=3$, $m_{k+1}=2$, $m_{k+2}=1$, $m_j=0$ otherwise. Then we have

$$R^{\alpha_1\alpha_2\alpha_3}=P'^{\alpha_1}_{k+1}P'^{\alpha_2}_{k+1}P'^{\alpha_3}_{k+2}+qP'^{\alpha_1}_{k+1}P'^{\alpha_2}_{k+2}P'^{\alpha_3}_{k+1}+q^2P'^{\alpha_1}_{k+2}P'^{\alpha_2}_{k+1}P'^{\alpha_3}_{k+1}\,.$$

[9]In Eq. (24), C, the polynomial in the crossed variables, was written in a factorized form. If we invert the ordering of vectors in both $L_{\alpha_1\alpha_2\cdots\alpha_l}$ and $R^{\alpha_1\alpha_2\cdots\alpha_l}$, the result will remain valid, i.e., we have

$$C=\tilde{L}_{\alpha_1\alpha_2\cdots\alpha_l}\hat{R}^{\alpha_1\alpha_2\cdots\alpha_l}, \tag{24'}$$

where

$$\tilde{L}_{\alpha_1\alpha_2\cdots\alpha_l}=\frac{(P_{1\alpha_1}P_{1\alpha_2}\cdots P_{1\alpha(n_2)})}{f(n_2)}\times\frac{(P_{2\alpha(n_2+1)}\cdots P_{2\alpha(n_2+n_3)})}{f(n_3)}\cdots\times\frac{(P_{k-1\alpha(l-n_k+1)}\cdots P_{k-1\alpha(l)})}{f(n_k)} \tag{20'}$$

and

$$\tilde{R}^{\alpha_1\alpha_2\cdots\alpha_l}=\sum_{dP}q^{T_P}(P'^{\alpha_1}_{P(N-1)}\cdots P'^{\alpha(m_N-1)}_{P(N-1)})\cdots\times(P'^{\alpha(l-m_{k+1}+1)}_{k+1}\cdots P'^{\alpha(l)}_{k+1}). \tag{25'}$$

The amplitude of Eq. (30) follows from the factorized form of Eq. (24′). For notational convenience, we have also made the change $n_i\to n_{i-1}$.

PHYSICAL REVIEW D VOLUME 5, NUMBER 12 15 JUNE 1972

Spontaneously Broken Gauge Symmetries. I. Preliminaries

Benjamin W. Lee
National Accelerator Laboratory, P. O. Box 500, Batavia, Illinois 60510
and Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790

and

Jean Zinn-Justin*
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790
(Received 10 March 1972)

This is the first of a series of papers addressed to the renormalizability question of spontaneously broken gauge theories. We give a brief outline of the motivation for such an investigation and describe the manner in which the renormalizability of such theories will be proved in the sequel. Put briefly, we will show that in an appropriate gauge, ultraviolet divergences of a spontaneously broken gauge theory are removed completely by the gauge-invariant counterterms in the Lagrangian which would make the Green's functions of the corresponding unbroken gauge theory finite, that the S matrix computed in this gauge is unitary, and that the S matrix is independent of the gauge chosen. In this paper, the renormalizability question of the unbroken gauge theory is considered. We derive the Ward-Takahashi identities of the theory. We discuss several ways of regulating divergent Feynman integrals of the theory without destroying gauge invariance. Infrared divergences are avoided by the device of intermediate renormalization, wherein we choose as subtraction points some points where external momenta are Euclidean. This suffices to establish that the Bogoliubov-Parasiuk-Hepp renormalization will give renormalized Green's functions which satisfy the Ward-Takahashi identities. The existence of finite, renormalized Green's functions satisfying the Ward-Takahashi identities provides us with the means of proving the renormalizability of the spontaneously broken symmetry case. The Ward-Takahashi identities were previously derived for the gauge bosons by Slavnov. We present here a new derivation. The discussions on regularization methods and intermediate renormalization procedure and the renormalization conditions for matter fields, we believe, are new contributions of the present paper.

I. INTRODUCTION

This is the first of a series of papers which will deal with the renormalizability of spontaneously broken gauge symmetries. The intriguing possibilities of unifying electromagnetic and weak interactions in terms of Yang-Mills gauge bosons,[1-9] whose masses are generated by spontaneous breakdown of gauge invariance of the second kind,[4,5] and of constructing a finite theory of weak interactions[4,7-9] prompt a closer examination of the quantization and renormalization questions of theories of this genre.

In the sequel of this series, we wish to examine the following questions: (1) We will discuss both the group-theoretic and field-theoretic problems associated with the Higgs phenomenon.[10-12] This entails a careful study of the stability of the physical system which possesses the freedom associated with the gauge invariance. (2) We will also study the perturbative treatment of such theories. Here our aim is to show that, in an appropriate gauge, ultraviolet divergences of a spontaneously broken gauge theory are removed completely by the counterterms in the Lagrangian which would make Green's functions of the corresponding unbroken gauge theory finite. Thus the renormalizability of the unbroken Yang-Mills theory (to be defined below) implies the same for the spontaneously broken gauge theory. The philosophy and methodology we shall follow are the same as those we employed in the study of the σ model.[13] (3) In the gauge in which the renormalizability can be proven, the unitarity of the S matrix is not manifest, since the quantization in that gauge implies the use of an indefinite-metric Hilbert space for the construction of Green's functions. We will show that the physical S matrix is nonetheless unitarity. (4) We shall also discuss the equivalence of the S matrix constructed in this gauge and in the gauge in which the unitarity of the S matrix is manifest (but not the renormalization).

In this paper, we shall give a discussion of the renormalization problem of the (unbroken) Yang-Mills field theory. It is not attempted in the present paper to establish that a renormalized Yang-Mills theory exists as a physically satisfactory theory of massless particles. Due to the infrared problem associated with massless quanta, such a theory may very well not exist at all. What we wish to demonstrate is that renormalized Green's functions of the theory exist (without implying the same for the S matrix), which satisfy the Ward-Takahashi identities which will be derived. The existence of renormalized Green's functions will prove to be a sufficient foundation for the discussion of the renormalizability of the spontaneously broken symmetry theory, which we shall discuss in the sequel.

We will proceed in the following manner. After a brief review of the quantization of the non-Abelian gauge theories, we shall derive the Ward-Takahashi identities. We will then discuss ways of regularizing divergent Feynman integrals in a gauge-invariant manner. The regularized Feynman amplitude then satisfies the identities automatically. The Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization procedure[14-17] requires specifying the values of primitively divergent vertices (A primitively divergent vertex is a proper vertex whose superficial degree of divergence is non-negative. Our definition here differs from the conventional usage of this term.) at some subtraction points. When these values are chosen in accordance with the Ward-Takahashi identities, and the cutoff parameters associated with the regularization are let go to infinity, the renormalized amplitudes are obtained which satisfy the Ward-Takahashi identities. Because of the infrared divergence, it is prudent to choose as subtraction points some points other than where all external momenta vanish. We shall describe in some detail this "intermediate" renormalization procedure.

The Ward-Takahashi identities were previously derived by Slavnov[18] for the gauge bosons. The derivation we shall present is somewhat different from his. The discussions on regularization methods and intermediate renormalization procedure, and renormalization conditions for matter fields, we believe, are new contributions of the present paper.

The organization of the paper is as follows: Sec. II: Quantization; Sec. III: Ward-Takahashi Identities I; Sec. IV: Ward-Takahashi Identities II; Sec. IV A: Two-Point Functions; Sec. IV B: Three-Point Vertices; Sec. IV C: Four-Point Vertices; Sec. V: Regularization; Sec. VI: Renormalization Conditions and Infrared Divergences; Sec. VII: Renormalization of Matter Fields; Appendix A: Derivation of Some Equations; Appendix B: Generating Functional of Proper Vertices; Appendix C: Ward-Takahashi Identity for the Generating Functional of Proper Vertices; Appendix D: Power Counting.

II. QUANTIZATION

Following the works of Feynman,[19] DeWitt,[20] Popov and Faddeev,[21] and 't Hooft[22] we define the generating functional $Z[\vec{J}_\mu]$ of connected Green's functions by

$$e^{Z[\vec{J}_\mu]}=\int[d\vec{A}]\,\Delta_L[\vec{A}]\exp i\left\{\mathcal{L}(x)-\frac{1}{2\alpha}(\partial_\mu\vec{A}^\mu)^2-\vec{J}_\mu(x)\cdot\vec{A}^\mu(x)\right\}, \tag{2.1}$$

where $[d\vec{A}]$ is the canonical functional metric for the vector fields

$$[d\vec{A}]=\prod_{a,\mu,x} dA_\mu^a(x), \tag{2.2}$$

where a is the internal-symmetry label. We shall assume the internal symmetry to be SU(2), so that $a=1,2,3$, but the generalizations to other groups are trivial and immediate. $\mathcal{L}(x)$ is the Lagrangian for the Yang-Mills fields

$$\mathcal{L}(x)=-\tfrac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu}, \tag{2.3}$$

$$\vec{F}_{\mu\nu}=\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu-g\vec{A}_\mu\times\vec{A}_\nu, \tag{2.4}$$

which is invariant under local gauge transformations, the infinitesimal version of which is

$$A_\mu^a \to A_\mu^a + g(\vec{\omega}\times\vec{A}_\mu)^a + \partial_\mu \omega^a = A_\mu^a + (D_\mu)^{ab}\omega^b, \tag{2.5}$$

D_μ being the covariant derivative:

$$D_\mu[\vec{A}]^{ab} = \delta^{ab}\partial_\mu - g(\vec{t}\cdot\vec{A}_\mu)^{ab}, \tag{2.6}$$

$$(t_c)^{ab} = \epsilon^{acb}. \tag{2.7}$$

The Jacobian $\Delta_L[\vec{A}_\mu]$ is essentially the determinant of the operator $\partial_\mu D^\mu$, and may be expressed as

$$\begin{aligned}\Delta_i[\vec{A}_\mu] &= \exp\left[\operatorname{Tr}\ln\left(1 - g\frac{1}{\partial^2}\partial^\mu \vec{A}_\mu\cdot\vec{t}\right)\right]\\ &= \exp\left\{-\sum_{n=2}^{\infty}\frac{(-g)^n}{n}\int dx_1\cdots dx_n\ \operatorname{tr}\tilde{D}_F(x_n-x_1)\partial_{x_1}^{\mu_1}\vec{A}_{\mu_1}(x_1)\cdot\vec{t}\cdots\tilde{D}_F(x_{n-1}-x_n)\partial_{x_n}^{\mu_n}\vec{A}_{\mu_n}(x_n)\cdot\vec{t}\right\}\\ &= \exp\left[\operatorname{Tr}\ln\left(1 - g\vec{t}\cdot\vec{A}^\mu\partial_\mu\frac{1}{\partial^2}\right)\right],\end{aligned} \tag{2.8}$$

where we have used the Feynman propagator $D_F(x-y)$ defined as

$$\tilde{D}_F(x-y) = \langle x|(-\partial^2+i\epsilon)^{-1}|y\rangle. \tag{2.9}$$

The symbol Tr denotes the trace operation over x and the isospin a; the trace operation over the isospin index is denoted by tr.

The Feynman rules for this theory are obtained in the usual manner if we regard

$$\int d^4x\left[\mathcal{L}(x) - \frac{1}{2\alpha}[\partial_\mu\vec{A}^\mu(x)]^2\right] - i\operatorname{Tr}\ln\left(1 - g\vec{t}\cdot\vec{A}^\mu\partial_\mu\frac{1}{\partial^2}\right) \tag{2.10}$$

as the effective Lagrangian. The bare vector-boson propagator is

$$-i\Delta_{\mu\nu}(k^2;\alpha) = -i\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+i\epsilon}(1-\alpha)\right]\frac{1}{k^2+i\epsilon}. \tag{2.11}$$

In Eqs. (2.8) and (2.11) the $i\epsilon$ prescription is dictated by the unitarity considerations which we shall discuss in the sequel. The term $(1/2\alpha)(\partial_\mu\vec{A}^\mu)^2$ specifies the gauge one is employing and depends on a parameter which can vary from $-\infty$ to ∞. For $\alpha=0$, we obtain the transverse or Landau gauge, and for $\alpha=1$, the Feynman gauge. The last, nonlocal term in Eq. (2.10) is the new feature of non-Abelian gauge theories. It may be viewed[19] diagrammatically as the sum of closed-loop contributions from fictitious complex massless scalar fields (ghosts) obeying Fermi statistics which are coupled to the gauge fields through the interaction

$$g\bar{c}(x)\partial^\mu[\vec{t}\cdot\vec{A}_\mu(x)c(x)]. \tag{2.12}$$

The connected Green's functions of $\vec{A}_\mu$'s are obtained as the functional derivatives of $Z[J_\mu]$:

$$\frac{i\delta^n Z[\vec{J}_\mu]}{\delta J_\mu^a(x)\delta J_\nu^b(y)\cdots} = (-i)^n\langle T^*(A_\mu^a(x)A_\nu^b(y)\cdots)\rangle_0^c. \tag{2.13}$$

The Feynman rules are summarized in Fig. 1. In addition, the following rules should be kept in mind: The ghost-ghost-vector vertex is "dotted," the dot indicating which ghost line is differentiated; a ghost line cannot be dotted at both ends; a ghost loop carries an extra minus sign.

III. WARD-TAKAHASHI IDENTITIES-I

The invariance of the Lagrangian under the local gauge transformation (2.5) gives rise to a hierarchy of identities among the Green's functions (2.13). Alternatively, these relations may be expressed globally as an equation satisfied by the generating functional $Z[\vec{J}_\mu]$.

We will first rewrite Eq. (2.1) as

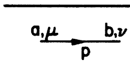
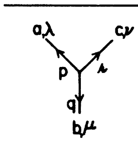
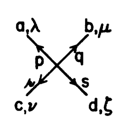
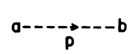
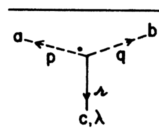
	VERTICES	BARE VERTICES
	$-i\delta^{ab}\Delta_{\mu\nu}(p)$	$-i\delta^{ab}[(g_{\mu\nu}-p_\mu p_\nu/p^2)1/p^2+\alpha p_\mu p_\nu(p^2)^{-2}]$
	$i\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)$ $p+q+r=0$	$\epsilon^{abc}[(p-q)_\nu g_{\lambda\mu} + (q-r)_\lambda g_{\mu\nu} + (r-p)_\mu g_{\nu\lambda}]$
	$i\Gamma^{abcd}_{\lambda\mu\nu\zeta}(p,q,r,s)$ $p+q+r+s=0$	$-i\epsilon^{abf}\epsilon^{cdf}(g_{\lambda\nu}g_{\mu\zeta}-g_{\lambda\zeta}g_{\mu\nu})$ $-i\epsilon^{acf}\epsilon^{bdf}(g_{\lambda\mu}g_{\nu\zeta}-g_{\lambda\zeta}g_{\mu\nu})$ $-\epsilon^{adf}\epsilon^{cbf}(g_{\lambda\nu}g_{\mu\zeta}-g_{\lambda\mu}g_{\zeta\nu})$
	$\delta^{ab}i\mathcal{D}(p)$	$i\delta^{ab}1/p^2$
	$i\gamma^{abc}_\lambda(p,q;r)$	$\epsilon^{abc}p_\lambda$

FIG. 1. Feynman rules of the Yang-Mills theory.

$$W[\vec{J}_\mu] = \exp iZ[\vec{J}_\mu] = \Delta_L[i\delta/\delta\vec{J}_\mu]W_0[\vec{J}_\mu], \tag{3.1}$$

where

$$W_0[\vec{J}_\mu] = \int [d\vec{A}] \exp i\left\{\mathcal{L}(x) - \frac{1}{2\alpha}[\partial^\mu \vec{A}_\mu(x)]^2 - \vec{J}^\mu(x)\cdot\vec{A}_\mu(x)\right\}. \tag{3.2}$$

We perform the gauge transformation (2.5) on the variables of integration $\vec{A}_\mu(x)$. Due to the invariance of the Lagrangian and the metric $[d\vec{A}]$, this transformation will affect only the source term and the gauge defining term:

$$\delta\int d^4x\left\{\frac{1}{2\alpha}[\partial_\mu\vec{A}^\mu]^2 + \vec{J}^\mu\cdot\vec{A}_\mu\right\} = \int d^4x\,\delta\vec{\omega}\cdot\left[\frac{1}{\alpha}D^\lambda\partial_\lambda\partial^\mu\vec{A}_\mu - D^\mu\vec{J}_\mu\right]. \tag{3.3}$$

Since a transformation of integration variables does not change the value of an integral, we may put the variation of W_0 with respect to $\delta\omega(x)$ equal to zero. We obtain thereby

$$\frac{1}{\alpha}D^\lambda[i\delta/\delta\vec{J}]^{ab}\partial_\lambda\partial_\mu\frac{i\delta W_0}{\delta J^b_\mu(x)} - D^\mu[i\delta/\delta\vec{J}]^{ab}J^b_\mu(x)W_0 = 0. \tag{3.4}$$

We note that (see Appendix A)

$$\Delta_L[i\delta/\delta\vec{J}]J^a_\lambda(x)\Delta_L^{-1}[i\delta/\delta\vec{J}] = J^a_\lambda(x) - ig\,\mathrm{tr}\,t^a[\partial_\lambda H(x,y;i\delta/\delta\vec{J})]_{x=y}, \tag{3.5}$$

where H is the solution of the equation

$$D_\lambda[\vec{A}]^{ab}\partial^\lambda H^{bc}(x,y;\vec{A}) = \delta^{ac}\delta^4(x-y), \tag{3.6}$$

satisfying the outgoing boundary condition; it has the representation

$$H^{ab}(x,y;\vec{A}) = -\langle x,a|[-\partial^2 + i\epsilon + g\vec{t}\cdot\vec{A}_\mu\partial^\mu]^{-1}|y,b\rangle. \tag{3.7}$$

Combining Eqs. (3.4) and (3.5), and recalling Eq. (3.1), we obtain

$$D^\lambda_x[i\delta/\delta\vec{J}]^{ab}\left\{\frac{i}{\alpha}\partial_\lambda\partial_\mu\frac{\delta}{\delta J^b_\mu(x)} - J^b_\lambda(x) + ig\,\mathrm{tr}\,t^b[\partial_\lambda H(x,y;i\delta/\delta\vec{J})]_{y=x}\right\}W = 0. \tag{3.8}$$

The last term on the left-hand side is equal to (see Appendix A)

$$igD^\lambda[i\delta/\delta\vec{J}]^{ab}t^{cbd}[\partial_\lambda H^{dc}(x,y;i\delta/\delta\vec{J})]_{y=x} = -ig\int d^4y\,\mathrm{tr}\,t^a[\partial_\mu H(x,y;i\delta/\delta\vec{J})]D^\mu_y[i\delta/\delta\vec{J}]\,\delta^4(x-y). \tag{3.9}$$

In showing this, one makes use of Eq. (3.6) and the Jacobi identity of the matrices t^a. Equation (3.8) may now be written as

$$D^\lambda[i\delta/\delta\vec{J}]^{ab}\partial_\lambda\left\{\frac{i}{\alpha}\partial_\mu\frac{\delta}{\delta J^b_\mu(x)} - \int d^4y : H^{bc}(x,y;i\delta/\delta\vec{J})D^\mu_y[i\delta/\delta\vec{J}]^{cd}J^d_\lambda(y):\right\}W = 0, \tag{3.10}$$

where the symbol : : denotes the normal product prescription that the $\delta/\delta J$ must stand to the right of the J.

We now define G by

$$\partial^\lambda D_\lambda[\vec{A}]^{ab}G^{bc}(x,y;\vec{A}) = \delta^{ac}\delta^4(x-y), \tag{3.11}$$

with the outgoing boundary condition, so that it may be represented as

$$G^{ab}(x,y;\vec{A}) = -\langle x,a|[-\partial^2 + i\epsilon + g\vec{t}\cdot\vec{A}_\mu\partial^\mu]^{-1}|y,b\rangle = H^{ba}(y,x;\vec{A}). \tag{3.12}$$

In terms of G, Eq. (3.10) may be considerably simplified. We finally obtain the desired identity:

$$\frac{i}{\alpha}\partial_\mu\frac{\delta W}{\delta J^a_\mu(x)} - \int d^4y\,J^c_\lambda(y)D^\lambda_y[i\delta/\delta\vec{J}]^{cb}G^{ba}(y,x;i\delta/\delta\vec{J})W = 0. \tag{3.13}$$

The above Ward-Takahashi identity was previously derived by Slavnov.[18] He considers a restricted class of gauge transformations which satisfy $\partial_\mu D^\mu\vec{\omega} = \vec{\chi}$ where $\vec{\chi}$ is an arbitrary function. He shows that the product $[d\vec{A}]\Delta_L[\vec{A}_\mu]$ remains invariant under the nonlinear gauge transformation generated by $\vec{\omega} = \vec{\omega}[\vec{A}_\mu, \vec{\chi}]$. The point of the above derivation is to show that Slavnov's form of the Ward-Takahashi identities is the most general form of the constraint on $W[\vec{J}_\mu]$ that follows from the gauge invariance of the Lagrangian.

For the purpose of renormalization, it is usually much more convenient to study the Ward-Takahashi identities connecting single particle irreducible (proper) vertices, as was done for the σ model[13,23] and the spontaneously broken Abelian gauge theories.[8] However, in the present instance, the Ward-Takahashi identities for the proper vertices are extremely complicated, being nonlinear relations among them. The Ward-Takahashi identity satisfied by the generating functional of the proper vertices is nevertheless derived and analyzed in Appendix C. The renormalization conditions will be analyzed on the basis of Eq. (3.13) in the following sections.

IV. WARD-TAKAHASHI IDENTITIES-II

We shall study the implications of Eq. (3.13) on the primitively divergent vertices.

A. Two-Point Functions

Differentiating Eq. (3.13) with respect to $J_\nu^b(y)$ and then letting $J_\mu=0$, we obtain

$$\frac{i}{\alpha}\frac{\partial}{\partial x_\mu}\frac{\delta^2 W}{\delta J_\mu^a(x)\delta J_\nu^b(y)}\bigg|_{\vec{J}_\mu=0} - D_y^\nu[i\delta/\delta\vec{J}]^{bc}G^{ca}(y,x;i\delta/\delta\vec{J})W\bigg|_{\vec{J}_\mu=0}=0. \tag{4.1}$$

Taking the divergence of the above equation with respect to y, and remembering the equation satisfied by G, we see that

$$\frac{i}{\alpha}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\frac{\delta^2 W}{\delta J_\mu^a(x)\delta J_\nu^b(y)}\bigg|_{\vec{J}_\mu=0}=\delta^{ab}\delta^4(x-y), \tag{4.2}$$

which shows that the longitudinal part of the propagator $\Delta_{\mu\nu}$,

$$\frac{\delta^2 Z[\vec{J}_\mu]}{\delta J_\mu^a(x)\delta J_\nu^b(y)}\bigg|_{\vec{J}_\mu=0}=\delta^{ab}\tilde{\Delta}_{\mu\nu}(x-y), \tag{4.3}$$

is not renormalized: The vector propagator has the form

$$\tilde{\Delta}_{\mu\nu}(x-y)=(g_{\mu\nu}-\partial_\mu\partial_\nu/\partial^2)\tilde{D}(x-y)+\alpha(\partial_\mu\partial_\nu/\partial^2)\tilde{D}_F(x-y). \tag{4.4}$$

In the momentum space, the inverse of the vector propagator, therefore, takes the form

$$[\Delta^{-1}(k)]_{\mu\nu}=(k^2g_{\mu\nu}-k_\mu k_\nu)J(k^2)+\frac{1}{\alpha}k_\mu k_\nu\,. \tag{4.5}$$

Were it not for the n-particle thresholds at $k^2=0$, $J(k^2)$ would be regular at $k^2=0$ (at least in perturbation theory), and the transverse part of the vector-meson propagator would have a simple pole at $k^2=0$.

Equation (4.1), when combined with Eq. (4.4) gives

$$\left\{\delta^{ab}\partial_\mu\tilde{D}_F(x-y)+\partial_\mu G^{ab}(x,y;i\delta/\delta\vec{J})-ig\left[\vec{t}\cdot\frac{\delta}{\delta\vec{J}^\mu(x)}\right]^{ac}G^{cb}(x,y;i\delta/\delta\vec{J})\right\}W\bigg|_{\vec{J}_\mu=0}=0 \tag{4.6}$$

or

$$\tilde{\mathcal{G}}^{ab}(x-y)=\delta^{ab}\tilde{D}_F(x-y)+ig\int d^4z\,\tilde{D}_F(x-z)\frac{\partial}{\partial z_\mu}\left[\vec{t}\cdot\frac{\delta}{\delta\vec{J}^\mu(z)}\right]^{ac}W^{-1}[\vec{J}_\mu]G^{cb}(z,y;i\delta/\delta\vec{J})W[\vec{J}_\mu]|_{\vec{J}_\mu=0}=0, \tag{4.7}$$

where $\tilde{\mathcal{G}}$ is the ghost propagator:

$$\tilde{\mathcal{G}}^{ab}(x-y)=-\{W^{-1}[\vec{J}_\mu]G^{ab}(x,y;i\delta/\delta\vec{J})W[\vec{J}_\mu]\}|_{\vec{J}_\mu=0}.$$

We may define the self-energy part Σ_g of the ghost by

$$\begin{aligned}&\delta^{ab}\mathcal{G}(k^2)=\int d^4x\,e^{ik\cdot x}\,\tilde{\mathcal{G}}^{ab}(x),\\&\mathcal{G}(k^2)=(k^2+i\epsilon)^{-1}[1-\Sigma_g(k^2)\mathcal{G}(k^2)]\,.\end{aligned} \tag{4.8}$$

The structure of Eq. (4.7) implies that $\Sigma_g(k^2)$ is of the form $-k_\mu\Sigma_g^\mu(k)$. Again, were it not for the fact that $k^2=0$ is the onsets of n-particle intermediate states, $\Sigma_g(k^2)$ would behave like k^2 near $k^2=0$, so that $\tilde{\mathcal{G}}(k^2)$ would have just a simple pole there.

B. Three-Point Vertices

From Eq. (3.13) we obtain

$$\left(\frac{i}{\alpha}\right)^2\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\frac{\delta^3 W}{\delta J_\mu^a(x)\delta J_\nu^b(y)\delta J_\lambda^c(z)}+D_z^\lambda[i\delta/\delta\vec{J}]^{cd}G^{da}(z,x;i\delta/\delta\vec{J})\frac{i}{\alpha}\frac{\partial}{\partial y^\nu}\frac{\delta W}{\delta J_\nu^b(y)}\bigg|_{\vec{J}_\mu=0}=0. \tag{4.9}$$

We note that

$$gW^{-1}[\vec{t}\cdot\delta/\delta\vec{J}_\lambda(z)]^{cd}G^{da}(z,x;i\delta/\delta\vec{J})\delta W/\delta J^b_\nu(y)|_{\vec{J}_\mu=0}$$

$$=g\int d^4x'd^4y'\tilde{\gamma}^{cab}_{\lambda\rho}(z,x';y')\mathcal{G}(x'-x)\tilde{\Delta}^{\rho\nu}(y'-y)+g\int d^4z'd^4z''d^4x'd^4y'\tilde{\Sigma}^\lambda_g(z-z')\mathcal{G}(z'-z'')\tilde{\gamma}^{cab}_\rho(z'',x';y')\times\mathcal{G}(x'-x)\tilde{\Delta}^{\rho\nu}(y'-y), \quad (4.10)$$

where

$$\tilde{\gamma}^{abc}_\rho(x,y;z)=-i\frac{\partial}{\partial x_\lambda}\tilde{\gamma}^{abc}_{\lambda\rho}(x,y;z),$$

and $\gamma^{abc}_\rho(p,q;r)$, defined by

$$\gamma^{abc}_\rho(p,q;r)(2\pi)^4\delta^4(p+q+r)=\int d^4x\,d^4y\,d^4z\,e^{i(p\cdot x+q\cdot y+r\cdot z)}\tilde{\gamma}^{abc}_\rho(x,y;z),$$

is the proper vertex for the coupling of a vector meson of momentum r, polarization ρ and isospin index c, with two ghosts, of momenta p and q and isospin indices a and b, respectively (we define the momenta outwardly from the vertex) (see Fig. 1). We have

$$\gamma^{abc}_\rho(p,q;r)=p^\lambda\gamma^{abc}_{\lambda\rho}(p,q;r). \quad (4.11)$$

The quantity $\tilde{\Sigma}^\lambda_g$ is defined by the equation

$$\tilde{\Sigma}_g(x-y)=-i\frac{\partial}{\partial x^\mu}\tilde{\Sigma}^\mu_g(x-y), \quad (4.12)$$

where

$$\Sigma_g(k^2)=\int d^4x\,e^{-ik\cdot x}\tilde{\Sigma}_g(x) \quad (4.13)$$

and satisfies the unrenormalized equation

$$i\delta^{ab}\tilde{\Sigma}^\lambda_g(x-y)=g\epsilon^{adc}\int(-i)\tilde{\Delta}^{\lambda\mu}(x-z)i\mathcal{G}(x-\omega)i\tilde{\gamma}^{dbc}_\mu(w,y;z)d^4\omega d^4z. \quad (4.14)$$

We consider the part which is transverse with respect to the index λ of Eq. (4.9). Noting that

$$W^{-1}[\vec{J}_\mu]\frac{\delta^3W[\vec{J}_\mu]}{\delta J^a_\mu(x)\delta J^b_\nu(y)\delta J^c_\lambda(z)}\bigg|_{\vec{J}_\mu=0}=-ig\int d^4x'd^4y'd^4z'\tilde{\Delta}^{\mu\mu'}(x-x')\tilde{\Delta}^{\nu\nu'}(y-y')\tilde{\Delta}^{\lambda\lambda'}(z-z')\tilde{\Gamma}^{abc}_{\mu'\nu'\lambda'}(x',y',z'),$$

where $\tilde{\Gamma}^{abc}_{\mu\nu\lambda}(x,y,z)$ is the proper three-point vertex of vector mesons, and taking the Fourier transform we obtain

$$(p^\lambda/p^2)(q^\mu/q^2)[r^2J(r^2)]^{-1}(g^{\rho\nu}-r^\rho r^\nu/r^2)\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)=(q^\mu/q^2)\mathcal{G}(p^2)(g^{\lambda\rho}-r^\lambda r^\rho/r^2)\gamma^{cab}_{\lambda\mu}(r,p;q),\quad p+q+r=0. \quad (4.15)$$

Equation (4.15) is a constraint among the propagators and three-point proper vertices. Equation (4.15) was first derived by Slavnov.[18]

C. Four-Point Vertices

It follows from Eq. (3.13) that

$$\frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\nu}\frac{\partial}{\partial w^\rho}\frac{\delta^4Z[\vec{J}_\mu]}{\delta J^a_\lambda(x)\delta J^b_\mu(y)\delta J^c_\nu(z)\delta J^d_\rho(w)}\bigg|_{\vec{J}_\mu=0}=0, \quad (4.16)$$

from which one obtains a constraint on the four-point vertex $\Gamma^{abc}_{\lambda\mu\nu\rho}$:

$$p^\lambda q^\mu r^\nu s^\rho\{\Gamma^{abcd}_{\lambda\mu\nu\rho}(p,q,r,s)+[\Gamma^{abe}_{\lambda\mu\sigma}(p,q,-p-q)\Delta^{\sigma\zeta}(p+q)\Gamma^{cde}_{\nu\rho\zeta}(r,s,p+q)+\text{two more terms}]\}=0,\quad p+q+r+s=0, \quad (4.17)$$

where $\Gamma^{abcd}_{\lambda\mu\nu\rho}$ is defined as

$$i\,\frac{\delta^4 Z[\tilde{J}_\mu]}{\delta J^a_\lambda(x)\delta J^b_\mu(y)\delta J^c_\nu(z)\delta J^d_\rho(w)}=(-i)^4 ig\int d^4x'd^4y'd^4z'd^4w'\,\tilde{\Delta}^{\lambda\lambda'}(x-x')\tilde{\Delta}^{\mu\mu'}(y-y')\tilde{\Delta}^{\nu\nu'}(z-z')\tilde{\Delta}^{\rho\rho'}(w-w')$$
$$\times\,\tilde{\Gamma}^{abcd}_{\lambda'\mu'\nu'\rho'}(x',y',z',w')+\text{ reducible parts} \qquad (4.18)$$

and

$$\int d^4x d^4y d^4z d^4w\; e^{i(p\cdot x+q\cdot y+r\cdot z+s\cdot w)}\,\tilde{\Gamma}^{abcd}_{\lambda\mu\nu\rho}(x,y,z,w)=(2\pi)^4\delta^4(p+q+r+s)\Gamma^{abcd}_{\lambda\mu\nu\rho}(p,q,r,s). \qquad (4.19)$$

The ghost-ghost-vector-vector vertex is superficially convergent and requires no discussion.

V. REGULARIZATION

The Feynman amplitudes constructed from the expression (2.1) would automatically satisfy the Ward-Takahashi identities discussed in the last two sections, if it were not for the ultraviolet divergences in their construction. A standard procedure of constructing the renormalized amplitudes satisfying the Ward-Takahashi identities is to regularize the Feynman integrals in a gauge-invariant manner; and then perform the R operation[14-17] of Bogoliubov, Parasiuk, and Hepp (BPH). The resulting amplitudes are cutoff independent, and if the values of primitively divergent vertices at subtraction points are chosen in accordance with the Ward-Takahashi identities, then the full amplitudes satisfy them too. Furthermore, under such circumstances, the R operation may be formally implemented by a gauge-invariant set of counter terms in the Lagrangian.

In this section we will discuss a few gauge-invariant regularization methods which can be implemented by adding gauge-invariant terms in the Lagrangian (e.g., Pauli-Villars regularization). 't Hooft[22] discussed a method which works for one-loop diagrams, but which does not appear to be implementable by modifying the Lagrangian.

We choose as regulator fields both scalar and spinor fields. They have all positive masses. They may belong to arbitrary representations (in general, reducible) of the symmetry group. They are coupled to the gauge fields by the minimal gauge-invariant coupling. They may, however, be quantized by the wrong spin-statistics connection (i.e., some scalar field multiplets may be quantized by the anticommutation relation). Let us show that the addition of these regulator fields to the Lagrangian renders the divergent Feynman integrals with one loop finite.

Let us first consider the self-energy of the gauge boson in the one-loop approximation. We will carry out the computations in the Feynman gauge. There are three diagrams (one is a ghost loop), and the sum of the contributions from these is

$$\Sigma_{\mu\nu}(p)=-4g_{\mu\nu}\frac{g^2}{16\pi^2}\int_0^\infty\frac{dz_1dz_2}{(z_1+z_2)^3}\left(-1+i\,\frac{z_1z_2}{z_1+z_2}\,p^2\right)\exp\left(i\,\frac{z_1z_2}{z_1+z_2}\,p^2\right)$$
$$+\frac{2g^2}{16\pi^2}\,i(g_{\mu\nu}p^2-p_\mu p_\nu)\int_0^\infty\frac{dz_1dz_2}{(z_1+z_2)^2}\,\frac{z_1^{\,2}+6z_1z_2+z_2^{\,2}}{(z_1+z_2)^2}\,\exp\left(i\,\frac{z_1z_2}{z_1+z_2}\,p^2\right). \qquad (5.1)$$

The first term on the right-hand side is gauge-noninvariant and quadratically divergent; the second term, which is gauge-invariant, is logarithmically divergent. We shall regulate $\Sigma_{\mu\nu}$ by the replacement

$$\Sigma_{\mu\nu}\to\Sigma^{(\;)}_{\mu\nu}=\Sigma_{\mu\nu}+\Sigma^{(1)}_{\mu\nu}+\Sigma^{(2)}_{\mu\nu}\,, \qquad (5.2)$$

where $\Sigma^{(1)}_{\mu\nu}$ is the sum of the scalar regulator contributions:

$$\Sigma^{(1)}_{\mu\nu}=\sum_i C_i\left\{-\frac{2g^2}{16\pi^2}g_{\mu\nu}\int_0^\infty\frac{dz_1dz_2}{(z_1+z_2)^2}\exp\left[i\left(\frac{z_1z_2}{z_1+z_2}\,p^2-(z_1+z_2)m_i^{\,2}\right)\right]\left[-1+i\left(\frac{z_1z_2}{z_1+z_2}\,p^2-(z_1+z_2)m_i^{\,2}\right)\right]\right.$$
$$\left.-\frac{g^2}{16\pi^2}\,i(p^2g_{\mu\nu}-p_\mu p_\nu)\int_0^\infty\frac{dz_1dz_2}{(z_1+z_2)^2}\exp\left(i\,\frac{z_1z_2}{z_1+z_2}\,p^2-i(z_1+z_2)m_i^{\,2}\right)\left(\frac{z_1-z_2}{z_1+z_2}\right)^2\right\}$$

and $\Sigma^{(2)}_{\mu\nu}$ is the sum of the spinor regulator contributions:

$$\Sigma^{(2)}_{\mu\nu} = -\sum_i D_i \int \frac{dz_1 dz_2}{(z_1+z_2)^2} \exp\left(i\,\frac{z_1 z_2}{z_1+z_2}p^2 - i(z_1+z_2)\mu_i^2\right)$$
$$\times\left\{-\frac{2g^2}{16\pi^2}\,g_{\mu\nu}\left[\frac{-1}{z_1+z_2} + i\left(\frac{z_1z_2}{(z_1+z_2)^2}\,p^2 - \mu_i^2\right)\right] - \frac{4g^4}{16\pi^2}\,i(g_{\mu\nu}p^2 - p_\mu p_\nu)\,\frac{z_1z_2}{z_1+z_2}\right\}.$$

The coefficients C_i and D_i depend on the representations to which the regulators belong, and also on whether they obey the normal or abnormal statistics. In any case, if we choose

$$2 + \sum_i C_i - \sum_j D_j = 0 \tag{5.3}$$

and

$$\sum C_i m_i^2 - \sum D_j \mu_j^2 = 0 \tag{5.4}$$

then the gauge-noninvariant term vanishes identically. Furthermore if we choose

$$10 - \sum_i C_i + 2\sum_j D_j = 0 \tag{5.5}$$

the logarithmic divergence in the gauge-invariant part may be eliminated. The introduction of two kinds of regulators is necessitated by the requirement that both quadratic and logarithmic divergences be eliminated.

Next, we consider the three-point vertex $\Gamma_{\lambda\mu\nu}(p,q,r)$ of three gauge bosons in the one-loop approximation. The integral is linearly divergent and has the asymptotic structure, in the Feynman gauge,

$$\sim \int \frac{d^4 l}{(2\pi)^4}\,\frac{l_\lambda l_\mu l_\nu}{(l^2)^3}\,, \tag{5.6}$$

when all diagrams, including the one with a ghost loop, are added. Again, by taking a suitable combination of scalar and spinor regulators, it is possible to eliminate all divergences from the Feynman integral for the three-point vertex.

The four-point vertex is logarithmically divergent and offers no special difficulty. We have not verified that this method works for higher-order diagrams. In any case, when there are matter fields present in the Lagrangian, the method presented above is insufficient and it becomes necessary to dampen the high-energy behavior of the gauge boson propagator itself. The method described below will do just this, and when combined with the spinor-scalar regulators, will render all Feynman integrals finite.

We will add gauge-invariant higher-derivative terms to the Lagrangian. Consider, for example, the Lagrangian

$$\mathcal{L}_\Lambda = -\tfrac{1}{4}\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu} - \frac{\alpha}{4\Lambda^2}(D^\sigma\vec{F}^{\mu\nu})\cdot(D_\sigma\vec{F}_{\mu\nu})$$
$$-\frac{\beta}{4\Lambda^4}(D^\sigma D_\sigma\vec{F}_{\mu\nu})\cdot(D^\rho D_\rho\vec{F}^{\mu\nu}). \tag{5.7}$$

The vector boson propagator is now

$$\Delta_{\mu\nu}(k;\Lambda^2) = (g_{\mu\nu} - k_\mu k_\nu/k^2)(k^2+i\epsilon)^{-1}\times\left[1+\alpha\frac{k^2}{\Lambda^2}+\beta\left(\frac{k^2}{\Lambda^2}\right)^2\right]^{-1}$$
$$+\text{gauge-dependent term}$$

and behaves like $(p^2)^{-3}$ asymptotically. The maximum dimension of various new couplings (in powers of mass) is eight. A power-counting argument (see Appendix D) shows that in this case only the two-, three-, and four-point vertices with *one loop* are primitively divergent (quadratically, linearly, and logarithmically, respectively). Other proper vertices, including two-, three-, and four-point vertices with more than one loop are at least superficially convergent. As one adds still higher gauge-covariant derivatives, the propagator becomes more convergent at large momentum, but the maximum dimensions of the interaction terms increase also, in such a way that the two-, three-, and four-point vertices with one loop remain always divergent (see Appendix D). Note also that ghosts loops for these vertices remain divergent.

Therefore, by the addition of the last two terms to the Lagrangian (5.7), the divergences of the theory have now been isolated to those diagrams for which the spinor-scalar regularization was shown to work.

The BPH R operation is to be applied to the entire two-, three-, and four-point proper vertices. The resulting vertices are cutoff-independent, in the sense that the limits $\Lambda^2\to\infty$ of these amplitudes are finite and independent of α and β of Eq. (5.7). This can be seen as follows. A proper amplitude with two, three, or four external lines which is proportional to some powers of α and β has in general an ambiguous limit as $\Lambda^2\to\infty$. However, the finite part of such an integral vanishes like $\Lambda^{-2}(\ln\Lambda^2)^m$ as $\Lambda^2\to\infty$. A proper Feynman diagram with n external lines, $n>4$, with one or more vertices proportional to α or β which are not contained in any subdiagrams with two, three, or four external lines vanishes at least as fast as $\Lambda^{-(n-4)}\times(\ln\Lambda^2)^m$ as $\Lambda^2\to\infty$, after the R operations are applied to the subdiagrams. (The above is a summary of a rather lengthy analysis.)

The results of regulating the Feynman integral by the method described above, applying the R operation and then letting the cutoff Λ^2 go to infinity is identical to applying the R operation directly to the Feynman integral. This shows that the BPH R operation is in fact a gauge-invariant procedure.

A similar regularization procedure has been applied to nonlinear chiral Lagrangians by Slavnov.[23] We understand from Jackiw and Faddeev[23a] that Slavnov has considered the regularization method of Eq. (5.7) for the gauge fields also. (After the completion of this work we received a report by Slavnov.[24]) This possibility has also been known to Johnson.[25]

VI. RENORMALIZATION CONDITIONS AND INFRARED DIVERGENCES

Let us first discuss briefly how the values of primitively divergent vertices are determined from the considerations of Sec. IV, ignoring the problems associated with infrared divergences. Under such circumstances, we may choose as subtraction points the points at which all external momenta vanish. Later we will discuss the nature of infrared divergences in gauge theories and give a set of renormalization conditions which avoid the infrared difficulties.

We may by convention choose, in Eq. (4.5),

$$J(0)=1,$$

which amounts to

$$\lim_{k^2\to 0}\Delta_{\mu\nu}(k)\sim\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}+\alpha\frac{k_\mu k_\nu}{(k^2)^2}\;. \tag{6.1}$$

The normalization of the ghost propagator is arbitrary. The ghost propagator has a simple pole at $k^2=0$ [see the discussion following Eq. (4.8)], and we write

$$\lim_{k^2\to 0}\mathcal{G}(k^2)\sim Z_g/k^2, \tag{6.2}$$

where Z_g is an arbitrary (finite) constant.

In the limit p, q, and $r=-p-q$ all go to zero, the three-point vertex $\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)$ has the form

$$\lim_{p,q,r\to 0}\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)\sim -iG\epsilon^{abc}[(p-q)_\nu g_{\lambda\mu}+(q-r)_\lambda g_{\mu\nu}+(r-p)_\mu g_{\nu\lambda}], \tag{6.3}$$

as follows from Lorentz covariance, isospin conservation, and Bose symmetry. Likewise, the low-energy form of the vertex $\gamma^{cab}_{\lambda\mu}(r,p;q)$ is given by

$$\lim_{p,q,r\to 0}\gamma^{cab}_{\lambda\mu}(r,p;q)=-iG'\epsilon^{abc}g_{\lambda\mu},$$

so that

$$\lim_{p,q,r\to 0}\gamma^{cab}_{\mu}(r,p;q)\sim -iG'\epsilon^{abc}r_\mu .$$

Equation (4.15) then tells us that

$$G=G'Z_g\,. \tag{6.4}$$

In the Green's functions, only the combination $G'Z_g$ enters, because the ghost never appears as an external line, so that it is convenient to set $Z_g=1$ and $G=G'$.

The low-energy form of the four-point vertex is given by

$$\lim_{p,q,r,s\to 0} i\Gamma^{abcd}_{\lambda\mu\nu\rho}(p,q,r,s)=-iF[\epsilon^{abe}\epsilon^{cde}(g_{\lambda\nu}g_{\mu\rho}-g_{\lambda\rho}g_{\mu\nu})+\epsilon^{ace}\epsilon^{bde}(g_{\lambda\mu}g_{\nu\rho}-g_{\lambda\rho}g_{\mu\nu})+\epsilon^{ade}\epsilon^{cbe}(g_{\lambda\nu}g_{\mu\rho}-g_{\lambda\mu}g_{\rho\nu})]$$
$$+F'[\delta^{ab}\delta^{cd}g_{\lambda\mu}g_{\nu\rho}+\delta^{ac}\delta^{bd}g_{\lambda\nu}g_{\mu\rho}+\delta^{ad}\delta^{bc}g_{\lambda\rho}g_{\mu\nu}]\,. \tag{6.5}$$

Equation (4.7) tells us that

$$F=G^2\ \text{ and }\ F'=0. \tag{6.6}$$

The conditions (6.1), (6.4), and (6.6) allow us to express all primitively divergent vertices in terms of only one constant G.

As we have stated before, the foregoing discussion is of heuristic value only because of the infrared divergences that the Feynman integrals experience when all external momenta are set equal to zero. More precisely, a Feynman integral of the theory becomes divergent if two or more external lines are set on the mass shell. We assert that a Feynman integral suffers no infrared catastrophe if all of the external lines are kept off the mass shell. One can easily see this for diagrams with one loop. As long as all of the external lines are off the mass shell, infrared divergences in any subintegrations can occur only in the measure zero of the space of all integration variables where the rest of the integrand is nonsingular. Therefore, by choosing subtraction points to be somewhere other than where all external lines are on the mass shell, we can circumvent the infrared difficulties in the construction of Green's functions (but not the S matrix) altogether.

A convenient convention for the subtraction points is given by Symanzik.[26] We choose as such the points where the squares of external momenta are all equal to a negative number, say, $-a^2$. Defining all external momenta outwardly from the vertex, we have at the subtraction point $p_i{}^2=-a^2$, $p_i\cdot p_j=(n-1)^{-1}a^2$. As an example, we will work out the renormalization conditions for two- and three-point vertices.

We shall normalize the fields $\vec{A}_\mu$ and $\vec{c}$ so that

$$J(-a^2)=1,$$

and

$$\Sigma_g(-a^2)=0.$$

Then we have

$$\lim_{k^2\to -a^2}\Delta_{\mu\nu}(k)\sim(g_{\mu\nu}-k_\mu k_\nu/k^2)(-a)^{-2}+\text{gauge-dependent terms},$$

$$\mathcal{G}(a^2)=-a^{-2}.$$

At the symmetric point $p^2=q^2=r^2=-a^2$, the three-point vertices $\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)$ and $\gamma^{cab}_{\lambda\mu}(r,p;q)$ have the structures

$$\lim_{p^2=q^2=r^2=-a^2} i\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)=\{G[(p-q)_\nu g_{\lambda\mu}+(q-r)_\lambda g_{\mu\nu}+(r-p)_\mu g_{\nu\lambda}]+H[(r-q)_\lambda r_\mu q_\nu+(p-r)_\mu p_\nu r_\lambda+(q-p)_\nu q_\lambda p_\mu]$$
$$+J[r_\lambda p_\mu q_\nu-q_\lambda r_\mu p_\nu]\}\epsilon^{abc}, \tag{6.7}$$

$$\lim_{p^2=q^2=r^2=-a^2} i\gamma^{cab}_{\lambda\mu}(r,p;q)=\epsilon^{abc}\{G'g_{\lambda\mu}+K_1p_\lambda r_\mu+K_2p_\lambda p_\mu+K_3p_\lambda q_\mu+L_1q_\lambda r_\mu+L_2q_\lambda p_\mu+L_3q_\lambda q_\mu+r_\lambda(\cdots)\}, \tag{6.8}$$

where the omitted terms $(\cdots)$ are of no interest to the present problem. The quantities G and G' require subtractions whereas the other form factors $J,J,K_i,\ldots$ are superficially convergent. Substituting Eqs. (6.7) and (6.8) into Eq. (4.15) and taking the limit $p^2=q^2=r^2=-a^2$, we obtain

$$G-\tfrac{1}{2}a^2(H+J)=G'+\tfrac{1}{2}a^2(L_1+L_2-2L_3-K_1-K_2+2K_3), \tag{6.9}$$

which gives G' in terms of G.

VII. RENORMALIZATION OF MATTER FIELDS

So far our discussion was based on the Lagrangian (2.3) which contains only the Yang-Mills quanta. As an illustration of the renormalization procedure in the presence of matter fields, we consider the case in which a triplet of real scalar fields ϕ^a is added to the Lagrangian by the minimal gauge-invariant coupling.

Let $K^a(x)$ be the sources of the scalar fields ϕ^a. It is not difficult to derive the generalization of Eq. (3.13). It is

$$\frac{i}{\alpha}\partial_\mu\frac{\delta W}{\delta J^a_\mu(x)}-\int d^4y J^c_\lambda(y)D^\lambda_y[i\delta/\delta\vec{J}]^{cb}G^{ba}(y,x;i\delta/\delta\vec{J})W+ig\int d^4yK^c(y)t^{cdb}\frac{\delta}{\delta K^b(y)}G^{da}(y,x;i\delta/\delta\vec{J})W=0. \tag{7.1}$$

In this theory the $A_\mu\phi^2$, $A_\mu{}^2\phi^2$, ϕ^2, and ϕ^4 vertices are primitively divergent. The renormalization of the ϕ^4 vertex has nothing to do with the local gauge invariance and presents no problem. In order to regularize Feynman integrals for these vertices, it becomes necessary to regularize the gauge boson propagators in a gauge-invariant manner, for example, by the use of the Lagrangian (5.7). The renormalization conditions for the ϕ^2, $A_\mu\phi^2$, and $A_\mu{}^2\phi^2$ vertices are obtained from Eq. (7.1). First, though, let us ignore the infrared complications. Later, we will detail the renormalization conditions which take due account of the infrared difficulties.

From Eq. (7.1) we obtain

$$\left(\frac{i}{\alpha}\right)^2\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\frac{\delta^4 iZ[\vec{J}_\mu,\vec{K}]}{\delta J^a_\mu(x)\delta J^b_\nu(y)\delta K^c(z)\delta K^d(w)}\Bigg|_{\vec{J}_\mu=\vec{K}=0}$$
$$=-ig\epsilon^{cfe}W^{-1}\frac{\delta}{\delta K^e(z)}G^{fa}(z,x;i\delta/\delta\vec{J})\frac{\delta}{\delta K^d(w)}\frac{i}{\alpha}\frac{\partial}{\partial y^\nu}\frac{\delta W}{\delta J^b_\nu(y)}\Bigg|_{\vec{J}_\mu=\vec{K}=0}+(c\leftrightarrow d,z\leftrightarrow w). \tag{7.2}$$

If we amputate the scalar propagators and go to the mass shells of two scalar particles $q_1{}^2=q_2{}^2=\mu^2$ in Eq. (7.2), the right-hand side of the equation vanishes. Let us define the primitively divergent vertices:

$$\frac{\delta^2 iZ}{\delta K^a(x)\delta K^b(y)}\bigg|_{\vec{J}_\mu=\vec{K}=0} = -\tilde{\Delta}^{ab}(x-y) = -\delta^{ab}\tilde{\Delta}(x-y),$$

$$\frac{\delta^3 iZ}{\delta J^a_\mu(x)\delta K^b(y)\delta K^c(z)}\bigg|_{\vec{J}_\mu=\vec{K}=0} = -g\int d^4x' d^4y' d^4z' \tilde{\Delta}_{\mu\nu}(x-x')\tilde{\Delta}(y-y')\tilde{\Delta}(z-z')\tilde{C}^{abc}_\nu(x';y,z),$$

$$\frac{\delta^4 iZ}{\delta J^a_\mu(x)\delta J^b_\nu(g)\delta K^c(z)\delta K^d(w)}\bigg|_{\vec{J}_\mu=\vec{K}=0} = ig^2\int d^4x' d^4y' d^4z' d^4w' \tilde{\Delta}_{\mu\rho}(x-x')\tilde{\Delta}_{\nu\sigma}(y-y')\tilde{\Delta}(z-z')\tilde{\Delta}(w-w') \times \tilde{C}^{abcd}_{\rho\sigma}(x',y';z',w'),$$

and

$$\int d^4x d^4y d^4z\, e^{i(px+q_1y+q_2z)}\tilde{C}^{abc}_\nu(x;y,z) = C^{abc}_\nu(p;q_1,q_2)(2\pi)^4\delta^4(p+q_1+q_2),$$

$$C^{abc}_\nu(p;q_1,q_2) = \epsilon^{abc}C_\nu(p;q_1,q_2),$$

and similarly for $C^{abcd}_{\mu\nu}$. Then we have from Eq. (7.2) and the subsequent discussion that

$$\lim_{q_1^2=q_2^2=\mu^2} p_1^\mu p_2^\nu\{C^{abcd}_{\mu\nu}(p_1,p_2;q_1,q_2) + \Gamma^{abf}_{\mu\nu\lambda}(p_1,p_2,-p_1-p_2)\Delta^{\lambda\rho}(q_1+q_2)C^{fcd}_\rho(p_1+p_2;q_1,q_2) - [C^{acf}_\mu(p_1;q_1,p_2+q_2)\Delta(p_2+q_2)C^{bdf}_\nu(p_2;q_2,p_1+q_1) + (c\leftrightarrow d, q_1\leftrightarrow q_2)]\} = 0. \tag{7.3}$$

We shall now consider the limit $p_1, p_2 \to 0$ while $Q = q_1 - q_2$ is kept finite. The low-energy forms of the vertices above are

$$\lim_{q_1^2=q_2^2=\mu^2} C_\mu(p;q_1,q_2) = -iCQ_\mu, \tag{7.4}$$

$$\lim_{p_1,p_2\to 0} \Gamma^{abc}_{\mu\nu\lambda}(p_1,p_2,r) \sim -iG[(p_1-p_2)_\lambda g_{\mu\nu} + (p_2-r)_\mu g_{\nu\lambda} + (r-p_1)_\nu g_{\mu\lambda}], \tag{7.5}$$

where G is defined previously and $r+p_1+p_2=0$;

$$\lim_{\substack{q_1^2=q_2^2=\mu^2\\ p_1,p_2\to 0}} C^{abcd}_{\mu\nu}(p_1,p_2;q_1,q_2) = \delta^{ab}\delta^{cd}(Ag_{\mu\nu}+BQ_\mu Q_\nu) + (\delta^{ab}\delta^{cd}+\delta^{ad}\delta^{cb})(A'g_{\mu\nu}+B'Q_\mu Q_\nu). \tag{7.6}$$

The factors C, A, and A' require subtractions. We may renormalize the ϕ fields so that

$$\lim_{p^2\to\mu^2} \Delta(p^2) \sim (p^2-\mu^2)^{-1}. \tag{7.7}$$

Substituting Eqs. (7.4)–(7.7) in Eq. (7.3) and isolating the part antisymmetric in a and b, we get

$$GC = C^2$$

or

$$G = C. \tag{7.8}$$

Next looking at the part symmetric in a and b, we obtain

$$B = B' = 0$$

and

$$A = -2G^2, \quad A' = G^2. \tag{7.9}$$

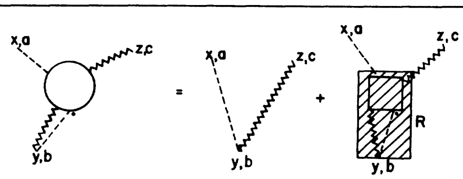


FIG. 2. Diagrammatic representation of Eq. (7.11).

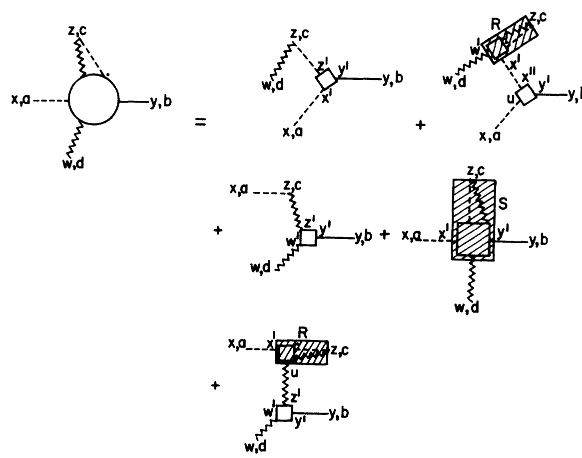


FIG. 3. Diagrammatic representation of Eq. (7.18).

The above treatment is careless, since the vertices $C_\mu, C_{\mu\nu}$ exhibit infrared divergence in the limit q_1^2 or $q_2^2=\mu^2$. We must, therefore, determine the renormalization conditions Δ, C_μ, and $C_{\mu\nu}$ without going to the mass shells of the scalar fields. For this purpose, we will return to Eq. (7.1). From Eq. (7.1) follows the relation

$$\frac{i}{\alpha}\frac{\partial}{\partial x^\mu}\frac{\delta^3 iZ}{\delta J_\mu^a(x)\delta K^b(y)\delta K^c(z)}\bigg|_{\vec{J}_\mu=\vec{K}=0}=-ig\left[W^{-1}\epsilon^{bfe}\frac{\delta}{\delta K^e(y)}\frac{\delta}{\delta K^c(z)}G^{fa}(y,x;i\delta/\delta\vec{J})W+(c\leftrightarrow d,\ y\leftrightarrow z)\right]_{\vec{J}_\mu=\vec{K}=0}. \tag{7.10}$$

In order to discuss this equation, it is necessary to define a new proper vertex R^{abc}. We write

$$\epsilon^{bfe}W^{-1}\frac{\delta^2}{\delta K^e(y)\delta K^c(z)}G^{fa}(y,x;i\delta/\delta\vec{J})W\bigg|_{\vec{J}_\mu=\vec{K}=0}=i\int d^4x'd^4y'\tilde{R}^{bac}(y,x',z')\tilde{\mathcal{G}}(x'-x)\tilde{\Delta}(z'-z)$$
$$+i\tilde{\mathcal{G}}(y-z)\tilde{\Delta}(y-z)\epsilon^{bac}, \tag{7.11}$$

$$R^{abc}(q,p,r)(2\pi)^4\delta(p+q+r)=\int d^4x\,d^4y\,d^4z\,\tilde{R}^{abc}(x,y,z)e^{i(px+qy+rz)},\qquad R^{abc}(p,q,r)=\epsilon^{abc}R(p,q,r)$$

(see Fig. 2). The new vertex does not arise in the perturbative construction of the Green's functions. It is, however, relevant to our discussion of the renormalization of Eq. (7.2). The vertex R^{abc} has the superficial degree of divergence equal to zero, and therefore, requires one subtraction. The value of this vertex at the subtraction point is related to that of C_μ^{abc} through Eq. (7.10). Let us again choose as subtraction point the point where $p^2=q^2=r^2=-a^2$, $p\cdot q=q\cdot r=r\cdot p=\frac{1}{2}a^2$. The quantity $R^{abc}(q_1,p,q_2)$ may be written in the neighborhood of the subtraction point as

$$R^{bac}(q_1,p,q_2)=\epsilon^{bac}[R+(q_1^2+a^2)r_1+(q_2^2+a^2)r_2+(p^2+a^2)r_3+\cdots], \tag{7.12}$$

where R requires a subtraction and r_1, r_2, r_3 are finite.

We now transform Eq. (7.10) into the momentum space. It reads then

$$[p^2\mathcal{G}(p^2)]^{-1}(-ip^\mu)C_\mu(p;q_1,q_2)=\Delta^{-1}(q_1^2)-\Delta^{-1}(q_2^2)+[R(q_1,p,q_2)\Delta^{-1}(q_1^2)-R(q_2,p,q_1)\Delta^{-1}(q_2^2)]. \tag{7.13}$$

Adopting the field normalization conventions

$$[p^2\mathcal{G}(p^2)]\,|_{p^2+a^2=0}=i, \tag{7.14}$$

$$\lim_{q_1^2=-a^2}\Delta^{-1}(q_1^2)\sim q_1^2+a^2-M^2, \tag{7.15}$$

and defining C by

$$\lim_{p^2=q_1^2=q_2^2=-a^2}C_\mu(p;q_1,q_2)\sim -iQ_\mu C,\quad Q_\mu=(q_1-q_2)_\mu \tag{7.16}$$

we have from Eq. (7.13)

$$C=1+R-M^2(r_1-r_2), \tag{7.17}$$

which is the required relation.

Now we turn to Eq. (7.2). We need again to define a new proper vertex. We write (see Fig. 3)

$$\epsilon^{cfe}W^{-1}\frac{\delta}{\delta K_e(z)}G^{fa}(z,x;i\delta/\delta\vec{J})\frac{\delta}{\delta K^d(w)}\frac{\delta W}{\delta J_\nu^b(y)}\bigg|_{\vec{J}_\mu=\vec{K}=0}$$
$$=-ig\epsilon^{cfd}\tilde{\Delta}(z-w)\int d^4z'd^4x'd^4y'\tilde{\mathcal{G}}(z-z')\tilde{\gamma}_\lambda^{fab}(z',x';y')\tilde{\Delta}_\nu^\lambda(y'-y)\tilde{\mathcal{G}}(x'-x)$$
$$-ig\int d^4x'd^4x''d^4y'd^4w'd^4u\tilde{R}^{cfd}(z,x',w')\tilde{\Delta}(w'-w)\tilde{\mathcal{G}}(x'-x'')\tilde{\gamma}_\lambda^{fab}(x'',u;y')\tilde{\Delta}_\nu^\lambda(y'-y)\tilde{\mathcal{G}}(u-x)$$
$$+ig\epsilon^{cae}\tilde{\mathcal{G}}(x-z)\int d^4y'd^4z'd^4w'\tilde{\Delta}(z-z')\tilde{\Delta}(w-w')\tilde{\Delta}_\nu^\mu(y-y')\tilde{C}_\mu^{bed}(y';z',w')$$
$$+ig\int \tilde{R}^{cae}(z,x',u)\tilde{\mathcal{G}}(x'-x)\tilde{\Delta}(u-z')\tilde{\Delta}(w-w')\tilde{\Delta}_\nu^\mu(y-y')\tilde{C}_\mu^{bed}(y';z',w')d^4x'd^4ud^4y'd^4z'd^4w'$$
$$+g\int d^4w'd^4x'd^4y'\tilde{S}_\mu^{abcd}(z,w',x',y')\tilde{\Delta}(w'-w)\tilde{\mathcal{G}}(x'-x)\tilde{\Delta}_\nu^\mu(y'-y), \tag{7.18}$$

which defines the vertex S_μ^{abcd}:

$$S_\mu^{abcd}(q_1,q_2,p_1,p_2)(2\pi)^4\delta^4(q_1+q_2+p_1+p_2)=\int d^4x d^4x' d^4y d^4y' \tilde{S}_\mu^{abcd}(y,y',x,x')e^{i(q_1y+q_2y'+p_1x+p_2x')}\,. \tag{7.19}$$

The fact that S_μ^{abcd} is superficially convergent is of importance in the ensuing discussion.

Now going to the momentum space, we rewrite Eq. (7.2) as

$$\begin{aligned} i[p_1^2\mathcal{G}(p_1^2)]^{-1}p_1^\mu p_2^\nu\{C_{\mu\nu}^{abcd}(p_1,p_2;q_1,q_2)+\cdots\} \\ = p_2^\nu\Delta^{-1}(q_1)\{\epsilon^{abf}\epsilon^{cfd}\mathcal{G}((p_1+p_2)^2)[1+R(q_1,p_1+p_2,q_2)]\gamma_\nu(q_1+q_2,p_1;p_2) \\ +\epsilon^{acf}\epsilon^{bfd}\Delta((p_1+q_1)^2)[1+R(q_1,p_1,p_2+q_2)]C_\nu(p_2,p_1+q_1,q_2)+iS_\nu^{cdab}(q_1,q_2;p_1,p_2)\} \\ +p_2^\nu\Delta^{-1}(q_2)\{c\leftrightarrow d, q_1\leftrightarrow q_2\}, \end{aligned} \tag{7.20}$$

where the expression in the curly bracket on the left-hand side is identical to that on the left-hand side of Eq. (7.3). Equation (7.20) is the generalization of Eq. (7.3) and it allows us to determine the values of the vertices $C_{\mu\nu}^{abcd}$ and C_μ^{abc} at some subtraction points, in terms of that of the vertex $\Gamma_{\mu\nu\lambda}^{abc}$. It is so, because the subtraction constant for γ_ν is known from Eq. (6.9); the subtraction constant for R is known in terms of the value of C_μ at the subtraction point through Eq. (7.17); and S_μ is superficially convergent, so that the right-hand side of the Eq. (7.20) contains the value we seek and no other unknowns.

APPENDIX A: DERIVATIONS OF EQS. (3.5) AND (3.9)

We recall the definition

$$\Delta_L[i\delta/\delta\vec{J}_\mu]=\exp\left[\mathrm{Tr}\ln\left(1-ig\vec{t}\cdot\frac{\delta}{\delta\vec{J}^\mu}\partial^\mu\frac{1}{\partial^2}\right)\right]. \tag{A1}$$

In evaluating

$$\Delta_L[i\delta/\delta\vec{J}_\mu]J_\lambda^a(x)\Delta_L^{-1}[i\delta/\delta\vec{J}_\mu]$$

it is convenient to make the following mapping:

$$\delta/\delta\vec{J}_\mu\to-\vec{\xi}_\mu,\quad \vec{J}_\mu\to\delta/\delta\vec{\xi}_\mu, \tag{A2}$$

which is canonical. We see that

$$\begin{aligned} \exp\left[\mathrm{Tr}\ln\left(1+ig\vec{t}\cdot\vec{\xi}^\mu\partial_\mu\frac{1}{\partial^2}\right)\right]\frac{\delta}{\delta\xi_\lambda^a(x)}\exp\left[-\mathrm{Tr}\ln\left(1+ig\vec{t}\cdot\vec{\xi}^\mu\partial_\mu\frac{1}{\partial^2}\right)\right] \\ =\frac{\delta}{\delta\xi_\lambda^a(x)}-ig\int d^4y d^4z\sum_{b,c}\left\langle y,b\left|\frac{1}{1+ig\vec{t}\cdot\vec{\xi}^\mu\partial_\mu 1/\partial^2}\right|z,c\right\rangle(t^a)^{cb}\delta^4(x-z)\partial_z^\lambda\left\langle z\left|\frac{1}{\partial^2}\right|y\right\rangle \\ =\frac{\delta}{\delta\xi_\lambda^a(x)}-ig\,\mathrm{tr}\,t^a[\partial_x^\lambda H(x,y;-i\vec{\xi})]_{y=x}, \end{aligned} \tag{A3}$$

which is Eq. (3.5).

As for Eq. (3.9), we begin by noting that

$$\begin{aligned} igD_x^\lambda[\vec{A}]^{ab}t^{cbd}[\partial_\lambda H^{dc}(x,y;\vec{A})]_{y=x} \\ =igt^{cad}\frac{\partial^2}{\partial x^\mu\partial x_\mu}H^{dc}(x,y;\vec{A})\bigg|_{y=x}+igt^{cad}\frac{\partial^2}{\partial x^\mu\partial y_\mu}H^{dc}(x,y;\vec{A})\bigg|_{x=y}-ig^2t^{aeb}t^{cbd}\frac{\partial}{\partial x_\mu}H^{dc}(x,y;\vec{A})\bigg|_{x=y}A_\mu^e(x). \end{aligned} \tag{A4}$$

Since

$$\frac{\partial^2}{\partial x_\mu\partial x^\mu}H^{dc}(x,y;\vec{A})=\delta^{dc}\delta^4(x-y)+gt^{def}A_\mu^e(x)\frac{\partial}{\partial x_\mu}H^{fc}(x,y;\vec{A}), \tag{A5}$$

we can write the right-hand side of Eq. (A4) as

$$ig^2(t^{cad}t^{def} - t^{aed}t^{cdf})\partial H^{fc}(x, y; \vec{A})/\partial x_\mu|_{x=y} A^e_\mu(x) + ig\, t^{cad}\partial^2 H^{dc}(x, y; \vec{A})/\partial x^\mu \partial y_\mu|_{y=x}$$
$$= ig \ \mathrm{Tr}\, t^a\left[\frac{\partial^2}{\partial x^\mu \partial y_\mu} H(x, y; \vec{A}) + \frac{\partial}{\partial x^\mu} H(x, y; \vec{A}) g\, \vec{t}\cdot \vec{A}^\mu(x)\right] . \tag{A6}$$

We have used the Jacobi identity

$$t^{cad}t^{def} + t^{aed}t^{dcf} = t^{adf}t^{ecd} .$$

Equation (A6) is precisely the right-hand side of Eq. (3.9).

APPENDIX B: GENERATING FUNCTIONAL OF PROPER VERTICES

In the usual manner, the generating functional $\Gamma[\vec{B}_\mu]$ may be obtained by a Legendre transformation from $Z[\vec{J}_\mu]$.

We define

$$-\vec{B}_\mu(x) = \delta Z[\vec{J}_\mu]/\delta \vec{J}_\mu(x) \tag{B1}$$

and

$$\Gamma[\vec{B}_\mu] = Z[\vec{J}_\mu] + \int d^4x\, \vec{J}^\mu(x)\cdot \vec{B}_\mu(x). \tag{B2}$$

It follows that

$$\vec{J}_\mu(x) = \delta\Gamma[\vec{B}_\mu]/\delta\vec{B}_\mu . \tag{B3}$$

The expansion coefficients of $\Gamma[\vec{B}_\mu]$ in terms of $\vec{B}_\mu$ are the proper vertices. The proof may be found in Jona-Lasinio.[27] It is possible to construct $\Gamma[\vec{B}_\mu]$ perturbatively by the functional integration technique. First consider

$$\exp iZ[\vec{J}_\mu] = \int [d\vec{A}] \exp\left\{iS_\alpha[\vec{A}_\mu] - i\int d^4x\, \vec{A}_\mu(x)\cdot\vec{J}^\mu(x) + \mathrm{Tr}\ln\left(1 - g\vec{A}^\mu\cdot\vec{t}\partial_\mu \frac{1}{\partial^2}\right)\right\},$$

where $S_\alpha[\vec{A}_\mu]$ is the gauge-dependent action

$$S_\alpha[\vec{A}_\mu] = \int d^4x\left[\mathcal{L}(x) - \frac{1}{2\alpha}(\partial^\mu \vec{A}_\mu(x))^2\right] .$$

We can perform the functional integration by the steepest-descent method. Let $\vec{A}^0_\mu$ be defined by

$$\delta S_\alpha[\vec{A}^0_\mu]/\delta\vec{A}^0_\mu(x) = \vec{J}_\mu(x),$$

i.e., $\vec{A}^0_\mu$ is the solution of the classical equation of motion in the presence of the external source, so that

$$\vec{A}^0_\mu = \vec{A}^0_\mu(x, \vec{J}).$$

Now we may write, after DeWitt,[20]

$$\exp iZ[\vec{J}_\mu] = \exp i\left\{S_\alpha[\vec{A}^0_\mu] - \int d^4x\, \vec{A}^0_\mu(x)\cdot \delta S_\alpha[\vec{A}^0_\mu]/\delta\vec{A}^0_\mu(x)\right\}$$
$$\times \int [d\vec{A}]\,\Delta_L[\vec{A}_\mu]\exp i\left\{S_\alpha[\vec{A}_\mu] - S_\alpha[\vec{A}^0_\mu] - \int d^4x\, \frac{\delta S_\alpha[\vec{A}^0_\mu]}{\delta\vec{A}^0_\mu(x)}[\vec{A}_\mu(x) - \vec{A}^0_\mu(x)]\right\} .$$

The tree approximation consists in approximating $Z[\vec{J}_\mu]$ by

$$Z^{\text{tree}}[\vec{J}_\mu] = S_\alpha[\vec{A}^0_\mu] - \int d^4x\, \delta S_\alpha[\vec{A}^0_\mu]/\delta\vec{A}^0_\mu(x)\cdot\vec{A}^0_\mu(x),$$

$$[\vec{B}_\mu(x)]^{\text{tree}} \equiv -\delta Z^{\text{tree}}[\vec{J}_\mu]/\delta\vec{J}_\mu(x) = \vec{A}^0_\mu(x).$$

Therefore, in the tree approximation we have

$$\Gamma^{\text{tree}}[\vec{B}_\mu] = S_\alpha[\vec{B}_\mu] .$$

The one-loop approximation consists in evaluating the functional integral by the steepest descent approximation. We then have

$$Z[\vec{J}_\mu]\simeq Z^{\text{tree}}[\vec{J}_\mu]+\tfrac{1}{2}i\,\text{Tr}\ln\frac{\delta^2 S_\alpha[\vec{A}^0_\mu]}{\delta\vec{A}_\mu\delta\vec{A}_\nu}-i\,\text{Tr}\ln\left(1-g\vec{t}\cdot\vec{A}^0_\mu\partial^\mu\frac{1}{\partial^2}\right)$$

and

$$\Gamma[\vec{B}_\mu]\simeq S_\alpha[\vec{A}^0_\mu]+\int d^4x\,\frac{\delta S_\alpha[\vec{A}^0_\mu]}{\delta\vec{A}^0_\mu(x)}[\vec{B}_\mu(x)-\vec{A}^0_\mu(x)]+\tfrac{1}{2}i\,\text{Tr}\ln\{\delta^2S_\alpha[\vec{B}_\mu]/\delta\vec{B}_\mu\delta\vec{B}_\nu\}-i\,\text{Tr}\ln(1-g\vec{t}\cdot\vec{B}^\mu\partial_\mu/\partial^2)$$
$$=S_\alpha[\vec{B}_\mu]+\tfrac{1}{2}i\,\text{Tr}\ln\{\delta^2S_\alpha[\vec{B}_\mu]/\delta\vec{B}_\mu\delta\vec{B}_\nu\}-i\,\text{Tr}\ln(1-g\vec{t}\cdot\vec{B}_\mu\partial^\mu/\partial^2).$$

APPENDIX C: WARD-TAKAHASHI IDENTITY FOR THE GENERATING FUNCTIONAL OF PROPER VERTICES

We begin by rewriting Eq. (3.8) as

$$D^\lambda_x[i\delta/\delta\vec{J}]^{ab}\left[\frac{i}{\alpha}\partial_\lambda\partial_\mu\frac{\delta}{\delta J^b_\mu(x)}-J^b_\lambda(x)\right]W-ig\int d^4y\;\text{tr}\,t^a[\partial_\mu H(x,y;i\delta/\delta\vec{J})]D^\mu_y[i\delta/\delta\vec{J}]\,\delta^4(x-y)W=0. \tag{C1}$$

The last term on the left-hand side may be written as

$$-igt^{bac}[\partial_\mu H^{cd}(x,y;i\delta/\delta\vec{J})]_{x=y}(-ig)t^{deb}\frac{\delta W}{\delta J^e_\mu(x)}+igt^{bac}[\partial^2H^{cb}(x,y;i\delta/\delta\vec{J})/\partial x_\mu\partial y^\mu]_{x=y}W. \tag{C2}$$

Noting that

$$(\partial^2-g\partial^\nu\vec{A}_\mu\cdot\vec{t})^{ab}_yH^{cb}(x,y;\vec{A})=\delta^{ac}\delta^4(x-y)$$

or

$$H^{cb}(x,y;\vec{A})=-\delta^{cb}\tilde{D}_F(x-y)-gt^{bad}\int d^4z\,\partial^\nu_y\tilde{D}_F(y-z)A^a_\nu(z)H^{cd}(x,y;\vec{A}), \tag{C3}$$

we can cast the second term of Eq. (C2) into

$$igt^{bac}[\partial^2H^{cb}(x,y;i\delta/\delta\vec{J})/\partial x_\mu\partial y^\mu]_{x=y}W=-(ig)^2t^{bac}t^{bed}\int d^4z\,\partial^\mu_y\partial^\nu_y\tilde{D}_F(y-z)\partial_\mu H^{cd}(x,y;\vec{A})\bigg|_{x=y}\frac{\delta W}{\delta J^e_\nu(z)}\;.$$

Thus the last term on the left-hand side of Eq. (C1) can be written as

$$-g^2t^{bac}t^{deb}\int d^4z\,\partial^\mu_xH^{cd}(x,y;i\delta/\delta\vec{J})g^{\text{tr}}_{\mu\nu}(x-z)\frac{\delta W}{\delta J^e_\nu(z)}\bigg|_{x=y},$$

where

$$g^{\text{tr}}_{\mu\nu}(x-y)=g_{\mu\nu}\delta^4(x-y)+\partial_\mu\partial_\nu\tilde{D}_F(x-y)\;.$$

So finally, we obtain

$$\frac{i}{\alpha}\partial^2\partial_\mu\frac{\delta W}{\delta J^a_\mu(x)}+\frac{1}{\alpha}t^{abc}\frac{\delta}{\delta J^b_\lambda(x)}\partial_\lambda\partial_\mu\frac{\delta W}{\delta J^c_\mu(x)}-\partial^\lambda J^a_\lambda(x)W-it^{abc}J^b_\lambda(x)\delta W/\delta J^c_\lambda(x)$$
$$+g^2t^{abc}t^{bde}\int d^4z\,\partial^\mu_xH^{cd}(x,y;i\delta/\delta\vec{J})g^{\text{tr}}_{\mu\nu}(x-z)\frac{\delta W}{\delta J^e_\nu(z)}\bigg|_{x=y}=0\;. \tag{C4}$$

Equation (C4) may be translated into an expression involving the generating functional of the proper vertices. We recall that

$$W[\vec{J}_\mu]=\exp iZ[\vec{J}_\mu] \tag{B1}$$

and

$$-\vec{B}_\mu=\frac{\delta Z[\vec{J}_\mu]}{\delta\vec{J}_\mu(x)};\quad \vec{J}_\mu=\frac{\delta\Gamma[\vec{B}_\mu]}{\delta\vec{B}_\mu}\;. \tag{B3}$$

Thus

$$\frac{1}{\alpha}\partial^2\partial^\mu B_\mu^a(x)-\frac{1}{\alpha}t^{abc}B_\lambda^b(x)\partial^\lambda\partial^\mu B_\mu^c(x)-\partial^\lambda\frac{\delta\Gamma[B_\mu]}{\delta B_\lambda^a(x)}+t^{abc}B_\lambda^b(x)\frac{\delta\Gamma[B_\mu]}{\delta B_\lambda^c(x)}$$
$$=\frac{1}{\alpha}t^{abc}\frac{\delta}{\delta J_\lambda^b(x)}\partial_\lambda\partial^\mu B_\mu^c(x)+ig^2t^{abc}t^{bde}\int d^4z\,\partial_x^\mu H^{cd}(x,y;\vec{B}-i\delta/\delta\vec{J})g_{\mu\nu}^{\mathrm{tr}}(x-z)B^{e\nu}(z)\Big|_{x=y}\equiv -G^a. \tag{C5}$$

This equation is somewhat simplified if we define Γ^0 by

$$\Gamma^0[\vec{B}_\mu]=\Gamma[\vec{B}_\mu]+\frac{1}{2\alpha}\int d^4x[\partial^\mu\vec{B}_\mu(x)]^2, \tag{C6}$$

which satisfies

$$[\partial_\lambda\delta^{ac}-t^{abc}B_\lambda^b]\frac{\delta\Gamma^0[\vec{B}_\mu]}{\delta B_\lambda^c(x)}=G^a(x). \tag{C7}$$

If $G^a(x)$ were identically zero, then Eq. (C7) would imply that $\Gamma^0[\vec{B}_\mu]$ is invariant under the local gauge transformation

$$\vec{B}_\mu\to\vec{B}_\mu+D_\mu[\vec{B}_\mu]\vec{w}.$$

By an explicit computation of $\delta^2G^a(x)/\delta B_\mu^b(y)\delta B_\nu^c(z)$, we have verified that $G^a(x)$ cannot be identically zero, however.

APPENDIX D: POWER COUNTING

Let $N_{n,m}$ be the number of vertices of the form $\partial^n\Phi^m$, i.e., m-point vertices with n derivatives, and let I and E be, respectively, the number of internal and external lines in a proper diagram. By L we denote the number of loops in the diagram.

We have two topological relations:

$$E+2I=\sum_{n,m}mN_{n,m}, \tag{D1}$$

$$L=I+1-\sum_{n,m}N_{n,m}. \tag{D2}$$

The superficial degree of divergence D of the diagram is given by

$$D=\sum_{n,m}nN_{n,m}+4L-2rI, \tag{D3}$$

where the propagator is assumed to have the asymptotic behavior $(p^2)^{-r}$. Eliminating L and I in favor of E by the use of Eqs. (D1) and (D2), we write Eq. (D3) as

$$D=E(n-2)+4+\sum N_{n,m}[n+(2-r)m-4]. \tag{D4}$$

For the gauge-invariant terms discussed in Sec. V, we have generally

$$n+m=2+2r. \tag{D5}$$

We, therefore, have

$$D=E(r-2)+4+(1-r)\sum_{n,m}N_{n,m}(m-2). \tag{D6}$$

The two statements in Sec. V, for which we referred the reader to this Appendix, can be justified on the basis of Eq. (D6).

*On leave of absence from SPT, CEN, Saclay, B.P. 2, 91 Gif-sur-Yvette, France.

[1]J. Schwinger, Ann. Phys. (N.Y.) 2, 407 (1957).
[2]S. L. Glashow, Nucl. Phys. 22, 579 (1961).
[3]A. Salam and J. Ward, Phys. Letters 13, 168 (1964).
[4]S. Weinberg, Phys. Rev. Letters 13, 1264 (1967).
[5]S. Salam, in *Elementary Particle Theory*, edited by N. Svartholm (Almquist and Forlag A. B., Stockholm, 1968).
[6]J. Schechter and Y. Ueda, Phys. Rev. D 2, 736 (1970).
[7]G. 't Hooft, Nucl. Phys. B35, 167 (1971).
[8]B. W. Lee, Phys. Rev. D 5, 823 (1972).
[9]S. Weinberg, Phys. Rev. Letters 27, 1688 (1971).
[10]P. Higgs, Physics 12, 132 (1966).

[11]T. W. B. Kibble, Phys. Rev. 155, 1554 (1967).
[12]Further references can be found in Ref. 8.
[13]B. W. Lee, Nucl. Phys. B9, 649 (1969); J.-L. Gervais and B. W. Lee, *ibid.* B12, 627 (1969); B. W. Lee, Chiral Dynamics (Gordon and Breach, New York, to be published).
[14]N. N. Bogoliubov and O. S. Parasiuk, Acta. Math. 97, 227 (1957).
[15]N. N. Bogoliubov and D. B. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959).
[16]K. Hepp, Commun. Math. Phys. 1, 95 (1965); *Théories de la Renormalisation* (Springer, Berlin, 1969).
[17]W. Zimmermann, in *Brandeis University Summer Institute in Theoretical Physics*, edited by M. Chretien *et al*. (MIT Press, Cambridge, Mass., 1970).
[18]A. Slavnov, Kiev Report No. ITP-71-83E (unpublished).
[19]R. P. Feynman, Acta. Phys. Polon. 26, 697 (1963); (unpublished).
[20]B. DeWitt, Phys. Rev., 162, 1195 (1967); 162, 1239 (1967).
[21]L. D. Faddeev and V. N. Popov, Phys. Letters 25B, 29 (1967); Kiev Report No. ITP-67-36 (unpublished).
[22]G. 't Hooft, Nucl. Phys. B33, 173 (1971).
[23]A. Slavnov, Nucl. Phys. B31, 301 (1971).
[23a]R. Jackiw and L. D. Faddeev (private communication).
[24]A. Slavnov, Kiev Report No. ITP-71-131E (unpublished).
[25]K. Johnson (private communication).
[26]K. Symanzik, Lett. Nuovo Cimento 2, 10 (1969); Commun. Math. Phys. 16, 48 (1970).
[27]G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).

PHYSICAL REVIEW D VOLUME 5, NUMBER 12 15 JUNE 1972

Spontaneously Broken Gauge Symmetries. II. Perturbation Theory and Renormalization

Benjamin W. Lee
National Accelerator Laboratory, P. O. Box 500, Batavia, Illinois 60510
and Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790

and

Jean Zinn-Justin*
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790
(Received 10 March 1972)

The second paper in this series is devoted to the formulation of a renormalizable perturbation theory of Higgs phenomena (spontaneously broken gauge theories). In Sec. II, we reformulate the renormalization prescription for massless Yang-Mills theories in terms of gauge-invariant renormalization counterterms in the action. Section III gives a group-theoretic discussion of Higgs phenomena. We discuss the possibility that an asymmetric vacuum is stable, and show how the symmetry of the physical vacuum determines the mass spectrum of the gauge bosons. We show further that in a special gauge (U gauge), all unphysical fields can be eliminated. Section IV discusses the quantization of a spontaneously broken gauge theory in the R gauge, where, as we show in Sec. V, Green's functions are made finite by the renormalization counterterms of the symmetric theory (in which the gauge invariance is not spontaneously broken). The R-gauge formulation makes use of redundant fields for the sake of renormalizability. Section VI is a discussion of the low-energy limits of propagators in the R-gauge formulation. In Sec. VII we show that the particles associated with redundant fields peculiar to the R-gauge formulation are unphysical, i.e., they do not contribute to the sum over intermediate states.

I. INTRODUCTION

In this paper we give a renormalization method and a proof of finiteness of renormalized Green's functions of spontaneously broken gauge theories. For definiteness we consider a very simple model in which SU(2) gauge bosons are coupled to a triplet of scalar mesons. There is an extra complication when chiral fermions are included in the model, as pointed out by Veltman,[1] and more recently by Gross and Jackiw.[2] This difficulty can be circumvented in a realistic model of electromagnetic and weak interactions. We shall not discuss this problem further in this paper, but postpone the discussion until we deal with the renormalizability of a realistic theory in a sequel to this paper.