[16]S. Weinberg, Phys. Rev. 133, B1318 (1964).
[17]C. L. Hammer and R. H. Tucker, J. Math. Phys. 12, 1327 (1971).
[18]See Ref. 11, Chap. 5, p. 193 or Ref. 13, Chap. 11, p. 323.
[19]See Ref. 11, p. 189.
[20]S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966), Chap. 26.
[21]C. N. Brown *et al.*, Phys. Rev. Letters 26, 991 (1971).

PHYSICAL REVIEW D VOLUME 5, NUMBER 12 15 JUNE 1972

Manifestly Conformal-Covariant Expansion on the Light Cone

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(Received 29 November 1971)

The isomorphism of the conformal algebra on space-time to the orthogonal O(4,2) algebra is exploited to derive in a manifestly covariant way an operator-product expansion on the light cone in terms of irreducible operator representations of the conformal algebra. The expansion provides for a solution of the causality problem for operator expansion on the light cone. Additional properties of the conformally covariant expansion, as well as its relation to the conformally invariant three-point function, are discussed.

I. INTRODUCTION

Wilson has advocated the relevance of scale invariance applied to an operator-product expansion.[1] The possible relevance of the stronger conformal invariance[2] for equal-time commutators[3] and operator-product expansions[4,5] has recently been proposed. In particular, in Ref. 5 the following "improved" light-cone expansion was derived:

$$A(x)B(0) \underset{x^2\to 0}{\sim} \sum_{n=0}^{\infty} c_n^{AB}\left(\frac{1}{x^2}\right)^{(l_A+l_B+n-l_n)/2} x^{\alpha_1}\cdots x^{\alpha_n}\,{}_1F_1(\tfrac{1}{2}(l_A-l_B+l_n+n);\, l_n+n;\, x\cdot\partial)\,O_{\alpha_1\cdots\alpha_n}(0)\,.$$

In the above equation $A(x)$ and $B(x)$ are local scalar (for simplicity) operators of dimensions l_A and l_B (in energy units), both annihilated by K_λ (the generator of special conformal transformations), i.e., satisfying $[K_\lambda, A(0)]=0$, $[K_\lambda, B(0)]=0$; $O_{\alpha_1\cdots\alpha_n}(0)$ (symmetric traceless tensors of dimension l_n) are those operators of the expansion basis which are annihilated by K_λ; c_n^{AB} are unknown constants; the hypergeometric function ${}_1F_1(a;c;z)$ arises from the structure of the conformal algebra.

In this paper we shall

(i) offer a manifestly conformal-covariant derivation of the improved expansion using the isomorphism of the conformal algebra to the orthogonal algebra O(4, 2). In a subsequent paper[6] the proof is extended (in view of later applications) to derive a conformally covariant operator-product expansion valid over the whole space-time;

(ii) present additional discussion on the properties of the improved expansion. Besides offering a solution of the important causality problem in operator expansions, and of translation invariance on a Hermitian basis, as extensively discussed in Ref. 5, the improved expansion is directly related to the conformally covariant expressions for the vacuum expectation value of a product of three local operators and for the vertex function.

II. CONFORMALLY COVARIANT FORMALISM

It is well known[2] that the conformal algebra on space-time is isomorphic to the orthogonal algebra O(4, 2), whose generators constitute an antisymmetric tensor J_{AB} ($A, B = 0, 1, 2, 3, 5, 6$) with

$$J_{\mu\nu} \equiv M_{\mu\nu}, \quad J_{65} = D, \quad J_{5\mu} = \tfrac{1}{2}(P_\mu - K_\mu), \quad J_{6\mu} = \tfrac{1}{2}(P_\mu + K_\mu), \tag{1}$$

where $M_{\mu\nu}$, D, P_μ, and K_μ are the usual conformal generators.

Let us consider the pseudo-Euclidean space in six dimensions with metric tensor $g_{AA} = (+---,-+)$, $g_{AB} = 0$ for $A \neq B$. On the hypercone $\eta^A\eta_A = 0$, $\eta_A \equiv (\eta_\mu, \eta_5, \eta_6)$, one defines spinor functions $\psi_{\{\alpha\}}(\eta)$ which transform as

$$\begin{aligned}\delta\psi_{\{\alpha\}}(\eta) &= -i\epsilon^{AB} J_{AB\{\alpha\}}{}^{\{\beta\}}\psi_{\{\beta\}}(\eta)\\ &= -i\epsilon^{AB}(L_{AB}\delta_{\{\alpha\}}{}^{\{\beta\}} + S_{\{\alpha\}}{}^{\{\beta\}})\psi_{\{\beta\}}(\eta),\end{aligned} \tag{2}$$

where $L_{AB} = i(\eta_A\partial_B - \eta_B\partial_A)$ and $S_{\{\alpha\}\{\beta\}}$ is an irreducible representation of the spinor group SU(2, 2) [which is locally isomorphic to O(4, 2)]. Assuming such functions to be homogeneous, i.e.,

$$\eta^A\partial_A\psi_{\{\alpha\}}(\eta) = \lambda\psi_{\{\alpha\}}(\eta), \tag{3}$$

one can show that the function ($k = \eta_5 + \eta_6$, $\pi_\mu = S_{6\mu} + S_{5\mu}$, $x_\mu = k^{-1}\eta_\mu$)

$$O_{\{\alpha\}}(x) = k^{-\lambda}(e^{-ix\cdot\pi}\psi)_{\{\alpha\}}(\eta) \tag{4}$$

transforms according to a representation of the conformal algebra on space-time, induced from a representation of the stability algebra at $x = 0$ by matrices

$$\Sigma_{\mu\nu} = S_{\mu\nu}, \quad \Delta = S_{65} - i\lambda, \quad K_\mu = S_{6\mu} - S_{5\mu} \tag{5}$$

or, in terms of spinor functions,

$$\begin{aligned}&[O_{\{\alpha\}}(x), P_\mu] = i\partial_\mu O_{\{\alpha\}}(x),\\ &[O_{\{\alpha\}}(x), M_{\mu\nu}] = \{i(x_\mu\partial_\nu - x_\mu\partial_\nu)\delta_{\{\alpha\}}{}^{\{\beta\}} + \Sigma_{\mu\nu\{\alpha\}}{}^{\{\beta\}}\}O_{\{\beta\}}(x),\\ &[O_{\{\alpha\}}(x), D] = (ix_\nu\partial^\nu\delta_{\{\alpha\}}{}^{\{\beta\}} + \Delta_{\{\alpha\}}{}^{\{\beta\}})O_{\{\beta\}}(x),\\ &[O_{\{\alpha\}}(x), K_\mu] = \{i(2x_\mu x_\nu\partial^\nu - x^2\partial_\mu)\delta_{\{\alpha\}}{}^{\{\beta\}} - 2ix^\nu(g_{\mu\nu}\Delta + \Sigma_{\mu\nu})_{\{\alpha\}}{}^{\{\beta\}} + K_{\mu\{\alpha\}}{}^{\{\beta\}}\}O_{\{\beta\}}(x).\end{aligned} \tag{6}$$

We shall be interested in irreducible representations of the conformal algebra which contain infinite towers of Lorentz tensors. It can be shown that such representations are uniquely specified from an irreducible representation with $K_\lambda = 0$ of the stability algebra; i.e., from an irreducible representation of $SL(2, C) \otimes D$, or, equivalently, from a Lorentz irreducible tensor $(\frac{1}{2}n, \frac{1}{2}n)$ of scale dimensions l_n.[7] In the covariant formalism in six dimensions such representations are specified in terms of an irreducible tensor representation of SU(2, 2), $\psi_{A_1\cdots A_n}(\eta)$, i.e., a representation for which n is the largest order of its Lorentz tensors, of degree of homogeneity $\lambda = -l_n$, and verifying the supplementary conditions[8]

$$\eta^{A_1}\psi_{A_1\cdots A_n}(\eta) = \partial^{A_1}\psi_{A_1\cdots A_n}(\eta) = 0. \tag{7}$$

III. MANIFESTLY COVARIANT LIGHT-CONE EXPANSION

The light-cone limit, $(x-x')^2 \to 0$, of the product $A(x)B(x')$ corresponds to the covariant limit of $A(\eta)B(\eta')$ for $\eta^A\eta'_A \to 0$, where $\eta^A\eta'_A = -\frac{1}{2}kk'(x-x')^2$. The most general expansion into irreducible tensor operators is of the form[9]

$$A(\eta)B(\eta') \underset{\eta^A\eta'_A \to 0}{\sim} \sum_{n=0}^{\infty} E_n(\eta^A\eta'_A)D^{(n)A_1\cdots A_n}(\eta, \eta')\psi_{A_1\cdots A_n}(\eta'), \tag{8}$$

where

$$E_n(\eta^A\eta'_A) = c_n(\eta^A\eta'_A)^{(\lambda_A + \lambda_B - \lambda_n - n)/2}, \tag{9}$$

with c_n constant and λ_A, λ_B, λ_n being the degrees of homogeneity of $A(\eta)$, $B(\eta)$, and $\psi_{A_1\cdots A_n}(\eta)$. The general structure of $D^{(n)A_1\cdots A_n}(\eta, \eta')$ is

$$D^{(n)A_1\cdots A_n}(\eta, \eta') = \sum_{m=0}^{n} \eta^{A_1}\cdots\eta^{A_{n-m}}\eta'^{A_{n-m+1}}\cdots\eta'^{A_n}D^{(n,m)}(\eta, \eta')c_{nm}, \tag{10}$$

where c_{nm} are constants and $D^{(n,m)}(\eta, \eta')$ is a differential operator defined on the cones $\eta^2 = \eta'^2 = 0$, finite for $\eta^A\eta'_A = 0$, and homogeneous of degree $h = \frac{1}{2}(\lambda_A - \lambda_B + \lambda_n - n) + m$ in k/k'. The last condition follows from the covariance of the expansion under the group of dilatations $\eta_A \to \rho\eta_A$ on the cone $\eta^2=0$. One can show that $D^{(n,m)}(\eta,\eta')$ is uniquely determined, and for $\eta^A\eta'_A=0$ it reduces to

$$D^{(n,m)}(\eta,\eta')=(\eta^A\partial'_A)^h, \tag{11}$$

where by $(\eta^A\partial'_A)^h$ we mean the application of $\eta^A\partial'_A$ h times, where h is an integer, and its analytic continuation for noninteger h (as specified later). By simple algebraic steps one can recognize that the $n+1$ tensor covariants in Eq. (10) are all proportional and therefore one can consistently take

$$D^{(n)A_1\cdots A_n}(\eta,\eta')=\eta^{A_1}\cdots\eta^{A_n}(\eta^A\partial'_A)^{(\lambda_A-\lambda_B+\lambda_n-n)/2}, \quad \eta^A\eta'_A=0. \tag{12}$$

To define the application of $(\eta^A\partial'_A)^{(\lambda_A-\lambda_B+\lambda_n-n)/2}$ on $\psi_{A_1\cdots A_n}(\eta')$, we introduce the coordinates (x_μ, k) for a vector on the hypercone, and write

$$\eta^A\partial'_A=\frac{k}{k'}\left[k'\frac{\partial}{\partial k'}+(x-x')\cdot\partial'\right] \quad \text{at} \quad (x-x')^2=0. \tag{13}$$

One has the following lemma[9]: For β integer,

$$(\eta^A\partial'_A)^\beta=\left(-\frac{k}{k'}\right)^\beta L^\beta\sum_{J=0}^{\beta}\binom{\beta}{J}L^{-J}(x-x')^{\alpha_1}\cdots(x-x')^{\alpha_J}\partial'_{\alpha_1}\cdots\partial'_{\alpha_J}(-1)^J, \tag{14}$$

where

$$L^J=l_n(l_n+1)\cdots(l_n+J-1)=\Gamma(l_n+J)/\Gamma(l_n) \quad \text{and} \quad \partial'_\mu=\partial/\partial x'^\mu. \tag{15}$$

One can write

$$(\eta^A\partial'_A)^\beta=\left(-\frac{k}{k'}\right)^\beta\frac{\Gamma(l_n+\beta)}{\Gamma(l_n)}\;{}_1F_1(-\beta;l_n;(x-x')\cdot\partial'), \tag{16}$$

where ${}_1F_1(a;c;z)$ is the confluent hypergeometric function and

$[(x-x')\cdot\partial']^J$ stands for $(x-x')^{\alpha_1}\cdots(x-x')^{\alpha_J}\partial'_{\alpha_1}\cdots\partial'_{\alpha_J}$.

Equation (16) defines $(\eta^A\partial'_A)^\beta$ also for β noninteger.

Making use of Eqs. (12) and (16), Eq. (8) can be rewritten as

$$A(x)B(x')\underset{(x-x')^2\to 0}{\sim}\sum_{n=0}^{\infty}\left[\frac{1}{(x-x')^2}\right]^{(l_A+l_B+n-l_n)/2}c_n^{AB}x^{A_1}\cdots x^{A_n}$$
$$\times\,{}_1F_1(\tfrac{1}{2}(l_A-l_B+l_n+n);l_n;(x-x')\cdot\partial')\tilde\psi_{A_1\cdots A_n}(x'), \tag{17}$$

where

$$\tilde\psi_{A_1\cdots A_n}(x')=(k')^{l_n}\psi_{A_1\cdots A_n}(\eta'), \quad x^A\equiv(x_\mu,\tfrac{1}{2}(1+x^2),\tfrac{1}{2}(1-x^2)). \tag{18}$$

We note that in the limit $(x-x')^2\to 0$ one can consistently put

$$\tilde\psi_{\alpha_1\cdots\alpha_J,xxx\cdots}(0)=0, \tag{19}$$

where $xxx\cdots$ stands for indices 5 or 6. In fact such components of $\tilde\psi$ correspond to less dominant contributions, of order $[(x-x')^2]^{n-J}$ with respect to the leading singularity of the particular tensor representation.

Making use of the fundamental integral representation[10]

$${}_1F_1(a;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 du\,u^{a-1}(1-u)^{c-a-1}e^{uz} \tag{20}$$

and of the identity

$$e^{-ix\cdot\pi}\partial_\mu e^{ix\cdot\pi}=\partial_\mu+i\pi_\mu, \tag{21}$$

one has

$$A(x)B(x')\underset{(x-x')^2\to 0}{\sim}\sum_{n=0}^{\infty}\left[\frac{1}{(x-x')^2}\right]^{(l_A+l_B+n-l_n)/2}c_n^{AB}x^{A_1}\cdots x^{A_n}e^{ix'\cdot\pi}$$

$$\times\,{}_1F_1(\tfrac{1}{2}(l_A-l_B+l_n+n);\,l_n;\,(x-x')\cdot(\partial'+i\pi))O_{A_1\cdots A_n}(x') \tag{22}$$

after having inserted Eq. (9) which reads as

$$\tilde{\psi}_{A_1\cdots A_n}(x')=(e^{ix'\cdot\pi}O)_{A_1\cdots A_n}(x'). \tag{23}$$

Taking $x'=0$ and inserting Eq. (20) one finds

$$\begin{aligned}x^{A_1}\cdots x^{A_n}\,{}_1F_1(a;c;x\cdot(\partial+i\pi))O_{A_1\cdots A_n}(0)&=x^{A_1}\cdots x^{A_n}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 du\,u^{a-1}(1-u)^{c-a-1}e^{iux\cdot\pi}O_{A_1\cdots A_n}(ux)\\&=x^{A_1}\cdots x^{A_n}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 du\,u^{a-1}(1-u)^{c-a-1}\tilde{\psi}_{A_1\cdots A_n}(ux).\end{aligned} \tag{24}$$

One can show that[11]

$$x^{A_1}\cdots x^{A_n}\tilde{\psi}_{A_1\cdots A_n}(x')=(x-x')^{\alpha_1}\cdots(x-x')^{\alpha_n}O_{\alpha_1\cdots\alpha_n}(x')$$

at $(x-x')^2=0$. Therefore

$$\begin{aligned}x^{A_1}\cdots x^{A_n}\tilde{\psi}_{A_1\cdots A_n}(ux)&=(x-ux)^{\alpha_1}\cdots(x-ux)^{\alpha_n}O_{\alpha_1\cdots\alpha_n}(ux)\\&=(1-u)^n x^{\alpha_1}\cdots x^{\alpha_n}O_{\alpha_1\cdots\alpha_n}(ux)\end{aligned} \tag{25}$$

and Eq. (22) can finally be written as

$$A(x)B(0)\underset{x^2\to 0}{\sim}\sum_{n=0}^{\infty}\tilde{c}_n^{AB}\left(\frac{1}{x^2}\right)^{(l_A+l_B+n-l_n)/2}x^{\alpha_1}\cdots x^{\alpha_n}\,{}_1F_1(\tfrac{1}{2}(l_A-l_B+l_n+n);\,l_n+n;\,x\cdot\partial)O_{\alpha_1\cdots\alpha_n}(0), \tag{26}$$

where $\tilde{c}_n^{AB}$ are new constants, related to those in Eq. (22). Equation (26) is the improved light-cone expansion exhibited in the Introduction and first derived in Ref. 5 (and its Errata, where an algebraic mistake is corrected).

IV. PROPERTIES OF THE IMPROVED LIGHT-CONE EXPANSION

Our earlier derivation, in Refs. 4 and 5, of the improved light-cone expansion, was directly obtained by commuting K_λ with both sides of the expansion

$$A(x)B(0)\underset{x^2\to 0}{\sim}\sum_{n=0}^{\infty}\left(\frac{1}{x^2}\right)^{(l_A+l_B+n-l_n)/2}\sum_{m=n}^{\infty}c_{n,m}^{AB}x^{\alpha_1}\cdots x^{\alpha_m}O^{n,m}_{\alpha_1\cdots\alpha_m}(0), \tag{27}$$

where

$$O^{n,m}_{\alpha_1\cdots\alpha_m}(0)=\left[\cdots[O^{n}_{\alpha_1\cdots\alpha_n}(0),P_{\alpha_{n+1}}],\cdots P_{\alpha_m}\right](i)^{m-n} \tag{28}$$

are the operators of the expansion basis and K_λ commutes with $O^{n,n}_{\alpha_1\cdots\alpha_n}(0)$. In this way one obtains the recurrence relation

$$C_{n,n+k-1}^{AB}[\tfrac{1}{2}(l_A-l_B+l_n+n)+k-1]=c_{n,n+k}^{AB}k(l_n+n+k-1) \tag{29}$$

giving the solution

$$c_{n,n+k}^{AB}=\frac{\Gamma(\frac{1}{2}(l_A-l_B+l_n+n)+k)\Gamma(l_n+n)}{k!\Gamma(\frac{1}{2}(l_A-l_B+l_n+n))\Gamma(l_n+n+k)}c_{n,n}^{AB}. \tag{30}$$

Equation (30) gives rise to the confluent hypergeometric function

$${}_1F_1(\tfrac{1}{2}(l_A-l_B+l_n+n),\,l_n+n;\,x\cdot\partial)=\sum_{k=0}^{\infty}\frac{c_{n,n+k}^{AB}}{c_{n,n}^{AB}}(x\cdot\partial)^k. \tag{31}$$

Translation invariance on a Hermitian basis (see Ref. 5) follows from the Kummer identity

$${}_1F_1(a;c;z)={}_1F_1(c-a;c;-z)e^z,$$

together with the property $c_n^{AB}=(-1)^n c_n^{BA}$. Kummer's identity is reflected in the properties of the integrand in Eq. (20) under the exchange $u\rightleftarrows(1-u)$. We note that there is a strict formal analogy with the analogous behavior that ensures the validity of crossing in the Veneziano representation.

For $l_A = l_B$ the function ${}_1F_1$ (Ref. 12) reduces essentially to a Bessel function I_ν and one thus obtains[5]

$$A(x)B(0) \underset{x^2\to 0}{\sim} \sum_{n=\text{even}} c_n\left(\frac{1}{x^2}\right)^{(2l_A+n-l_n)/2} x^{\alpha_1}\cdots x^{\alpha_n}(x\cdot\partial)^{(1-l_n-n)/2} I_{(l_n+n-1)/2}(\tfrac{1}{2}x\cdot\partial)O^n_{\alpha_1\cdots\alpha_n}(\tfrac{1}{2}x)\,. \tag{32}$$

We also note that for $\frac{1}{2}(l_A - l_B + n + l_n) = -h$, with h a non-negative integer, ${}_1F_1$ reduces essentially to a Laguerre polynomial and for given n only a finite number of tensors $O^{n,m}$ contribute to the expansion.[13]

An important property, discussed in Ref. 5, is the manifest causality of the contribution to the expansion from each irreducible representation

$$\left(\frac{1}{x^2}\right)^{(l_A+l_B+n-l_n)/2} x^{\alpha_1}\cdots x^{\alpha_n}\int_0^1 du\, u^{(l_A-l_B+l_n+n)/2-1}(1-u)^{(l_B-l_A+l_n+n)/2-1}O_{\alpha_1\cdots\alpha_n}(ux)\,. \tag{33}$$

In fact the vanishing of $[A(x)B(0), C(y)]$ for $y^2<0$, $(x-y)^2<0$, $x^2=0$ is entirely equivalent to the vanishing of $[O_{\alpha_1\cdots\alpha_n}(ux), C(y)]$ for $(ux-y)^2<0$, $y^2<0$, $x^2=0$, provided $0\le u\le 1$.

In Eq. (33) the functions

$$f_n^{AB}(u) = u^{(l_A-l_B+l_n+n)/2-1}(1-u)^{(l_B-l_A+l_n+n)/2-1}$$

are nothing but the Clebsch-Gordan coefficients for the decomposition of the product of two representations into irreducible components on a continuous basis, whereas the coefficients $c^{AB}_{n,m}$ refer to a discrete basis. The relation between the two sets is

$$\int_0^1 du\, f_n^{AB}(u)u^k = k!\,\frac{c^{AB}_{n,n+k}}{c^{AB}_{n,n}}\,\frac{\Gamma(\frac{1}{2}(l_A-l_B+l_n+n))\Gamma(\frac{1}{2}(l_B-l_A+l_n+n))}{\Gamma(l_n+n)}\,. \tag{34}$$

The relation analogous to the recurrence relation (29) in the case of the discrete basis for the Clebsch-Gordan coefficients $f_n^{AB}(u)$ of the continuous basis, is now a differential equation,

$$u(1-u)\frac{d}{du}f_n^{AB}(u) = [\tfrac{1}{2}(l_A-l_B+l_n+n) - 1 + u(2-l_n-n)]f_n^{AB}(u)\,, \tag{35}$$

which can be deduced exactly like Eq. (29). The causality condition is essentially implicit in the support properties of the Clebsch-Gordan coefficients $f_n^{AB}(u)$.

By similar reasoning one can deduce the most singular contribution on the light-cone to the product of two arbitrary tensors, irreducible under the conformal algebra. One obtains, for the expansion of $J^A_{\alpha_1\cdots\alpha_{n_A}}(x)J^B_{\beta_1\cdots\beta_{n_B}}(0)$, the solution

$$f_n^{AB}(u) \sim u^{(l_A-l_B+l_n+n+n_B-n_A)/2-1}(1-u)^{(l_B-l_A+l_n+n+n_A-n_B)/2-1}\,,$$

or equivalently

$$\begin{aligned}J^A_{\alpha_1\cdots\alpha_{n_A}}(x)J^B_{\beta_1\cdots\beta_{n_B}}(0) \underset{x^2\to 0}{\sim} \sum_n \left(\frac{1}{x^2}\right)^{(l_A+l_B+n_A+n_B+n-l_n)/2} c_n^{AB}\\ \times x_{\alpha_1}\cdots x_{\alpha_{n_A}}x_{\beta_1}\cdots x_{\beta_{n_B}}x^{\mu_1}\cdots x^{\mu_n}\,{}_1F_1(\tfrac{1}{2}(l_A-l_B+l_n+n+n_B-n_A); l_n+n; x\cdot\partial)\\ \times O^{(n)}_{\mu_1\cdots\mu_n}(0)\,.\end{aligned} \tag{36}$$

V. RELATION TO VACUUM EXPECTATION VALUES AND TO VERTEX FUNCTIONS

It is well known that in an exactly conformally invariant theory the vacuum expectation value $\langle 0|C(y)A(x)B(z)|0\rangle$ of three conformal scalars of dimensions l_C, l_A, and l_B is uniquely fixed apart from a multiplicative constant[14]:

$$\langle 0|C(y)A(x)B(z)|0\rangle = c_{ABC}\left[\frac{1}{(x-y)^2}\right]^{(l_A+l_C-l_B)/2}\left[\frac{1}{(x-z)^2}\right]^{(l_A+l_B-l_C)/2}\left[\frac{1}{(y-z)^2}\right]^{(l_B+l_C-l_A)/2}. \tag{37}$$

We put $z=0$ in Eq. (37) and take $(x-z)^2\to 0$, obtaining

$$\langle 0|C(y)A(x)B(0)|0\rangle = \underset{x^2\to 0}{\sim} c_{ABC}\left(\frac{1}{x^2}\right)^{(l_A+l_B-l_C)/2}\left(\frac{1}{y^2}\right)^{(l_B+l_C-l_A)/2}\left(\frac{1}{y^2-2xy}\right)^{(l_A+l_C-l_B)/2}, \tag{38}$$

or by using a Feynman parametrization,[15] following Migdal,[16] one finds

$$\langle 0|C(y)A(x)B(0)|0\rangle \underset{x^2\to 0}{\sim} c_{ABC}\left(\frac{1}{x^2}\right)^{(l_A+l_B-l_C)/2}\int_0^1 du\, u^{(l_A-l_B+l_C)/2-1}(1-u)^{(l_B-l_A+l_C)/2-1}\langle 0|C(y)C(\lambda x)|0\rangle\,. \tag{39}$$

Alternatively, one can take the contribution to $A(x)B(0)$ from the term proportional to

$$\left(\frac{1}{x^2}\right)^{(l_A+l_B-l_C)/2}{}_1F_1(\tfrac{1}{2}(l_A-l_B+l_C),\,l_C;\,x\cdot\partial)\,C(0) \tag{40}$$

on the light cone, $x^2=0$, insert it into $\langle 0|C(y)A(x)B(0)|0\rangle$, and make use of the selection rule[17]

$$\langle 0|C(y)O(0)|0\rangle \sim \left(\frac{1}{y^2}\right)^{l_C} \quad \text{for } l_C=l_O$$
$$=0 \qquad \text{for } l_C\neq l_O\,. \tag{41}$$

One then gets exactly Eq. (39).

Finally we note that one can calculate the contribution of a given tensor representation of the conformal algebra, contained in $A(x)B(0)$, to the off-mass-shell vertex function. One has

$$\begin{aligned}V_n(x^2,x\cdot p)&=\langle 0|A(x)B(0)|p\rangle_n\\&=\text{const}\left(\frac{1}{x^2}\right)^{(l_A+l_B+n-l_n)/2}(x\cdot p)^n\,{}_1F_1(\tfrac{1}{2}(l_A-l_B+l_n+n);\,l_n+n;\,-ix\cdot p)\,.\end{aligned} \tag{42}$$

In Eq. (42) $|p\rangle$ is a scalar state (or spin averaged) and the subscript n applies to the given representation.

VI. CONCLUSIONS

We have considered an expansion of a product of two local operators on the light cone into irreducible representations of the conformal algebra. Such an expansion has been derived by different independent methods, and, perhaps most elegantly, by exploiting the isomorphism of the conformal algebra to the orthogonal O(4, 2) algebra. A most remarkable property of the obtained expansion is its explicit causality, which appears here as a property of support of the Clebsch-Gordan coefficients of the conformal algebra on a continuous basis. Each irreducible representation contributes a term which explicitly satisfies the causality requirement. The conformally covariant expansion also directly satisfies the requirements of translation invariance on a Hermitian basis, a requirement which in an ordinary Wilson's type expansion is equivalent to an infinite set of algebraic conditions that the covariant expansion automatically takes into account. The relation of the covariant operator-product expansion to the covariant three-point vacuum expectation value is also discussed in this paper and the two problems are seen to be directly related to one another.

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[1]K. Wilson, Phys. Rev. 179, 1499 (1969); R. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971); Y. Frishman, Phys. Rev. Letters 25, 966 (1970).

[2]See, e.g., G. Mack and A. Salam, Ann. Phys. (N.Y.) 53, 174 (1969); C. G. Callan, Jr., S. Coleman, and R. Jackiw, *ibid.* 59, 42 (1970); B. Zumino, in *Lectures on Elementary Particles and Quantum Field Theory,* 1970 Brandeis Summer Institute in Theoretical Physics, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1971).

[3]S. Ciccariello, R. Gatto, G. Sartori, and M. Tonin, Ann. Phys. (N.Y.) 65, 265 (1971).

[4]S. Ferrara, R. Gatto, and A. F. Grillo, Nucl. Phys. B34, 349 (1971).

[5]S. Ferrara, R. Gatto, and A. F. Grillo, Phys. Letters 36B, 124 (1971); 38B, 188 (1972).

[6]S. Ferrara, R. Gatto, and A. F. Grillo, Lett. Nuovo Cimento 2, 1363 (1971); in Springer Tracts in Modern Physics, edited by G. Höhler (Springer, New York, to be published).

[7]The three independent Casimir $C_{\rm I}$, $C_{\rm II}$, $C_{\rm III}$ operators of O(4,2) are fixed for such representations from the order n of the irreducible tensor and from l_n :

$$C_{\rm I}\equiv J_{AB}J^{AB}=2n(n+2)+2l_n(l_n-1)\,,$$
$$C_{\rm II}\equiv\epsilon_{ABCDEF}J^{AB}J^{CD}J^{EF}=0\,,$$
$$C_{\rm III}\equiv J_A^BJ_B^CJ_C^DJ_D^A=n(n+2)[l_n(l_n-4)+3]\,.$$

[8]The condition $\eta^{A_1}\psi_{A_1\ldots A_n}(\eta)=0$ ensures the property that the Lorentz tensor of highest order contained in the representation is annihilated by K_λ at $x=0$. The condition $\partial^{A_1}\psi_{A_1\ldots A_n}(\eta)=0$ fixes the components $O_{\alpha_1\cdots\alpha_{n-J},xx\cdots}$ (where $x=5$ or 6) of the tensor in Eq. (4) in terms of the divergences $\partial_{\beta_1}\cdots\partial_{\beta_J}O_{\alpha_1\cdots\alpha_{n-J}}{}^{\beta_1\cdots\beta_J}$.

[9]S. Ferrara, A. F. Grillo, and R. Gatto, Ann. Phys. (N.Y.) (to be published).

[10]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, Chap. VI, pp. 19–255.

[11]This result follows from the identity

$$x^{A_1}\cdots x^{A_n}(e^{ix'\cdot\pi})^{\alpha_1\cdots\alpha_n}_{A_1\cdots A_n}=(x-x')^{\alpha_1}\cdots(x-x')^{\alpha_n}\,,$$

expressing the action of π_μ as generator of coordinate translations (Ref. 2).

[12]${}_1F_1(a\,,2a;z)=\Gamma(a+\frac{1}{2})(\frac{1}{4}z)^{1/2-a}e^{z/2}I_{a-1/2}(\frac{1}{2}z)$ [*Higher Transcendental Functions* (Bateman Manuscript Project), Ref. 10, Vol. 1, Chap. VI, pp. 19–265].

[13]${}_1F_1(-h,c;z)=\Gamma(1+h)L_h^{(c-1)}(z)$ [*Higher Transcendental Functions* (Bateman Manuscript Project), Ref. 10, Vol. 1, Chap. VI, pp. 19–268].

[14]Equation (37) can be obtained in a straightforward and simple way by the six-dimensional covariant formalism.

[15]The integral representation in (39) is nothing but a Riemann-Liouville fractional integral [*Tables of Integral Transforms* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. II, pp. 19–186].

[16]A. A. Migdal, Phys. Letters 37B, 98 (1971).

[17]This rule is a special case of the selection rule of the conformal algebra on the light cone, $\langle 0|O_n(x)O_m(0)|0\rangle\neq 0$ if and only if $l_n-l_m=n-m$. This result can be proven by requiring the conformal covariance of the identity representation (c-number) of the conformal algebra, contained in the operator product $O_{\alpha_1\cdots\alpha_n}(x)O_{\beta_1\cdots\beta_m}(0)$ for $x^2=0$.

PHYSICAL REVIEW D VOLUME 5, NUMBER 12 15 JUNE 1972

First and Second Factorization in a Dual Multiparticle Theory with Nonlinear Trajectories*

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(Received 22 February 1972)

The N-point tree graphs of a dual-resonance theory with nonlinear trajectories are factorized. The factorization procedure applies equally well with the external lines on or off the mass shell. The particle spectrum of the theory is found to have a degeneracy of approximately 6^l for the lth level. Since the trajectory is logarithmic, the asymptotic density of levels grows only as some power of the mass. The techniques used in first factorization are applied to second factorization to arrive at an amplitude for processes involving two external particles with nonzero spins. Two sets of linear-dependence relations among the amplitudes obtained by factorization are proven.

I. INTRODUCTION

In this paper and a related one,[1] we derive the following properties for a dual-resonance theory with nonlinear trajectories[2]:

(i) first and second factorization,
(ii) generalized Ward-like identities,
(iii) the twist relation,
(iv) third factorization and the general three-point vertex.

Some of these results have been obtained using an operator formalism.[3] In these papers, direct c-number techniques will be developed, which, aside from giving new understanding of the nature of factorization, allow us to obtain results which have not yet been obtained with the operator method.

From these and previous results,[2] it is clear that the nonlinear theory closely parallels the Veneziano model.[4] Thus, we have found a larger class of theories with many of the interesting and perhaps desirable properties of the Veneziano model. In addition to the intercept and slope parameters $\alpha(0)$ and $\alpha'(0)$, which characterize the Veneziano model, the nonlinear theory has another parameter q (where $0<q<1$). The particles lie on a logarithmic trajectory, and q determines the degree of nonlinearity of the trajectory. The $q\to 1$ limit reduces to the Veneziano model,[2] while the $q\to 0$ limit gives a nondual ϕ^3 theory.[5] From a purely mathematical viewpoint, the nonlinear theory is a natural extension of the Veneziano model because the functions of the theory appear in the mathematical literature[6] as the natural one-param-