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Field Theory for Stable and Unstable Particles*

C. L. Hammer and T. A. Weber
Institute for Atomic Research and Department of Physics, Iowa State University, Ames, Iowa 50010
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A completely covariant field theory is developed which includes both stable and unstable particle fields. Exact single-particle propagators for both the unstable and stable cases are derived for arbitrary spin in terms of matrix elements of the basic interaction. The free-particle approximation to these propagators does not contain the unphysical terms which are usually present in the propagators derived in the interaction picture. The relationship to the Lehmann spectral representation is established and general equations for the various renormalization constants are given. Based upon general considerations it is shown that in the limit of high momentum transfer an extra factor t^{-2} occurs in the cross section for 2-particle-to-2-particle scattering, more in line with experimental observations.

I. INTRODUCTION

In a previous paper,[1] in an investigation of the Lee model, the authors showed that a V-particle state is well defined even though no stable V-particle state exists as an in- or out-state. This state, which can be simply described as

$$|V(p,t)\rangle = e^{-iHt}|p\rangle, \tag{1}$$

where H is the exact Hamiltonian and $|p\rangle$ is the renormalized "mathematical" V-particle state, is shown to be the scattered-wave part of the exact N, θ scattering solution thereby relating the unstable state to the production process and, therefore, to the stable in-states of the model. In the large-time limit, corresponding to an out-state for the stable case, $|V(p,t)\rangle_{t\to\infty}$ approaches the exact V-particle eigenstate of H. In the unstable case, for large mean life Γ and $\Gamma t \ll 1$,

$$\begin{aligned} &\lim_{t\to\infty} |\langle p|V(p,t)\rangle|^2 \sim e^{-\Gamma t} + O(t^{-3/2}), \\ &\lim_{t\to\infty} \int \frac{d\vec{k}}{2\omega} |\langle k, p-k|V(p,t)\rangle|^2 \sim (1-e^{-\Gamma t}) + O(t^{-3/2}), \end{aligned} \tag{2}$$

where $|k, p-k\rangle$ is the N, θ in-state, precisely what one expects for the time dependence for an unstable state and its decay products. Thus, for the Lee model there appears to be no difficulty in extending the usual field-theoretical approach to include a discussion of "asymptotic" states, rather than just in- or out-states, thereby including the possibility for a description of an unstable particle within the framework of the theory.

In this paper the authors extend the analysis used on the Lee model to a general relativistically invariant field theory that includes stable as well as unstable particles. Covariant Heisenberg field operators are defined which create single-particle states with the properties of the V-particle state given in Eqs. (1) and (2). Expressions are derived, in terms of the basic matrix elements of the interaction, for exact propagators for particles of arbitrary spin for both the stable and unstable cases. Since the calculations are carried out in the Heisenberg representation rather than the interaction representation, no unphysical contact terms arise in these expressions. Along the way expressions are also obtained for the various renormalization constants.

A sample calculation of the S matrix for the process 2 particles in and 2 particles out is done as an illustration. In the limit of high momentum transfer, for both stable- and unstable-particle exchange, extra factors of the momentum transfer appear which depress the cross section over the usual Born-approximation results.

The approach presented differs from that of other authors[2,3] who describe a field theory of unstable particles, in that here, rather than the axiomatic approach, all fields are constructed from the primitive, mathematical creation and annihilation operators. The relationship to the axiomatic approach becomes obvious through the construction of in and out operators parallel to Lehmann, Symanzik, and Zimmermann (LSZ).[4] Also, the connection to the Lehmann spectral representation[5] is made apparent by the direct calculation of the vacuum expectation value of the field operators and their relationship to the particle propagators. Consequently, for the stable case, the propagators are shown to agree with those obtained via the usual treatment. In the unstable case, if the lowest-order approximation is made, the propagators agree with the results obtained by Schwinger.[2]

II. BASIC ASSUMPTIONS AND NOTATION

In the following it is assumed that a time-independent 4-vector $P_\mu \equiv (\vec{P}, iH)$ exists in the Heisenberg representation that transforms under the homogeneous Lorentz transformation as

$$U^\dagger(L) P_\mu U(L) = L_{\mu\nu} P_\nu , \tag{3}$$

where $\vec{P}$ and H represent the total linear momentum and Hamiltonian operators of the system, and greek indices run from one to four. Similarly for the homogeneous transformations, the generators $M_{\mu\nu}$ are assumed to transform as

$$U^\dagger(L) M_{\mu\nu} U(L) = L_{\mu\sigma} L_{\nu\tau} M_{\sigma\tau} , \tag{4}$$

where the angular momentum and the boost generators are defined by

$$\begin{aligned} (\vec{J})_i &= \tfrac{1}{2}\epsilon_{ijk} M_{jk} , \\ G_i &= i M_{i4} , \\ M_{44} &= 0 , \end{aligned} \tag{5}$$

where latin indices run from one to three.

It has been shown by Fleming[6] that the exact Poincaré generators, although themselves hyperplane-independent, can be written as the sum of a noninteracting term and an interacting term,

$$\begin{aligned} P_\mu &= P_\mu^0(\eta) + P_\mu'(\eta), \\ M_{\mu\nu} &= M_{\mu\nu}^0(\eta) + M_{\mu\nu}'(\eta), \end{aligned} \tag{6}$$

where the hyperplane parameter η is defined by

$$\begin{aligned} \eta_\mu &= L_{\mu\nu}\eta_\nu^0 , \\ \eta_\mu^0 &\equiv (\vec{0}, i). \end{aligned} \tag{7}$$

The hyperplane determined by η^0 is called the instantaneous hyperplane and the noninteracting generators transform as

$$\begin{aligned} U^\dagger(L) P_\mu^0(\eta) U(L) &= L_{\mu\nu} P_\nu^0(L^{-1}\eta), \\ U^\dagger(L) M_{\mu\nu}^0(\eta) U(L) &= L_{\mu\sigma} L_{\nu\tau} M_{\sigma\tau}^0(L^{-1}\eta), \\ U^\dagger(L) N^0(\eta) U(L) &= N^0(L^{-1}\eta), \end{aligned} \tag{8}$$

where $N^0(\eta)$ is the noninteracting number of particles operator.

It was first shown by Dirac[7] and later by Fleming[6] that the dynamics are contained only in P_4 and M_{i4}. Consequently the Poincaré generators can be chosen[6] in a particular hyperplane in such a way that

$$\begin{aligned} \vec{P} &= \vec{P}^0(\eta^0), \\ \vec{J} &= \vec{J}^0(\eta^0), \\ H &= H^0(\eta^0) + V(\eta^0), \\ G &= G^0(\eta^0) + U(\eta^0), \end{aligned} \tag{9}$$

where here the instantaneous hyperplane has been singled out.

Since in addition to the usual commutation rules for the free generators,

$$[\vec{P}^0(\eta^0), V(\eta^0)] = [\vec{J}^0(\eta^0), V(\eta^0)] = 0, \tag{10}$$

it is possible to construct simultaneous eigenstates of $\vec{P}$, $\vec{S}$, $H^0(\eta)$, and $N^0(\eta^0)$ in the instantaneous hyperplane, where $\vec{S}$ is a polarization operator that describes the intrinsic angular momentum of the state. Let these states be given by

$$\begin{aligned} \vec{P}|npk\hat{e}\eta^0\rangle &= \vec{p}|npk\hat{e}\eta^0\rangle, \\ \vec{S}\cdot\hat{e}|npk\hat{e}\eta^0\rangle &= k|npk\hat{e}\eta^0\rangle, \\ H^0(\eta^0)|npk\hat{e}\eta^0\rangle &= E_p|npk\hat{e}\eta^0\rangle, \\ N^0(\eta^0)|npk\hat{e}\eta^0\rangle &= n|npk\hat{e}\eta^0\rangle, \end{aligned} \tag{11}$$

where n indicates the number of particles in the state, $p_\mu = (\vec{p}, iE_p)$, $E_p = (\vec{p}^{\,2} + M^2)^{1/2}$, where M is the rest energy of the system of n particles.

Considerable mathematical simplicity is obtained if the instantaneous hyperplane is chosen as the frame where $\vec{p} = 0$. It then follows from Eq. (7) that

$$\eta_\mu = -(p_\mu / M), \tag{12}$$

so that the hyperplane is completely specified by the velocity $\vec{\beta} = (\vec{p}/E_p)$. To indicate this special choice for the instantaneous hyperplane, the free-particle generators will be written as a function of the 4-vector velocity $\beta_\mu \equiv (\vec{\beta}, i)$ rather than the hyperplane parameter η_μ. Thus we can write, for example, $P_\mu^0(\eta) \equiv P_\mu^0(\beta)$, whereas for the instantaneous hyperplane $P_\mu^0(\eta^0) \equiv P_\mu^0(\beta^0)$, $\beta^0 = (\vec{0}, i)$. In terms of this notation

$$\begin{aligned} U^\dagger(L) P_\mu^0(\beta) U(L) &= L_{\mu\nu} P_\nu^0(L^{-1}\beta), \\ U^\dagger(L) M_{\mu\nu}^0(\beta) U(L) &= L_{\mu\sigma} L_{\nu\tau} M_{\sigma\tau}^0(L^{-1}\beta), \end{aligned} \tag{13}$$

$$U^\dagger(L)N^0(\beta)U(L)=N^0(L^{-1}\beta),$$

and

$$\begin{aligned}
&\vec{P}|nMk\hat{e}\rangle=\vec{P}^0(\beta^0)|nMk\hat{e}\rangle=0,\\
&\vec{S}\cdot\hat{\vec{e}}|nMk\hat{e}\rangle=\vec{S}^0(\beta^0)\cdot\hat{\vec{e}}|nMk\hat{e}\rangle=k|nMk\hat{e}\rangle,\\
&H^0(\beta^0)|nMk\hat{e}\rangle=M|nMk\hat{e}\rangle,\\
&N^0(\beta^0)|nMk\hat{e}\rangle=n|nMk\hat{e}\rangle.
\end{aligned}\tag{14}$$

For the single-particle states, $n=1$, $\vec{S}$ is the intrinsic spin operator, and $M=m$, the mass of the particle. In the analysis presented here, m is chosen to be the observed mass for the stable particles or the center of the observed resonance spectrum for the states that represent "undressed" unstable particles.

Because of the transformation rule expressed by Eq. (13), eigenstates of the generators $P_\mu^0(\beta)$ and $N^0(\beta)$ can be obtained from Eq. (14). This transformation gives

$$|npk\hat{e}\rangle=U(L(-\vec{\beta}))|nMk\hat{e}\rangle,\tag{15a}$$

$$\begin{aligned}
&P_\mu^0(\beta)|npk\hat{e}\rangle=p_\mu|npk\hat{e}\rangle,\\
&N^0(\beta)|npk\hat{e}\rangle=n|npk\hat{e}\rangle.
\end{aligned}\tag{15b}$$

However these states are no longer eigenstates of $\vec{P}$ and $\vec{S}$ since the equalities expressed by Eq. (14) apply only in the instantaneous hyperplane. They remain eigenstates of the exact space and intrinsic angular displacement operators [6,8]

$$\begin{aligned}
&\vec{P}_\mu=L_{\mu i}(-\vec{\beta})U(L(-\vec{\beta}))P_iU^\dagger(L(-\vec{\beta})),\\
&S_i=U(L(-\vec{\beta}))(\vec{S})_iU^\dagger(L(-\vec{\beta})).
\end{aligned}\tag{16}$$

Specifically,

$$\vec{P}_\mu=P_\mu+(p_\mu/M^2)(pP),\tag{17}$$

which, combined with Eqs. (14) and (15), gives

$$\begin{aligned}
&P_\mu(pP)^{-1}|npk\hat{e}\rangle=-(p_\mu/M^2)|npk\hat{e}\rangle,\\
&\sum_i S_i\hat{e}_i|npk\hat{e}\rangle=k|npk\hat{e}\rangle.
\end{aligned}\tag{18}$$

The space and time parts of the first of these equations can be combined to give

$$(\vec{P}/H)|npk\hat{e}\rangle=\vec{\beta}|npk\hat{e}\rangle.\tag{19}$$

Thus these states are eigenstates of the exact velocity operator. This operator is Hermitian, so states of different velocities can be chosen orthogonal. For states of given n with the same rest energy M, this orthogonality implies

$$\langle npk\hat{e}|nql\hat{e}\rangle=\delta_{k,l}\rho^{-1}(E_p)\delta(\vec{p}-\vec{q}),$$

where $\rho(E_p)$ is $(2E_p)^{-1}$ for bosons and $(M/E_p)^{-1}$ for fermions. However states of different n which have the same velocity are not necessarily orthogonal by this argument. That they are orthogonal is shown from the fact that they are eigenstates of the Hermitian operator $N^0(\beta)$ with different eigenvalues. Consequently, it follows that

$$\langle n'pk\hat{e}|nql\hat{e}\rangle=\delta_{kl}\delta_{n',n}\rho^{-1}(E_p)\delta(\vec{p}-\vec{q}).\tag{20}$$

It is assumed that these states form a complete set, that is, we are interested in discussing those physical systems which can be described by these basis vectors. They are not, however, the most general set of vectors since each state $|npk\hat{e}\rangle$ is only one member of the general class of states $|npk\hat{e}\eta\rangle$.

The general homogeneous transformation of these states can be implied from Eq. (15a) which can be written as

$$U(L)|npk\hat{e}\rangle=U(L)U(L(-\beta))|nMk\hat{e}\rangle.\tag{21}$$

This can be used to derive the result

$$U(L)|npk\hat{e}\rangle=\sum_{k'}|n(Lp)k'\hat{e}\rangle D^s_{k'k}(\vec{\theta}),\tag{22}$$

where $D^s_{k'k}(\vec{\theta})$ is the Wigner rotation. It should be emphasized that although this result is familiar looking there are important differences between these states and the usual "free" particle states. Firstly, these states are eigenstates of $\vec{P}$ and $\vec{S}$ only in the center-of-momentum system and secondly, the generators of the Lorentz transformation are the exact generators, not the free-particle generators.

III. CREATION AND ANNIHILATION OPERATORS

The free-particle states can be constructed from the vacuum by the successive application of creation operators $a_k^\dagger(p)$ that obey the commutation rules for bosons

$$\begin{aligned}
&[a_k(p),a_l(q)]=0,\\
&[a_k(p),a_l^\dagger(q)]=2E_p\delta_{kl}\delta(\vec{p}-\vec{q}),
\end{aligned}\tag{23}$$

or the anticommutation rules for fermions

$$\begin{aligned}
&\{a_k(p),a_l^\dagger(q)\}=(E_p/m)\delta_{kl}\delta(\vec{p}-\vec{q}),\\
&\{a_k(p),a_l(q)\}=0.
\end{aligned}\tag{24}$$

Similar rules would also apply for antiparticle operators $b_k(p)$. The single-particle specialization of Eq. (22) can be used to show that these operators transform under the homogeneous transformations like

$$U^\dagger(L)a_k(p)U(L)=\sum_{k'}D^s_{kk'}(\vec{\theta})a_{k'}(L^{-1}p)\tag{25}$$

for either particle or antiparticle operators. Again, although familiar looking, these operators do not, in general, create eigenstates of $\vec{P}$ and $\vec{S}$.

These operators can be used to define Heisenberg field operators $\psi(x)$ that are covariant with respect to the homogeneous transformations. These definitions of $\psi(x)$ and the adjoint $\bar{\psi}(x)$ are

$$\psi(x)=\psi_+(x)+\psi_-(x);\ \bar{\psi}(x)=\bar{\psi}_+(x)+\bar{\psi}_-(x),$$

$$\psi_+(x)=(2\pi)^{-3/2}\sum_k\int d\vec{p}\,\rho(E_p)u_k(p)[e^{-iPx}a_k(p)e^{iPx}],$$

$$\psi_-(x)=(2\pi)^{-3/2}\sum_k\int d\vec{p}\,\rho(E_p)v_k(p)[e^{-iPx}b_k^\dagger(p)e^{iPx}],\tag{26}$$

$$\bar{\psi}_+(x)=(2\pi)^{-3/2}\sum_k\int d\vec{p}\,\rho(E_p)\bar{v}_k(p)[e^{-iPx}b_k(p)e^{iPx}],$$

$$\bar{\psi}_-(x)=(2\pi)^{-3/2}\sum_k\int d\vec{p}\,\rho(E_p)\bar{u}_k(p)[e^{-iPx}a_k^\dagger(p)e^{iPx}],$$

where $u_k(p)$ and $v_k(p)$ are any spinors which satisfy the free-particle c-number equations [9,10] for arbitrary mass and spin for particles and antiparticles, respectively. If relativistic normalization is used

$$\bar{u}_k u_l=\bar{v}_k v_l=\delta_{k,l},\ \bar{v}_k u_l=0,$$
$$\bar{u}_k=u_k^\dagger\gamma_4,\ \bar{v}_k=v^\dagger\gamma_4,\tag{27}$$

where γ_4 is the $2(2s+1)$ generalization of the Dirac γ_4 matrix.[8]

It is well known [8-10] that a generalized Foldy-Wouthuysen transformation can be used to generate such spinors from rest-system spinors (corresponding to spinors in the hyperplane β^0), defined by

$$\gamma_4 u_{Rk}=u_{Rk}\ ,\ \ \gamma_4 v_{Rk}=-v_{Rk}\ ,$$
$$\vec{s}\cdot\hat{\vec{e}}u_{Rk}=ku_{Rk}\ ,\ \ \vec{s}\cdot\hat{\vec{e}}v_{Rk}=-kv_{Rk}\ .\tag{28}$$

Here $\vec{s}$ are the $2(2s+1)$ spin matrices. These spinors have the transformation rule [9,10]

$$S(L)u_k(L^{-1}p)=\sum_{k'}u_{k'}(p)D^s_{k'k}(\vec{\theta}),\tag{29}$$

for both u_k and v_k, where $S(L)$ is the homogeneous Lorentz transformation on the spinor indices.

The transformation properties of $\psi(x)$ are established by Eqs. (3), (25), and (29) to be

$$U^\dagger(L)\psi(x)U(L)=\psi'(x)$$
$$=S(L)\psi(L^{-1}x),\tag{30a}$$

which leads to the appropriate infinitesimal transformation

$$[\psi(x),M_{\mu\nu}]=\mathfrak{M}_{\mu\nu}\psi(x),\tag{30b}$$

where $\mathfrak{M}_{\mu\nu}$ are the c-number generators of the homogeneous transformations. Furthermore Eq. (26) can be used to show directly that

$$[\psi(x),\ P_\mu]=-i\frac{\partial}{\partial x_\mu}\psi(x).\tag{31}$$

Thus $\psi(x)$ is a Heisenberg field which transforms covariantly under Poincaré transformations.

In the following, it will be convenient to define operators $A_k(p,t)$,

$$A_k(p,t)=(e^{iHt})a_k(p)(e^{-iHt})\tag{32}$$

which also satisfy Eq. (24). Equivalently, the Cauchy integral representation,

$$e^{-iHt}=(-2\pi i)^{-1}\int_{\mathfrak{C}}dE\,\frac{e^{-iEt}}{E-H}\ ,\tag{33a}$$

can be used in Eq. (32), where $\mathfrak{C}$ is a contour in the negative direction which encircles all the singularities on the physical sheet of the exact Green's function $(E-H)^{-1}$.

If $\mathfrak{C}$ encircles only a subset of the singularities of $(E-H)^{-1}$, then

$$e^{-iHt}\sum_i|\alpha_i\rangle\langle\alpha_i|=(-2\pi i)^{-1}\int_{\mathfrak{C}}dE\,\frac{e^{-iEt}}{E-H}\,,\tag{33b}$$

where $|\alpha_i\rangle$ is the corresponding subset of exact eigenstates of H. Since the number and types of particles are unaffected by the Poincaré transformation, subsets of exact states that represent particular numbers of particles of a given type will transform into themselves,

$$U(L)\Big(\sum_i|\alpha_i\rangle\langle\alpha_i|\Big)U^\dagger(L)=\sum_i|\alpha_i\rangle\langle\alpha_i|\,.$$

Therefore, in the definitions for $\psi(x)$, e^{-iHt} can be replaced everywhere by its integral representation with $\mathfrak{C}$ encircling either all or a particular subset of singularities, as indicated above, without altering the already established transformation properties of the theory.

IV. SINGLE-PARTICLE CONFIGURATION STATES

In a theory for stable particles it is reasonable to expect that the exact single-particle states can be chosen equal to the free-particle states except perhaps for a normalization constant. This choice cannot be made for unstable particles since an unstable state is not an eigenstate of H. The natural generalization is to require the configuration representation of single-particle states to be solutions of the Klein-Gordon equation since all particles are observed to satisfy the Klein-Gordon dispersion relationship.

It is easy to show that the single-particle configuration states satisfy this requirement for this theory. The configuration fields can be used to define single-particle position states $|x\epsilon\rangle$ where ϵ is positive for particle and negative for antiparticle. For example,

$$\langle\psi_0|\psi(x)=\langle x+|,$$

where $|\psi_0\rangle$ is the vacuum state. This state satisfies the differential equation

$$\langle x+|P_\mu = \langle\psi_0|[\psi(x), P_\mu]$$
$$= -i(\partial/\partial x_\mu)\langle x+|,$$

as can be seen from Eq. (31) and consequently satisfies the exact Schrödinger equation

$$\langle x+|H = i(\partial/\partial t)\langle x+|.$$

Therefore given any Heisenberg state $|\psi_H\rangle$, the single-particle configuration representation for that state satisfies the Schrödinger equation

$$\langle x+|H|\psi_H\rangle = i\frac{\partial}{\partial t}\langle x+|\psi_H\rangle \tag{34}$$

in the configuration representation. It also follows that

$$\langle x+|P^2|\psi_H\rangle = -\frac{\partial^2}{\partial x_\mu \partial x_\mu}\langle x+|\psi_H\rangle.$$

Thus if $|\psi_H\rangle$ is chosen to be an eigenstate of P_μ,

$$P^2|\psi_M\rangle = -M^2|\psi_M\rangle,$$

then the single-particle configuration representation of that exact eigenstate satisfies the free-particle Klein-Gordon equation

$$\frac{\partial^2}{\partial x_\mu \partial x_\mu}\langle x+|\psi_M\rangle - M^2\langle x+|\psi_M\rangle = 0. \tag{35}$$

For stable particles, it will be shown that the single-particle in- or out-state is equivalent to an eigenstate of H. Therefore

$$\frac{\partial^2}{\partial x_\mu \partial x_\mu}\langle x+|1, \text{in}\rangle - m^2\langle x+|1, \text{in}\rangle = 0,$$

$$\frac{\partial^2}{\partial x_\mu \partial x_\mu}\langle x+|1, \text{out}\rangle - m^2\langle x+|1, \text{out}\rangle = 0,$$

as it should. For the unstable states these are no single-particle in- or out-states. However there will be exact continuum states of H, which in the asymptotic limit evolve into the decay products of the unstable states. If this state is chosen as $|\psi_M\rangle$, then M^2 will have a continuum of values corresponding to the unstable particle spectrum. Consequently in this theory both stable and unstable particles will have the appropriate Klein-Gordon dispersion.

V. RENORMALIZATION

The renormalization process follows directly parallel to that previously described for the Lee model. In that paper the renormalization constants are obtained from matrix elements of the exact Green's function using methods described by Goldberger and Watson [11] and Mower.[12] These matrix elements arise most simply in a calculation of the momentum representation of the single-particle configuration states.

Consider the matrix element

$$\langle x+|1pk\hat{e}\rangle = \langle\psi_0|\psi_+(x)|1pk\hat{e}\rangle$$
$$= (2\pi)^{-3/2}\sum_l \int d\vec{q}\,\rho(E_q)u_l(q)$$
$$\times\langle 1ql\hat{e}|e^{iPx}|1pk\hat{e}\rangle.$$

Since (Px) commutes with the operators S_i and $P_\mu(pP)^{-1}$ the integration and spin sums can be carried out giving

$$\langle x+|1pk\hat{e}\rangle = (2\pi)^{-3/2}u_k(p)\langle 1pk\hat{e}\|e^{iPx}\|1pk\hat{e}\rangle,$$

where the reduced matrix element is defined by

$$\langle 1ql\hat{e}|e^{iPx}|1pk\hat{e}\rangle = \rho^{-1}(E_p)\delta_{kl}\delta(\vec{p}-\vec{q})$$
$$\times\langle 1pk\hat{e}\|e^{iPx}\|1pk\hat{e}\rangle. \tag{36}$$

The reduced matrix element can be evaluated most easily in the rest system since there the single-particle states are also eigenstates of $\vec{P}$. Equations (3) and (15) can be used to show

$$\langle 1pk\hat{e}\|e^{iPx}\|1pk\hat{e}\rangle = \langle 1mk\hat{e}\|e^{-iH\tau}\|1mk\hat{e}\rangle,$$

where the proper time is

$$\tau = -(px)/m. \tag{37}$$

The Cauchy integral representation shown in Eq. (33) is useful in relating these matrix elements to those of the exact Green's function. Evaluated in this manner,

$$\langle 1pk\hat{e}\|e^{iPx}\|1pk\hat{e}\rangle$$
$$= (-2\pi i)^{-1}\int_{\mathfrak{C}} dm'\langle 1mk\hat{e}\|(m'-H)^{-1}\|1mk\hat{e}\rangle e^{-im'\tau}$$
$$= (-2\pi i)^{-1}\int_{\mathfrak{C}} \frac{dm'\,e^{-im'\tau}}{m'-m-R(m')}, \tag{38}$$

where

$$R(m') = \langle 1mk\hat{e}\|R\|1mk\hat{e}\rangle,$$
$$R = V + V\Lambda(m'-\Lambda H\Lambda)^{-1}\Lambda V. \tag{39}$$

Here Λ is a projection operator which can be conveniently chosen [11,12] to eliminate the single-particle states, i.e.,

$$\Lambda|1mk\hat{e}\rangle = 0,$$

and $\mathfrak{C}$ encircles all the singularities of $[m'-m-R(m')]^{-1}$. In general these singularities will be poles on the real m' axis corresponding to stable-particle states or poles in the unphysical sheet corresponding to unstable-particle states, and branch lines corresponding to the multiparticle states. If $|1mk\hat{e}\rangle$ is to represent a single stable particle, the interaction V will be such that there will be just one simple pole and the branch lines which correspond to those multiparticle states which involve the particle, as shown in Fig. 1(a).

From this point on, the analysis follows precisely parallel to the analysis used to study the Lee model.[1] The position of the pole in the m' plane is then determined from

$$m'-m-R(m')=0. \tag{40a}$$

If m is to correspond to the physical mass, the pole is at $m'=m$, which means

$$R(m)=0. \tag{40b}$$

Goldberger[11] defines the dispersion relation

$$\begin{aligned}R(m')&=\langle 1mk\hat{e}\|V\|1mk\hat{e}\rangle-\pi^{-1}\int_{\mathfrak{C}'}\frac{dm''I(m'')}{m''-m'}\\&=D(m')-iI(m'),\end{aligned} \tag{41a}$$

where $\mathfrak{C}'$ extends from threshold to infinity below the pole at $m''=m'$. It then follows from Eq. (40b) that

$$\langle 1mk\hat{e}\|V\|1mk\hat{e}\rangle=\pi^{-1}\int_{\mathfrak{C}'}dm''\frac{I(m'')}{m''-m}. \tag{41b}$$

If V contains the mass counter terms, then this equation implies

$$\begin{aligned}\delta m&=m_0-m\\&=\langle 1mk\hat{e}\|V'\|1mk\hat{e}\rangle-\pi^{-1}\int_{\mathfrak{C}'}\frac{dm''I(m'')}{m''-m},\end{aligned} \tag{42}$$

where V' is the basic interaction. This can be written in terms of matrix elements by using Goldberger and Watson's result[11]

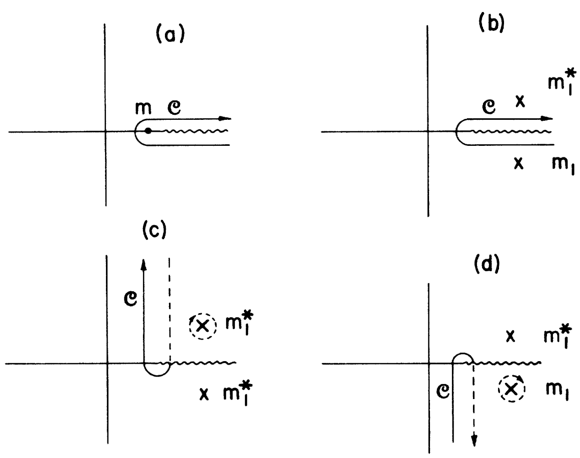

FIG. 1. The m' plane showing the singularities corresponding to the stable- and unstable-particle states. The poles shown at m_1 are in the unphysical sheet.

$$I(m')=\pi\langle 1mk\hat{e}\|R\Lambda\delta(m'-H_0)\Lambda R\|1mk\hat{e}\rangle. \tag{43}$$

This result is a general equation for mass renormalization, independent of the specific theory. In most theories the reduced matrix element of V' is identically zero.

A general result for wave function renormalization can also be derived. The quantity $R(m')$ can be reexpressed as

$$R(m')=\pi^{-1}(m-m')\int_{\mathfrak{C}'}dm''\frac{I(m'')}{(m''-m)(m''-m')} \tag{44}$$

by using the result given in Eq. (41b). Here $\mathfrak{C}'$ must go from threshold to infinity below both poles in the integrand. The denominator of Eq. (38) now can be written as

$$\begin{aligned}m'-m-R(m')&=(m'-m)Z^{-1}(m',m),\\Z^{-1}(m',m)&=1+\pi^{-1}\int_{\mathfrak{C}'}dm''\frac{I(m'')}{(m''-m')(m''-m)},\end{aligned} \tag{45}$$

thereby showing explicitly the pole and cut contributions. Thus the configuration representation for the single-particle state becomes

$$\begin{aligned}\langle x+|1pk\hat{e}\rangle&=(2\pi)^{-3/2}u_k(p)\\&\quad\times(-2\pi i)^{-1}\int_{\mathfrak{C}}dm'\frac{Z(m',m)e^{-im'\tau}}{m'-m}.\end{aligned} \tag{46}$$

In the asymptotic limit for large $|\tau|$ (the in- and out-state limit) the cut contribution is negligible, only the pole contribution survives.[1] Therefore

$$\begin{aligned}\langle x+|1pk\hat{e}\rangle&=(2\pi)^{-3/2}u_k(p)Z(m,m)e^{-im\tau}+O(\tau^{-\gamma})\\&=(2\pi)^{-3/2}u_k(p)Z(m,m)e^{ipx}+O(\tau^{-\gamma}),\\&\qquad\gamma>1.\end{aligned} \tag{47}$$

The renormalization constant $Z(m,m)$ which occurs here is usually[13] absorbed into the field operators by the redefinition

$$\begin{aligned}a_{Dk}(p)&=Z^{-1/2}(m,m)a_k(p),\\\psi_D(x)&=Z^{-1/2}(m,m)\psi(x),\end{aligned} \tag{48}$$

from which follows, since $Z(m,m)$ is real for m below threshold,

$$[a_{Dk}(p),a^\dagger_{Dl}(q)]=Z^{-1}(m,m)\rho^{-1}(E_p)\delta_{kl}\delta(\vec{p}-\vec{q}). \tag{49}$$

This procedure "dresses" both states $|x+\rangle$ and $|1pk\hat{e}\rangle$ so that

$${}_D\langle x+|1pk\hat{e}\rangle_D=(2\pi)^{-3/2}u_k(p)e^{ipx}+O(\tau^{-\gamma}). \tag{50}$$

A parallel treatment can be made for the unstable particle. If the poles in the unphysical sheet are at the complex value of m_1 and m_1^* as shown in Fig. 1(b), then the denominator of Eq.

(36) vanishes when

$$m_1 - m - R(m_1) = 0. \tag{51}$$

Since m is to correspond to the center of the resonance spectrum,

$$m_1 = m - i(\tfrac{1}{2}\Gamma), \tag{52}$$

where Γ is the usual resonance width. Combining this definition with Eq. (51) gives

$$i(\tfrac{1}{2}\Gamma) = -R(m_1)$$

or

$$\begin{aligned} &\mathrm{Re}R(m_1) = 0\,,\\ &\mathrm{Im}\,R(m_1) = -(\tfrac{1}{2}\Gamma). \end{aligned} \tag{53}$$

The dispersion rule given in Eq. (41a) can then be used to rewrite Eq. (53) as

$$\begin{aligned} &\delta m = \langle 1mk\hat{e}\|V'\|1mk\hat{e}\rangle - \pi^{-1}\,\mathrm{Re}\int_{\mathcal{C}'}\frac{dm''I(m'')}{m''-m_1}\,,\\ &(\tfrac{1}{2}\Gamma) = \pi^{-1}\,\mathrm{Im}\int_{\mathcal{C}'}\frac{dm''I(m'')}{m''-m_1}\ . \end{aligned} \tag{54}$$

The first of these equations gives the general mass renormalization result for unstable particles whereas the second gives the half-width in terms of the basic reduced matrix elements.

The denominator of Eq. (38) can now be rewritten as

$$\begin{aligned} m' - m - R(m') &= m' - m_1 - R(m') + R(m_1)\\ &= (m' - m_1)Z^{-1}(m', m_1), \end{aligned} \tag{55}$$

where Eq. (51) has been used to eliminate m. Since $R(m') = D(m') - iI(m')$, where D and I are real functions of m', it follows that

$$\begin{aligned} &D(m) = -(\tfrac{1}{2}\Gamma)\,\mathrm{Im}\,Z^{-1}(m, m_1),\\ &I(m) = (\tfrac{1}{2}\Gamma)\,\mathrm{Re}\,Z^{-1}(m, m_1). \end{aligned} \tag{56}$$

In terms of the matrix elements the last equation gives the half-width as

$$(\tfrac{1}{2}\Gamma) = I(m)\left(1 + \pi^{-1}\,\mathrm{Re}\int_{\mathcal{C}'}\frac{dm''I(m'')}{(m''-m)(m''-m_1)}\right)^{-1}. \tag{57}$$

Thus to lowest order in the matrix elements

$$(\tfrac{1}{2}\Gamma) \cong I(m)$$

in agreement with Goldberger and Watson.[11] The renormalized "golden rule" then follows immediately from this equation and Eq. (43) as

$$\begin{aligned} \Gamma &= 2\pi[\mathrm{Re}\,Z^{-1}(m, m_1)]^{-1}\langle 1mk\hat{e}\|R\Lambda\delta(m' - H_0)\Lambda R\|1mk\hat{e}\rangle\\ &\cong 2\pi\langle 1mk\hat{e}\|R\Lambda\delta(m' - H_0)\Lambda R\|1mk\hat{e}\rangle. \end{aligned}$$

If the definition $(\tfrac{1}{2}\Gamma_0) \equiv I(m)$ is made,

$$\begin{aligned} (\tfrac{1}{2}\Gamma) &= (\tfrac{1}{2}\Gamma_0)[\mathrm{Re}\,Z^{-1}(m, m_1)]^{-1}\\ &\cong (\tfrac{1}{2}\Gamma_0)|Z(m, m)| \end{aligned} \tag{58}$$

gives the half-width renormalization parallel to that of the Lee model.[1]

Finally Eq. (38) can be written as

$$\begin{aligned} \langle x+|1pk\hat{e}\rangle = (2\pi)^{-3/2}u_k(p)\\ \times(-2\pi i)^{-1}\int_{\mathcal{C}}\frac{dm'Z(m', m_1)e^{-im'\tau}}{m'-m_1}\,. \end{aligned} \tag{59}$$

The asymptotic limits are again similar for $\tau\to\pm\infty$. For the in-state limit, $\tau\to-\infty$, the contour can be rotated to the line of steepest descent in the upper m' plane as shown in Fig. 1(c) whereas in the out-state limit, $\tau\to\infty$, the contour can be rotated to the line of steepest descent in the lower m' plane as shown in Fig. 1(d). In each case there is a contribution due to a pole in the unphysical sheet and a cut contribution which gives

$$\begin{aligned} \langle x+|1pk\hat{e}\rangle \underset{\tau\to\infty}{\sim} &(2\pi)^{-3/2}u_k(p)Z(m_1, m_1)e^{-im_1\tau}\\ &+O(\tau^{-\gamma})\\ &= (2\pi)^{-3/2}u_k(p)Z(m_1, m_1)e^{ipx}e^{-\Gamma\tau/2}\\ &+O(\tau^{-\gamma}),\\ \langle x+|1pk\hat{e}\rangle \underset{\tau\to-\infty}{\sim} &(2\pi)^{-3/2}u_k(p)Z(m_1^*, m_1^*)e^{ipx}e^{\Gamma\tau/2}\\ &+O(\tau^{-\gamma}). \end{aligned} \tag{60}$$

Thus for large $|\tau|$ but $\Gamma|\tau|$ small, the probability for finding the unstable particle in all space initially ($\tau = 0$) damps exponentially,

$$\int d\vec{x}|\langle x+\|1pk\hat{e}\rangle|^2 \cong |Z(m_1, m_1)|^2e^{-\Gamma|\tau|}, \tag{61}$$

and in the distant past or future, damps as an inverse power of τ. Again the symbol $\|$ indicates that $\rho^{-1}(E_p)\delta(\vec{p}-\vec{q})$ has been factored from the matrix element. As expected, an out-state exists for an unstable particle only in the sense of Eq. (61).

It would also be possible to renormalize the field operators for the unstable particles in the same sense as those of the stable-particle field operators. Thus for example

$$\begin{aligned} &a_{Dk}(p) = Z^{-1/2}(m_1, m_1)a_k(p),\\ &[a_{Dk}(p), a_{Dl}^\dagger(q)] = |Z(m_1, m_1)|^{-1}\delta_{k,l}\rho^{-1}(E_p)\delta(\vec{p}-\vec{q}),\\ &\psi_{D+}(x) = Z^{-1/2}(m_1, m_1)\psi_+(x),\\ &\psi_{D-}(x) = Z^{*-1/2}(m_1, m_1)\psi_-(x), \end{aligned} \tag{62}$$

$$\int d\vec{x}|_D\langle x+\|1pk\hat{e}\rangle_D|^2 \cong e^{-i\Gamma|\tau|}\,. \tag{63}$$

VI. SINGLE-PARTICLE PROPAGATORS

Single-particle propagators are usually defined[2,14] as the time-ordered product

$$G(x-y)=\langle\psi_0|T\psi(x)\overline{\psi}(y)|\psi_0\rangle$$
$$=\langle\psi_0|\psi_+(x)\overline{\psi}_-(y)|\psi_0\rangle_{x_0>y_0}+\lambda\langle\psi_0|\overline{\psi}_+(y)\psi_-(x)|\psi_0\rangle_{y_0>x_0}, \quad (64)$$

where $\lambda=+1$ (-1) for bosons (fermions). These matrix elements can be calculated using the results of Eqs. (26)–(38). Direct substitution into Eq. (64) from Eq. (26) gives

$$\langle\psi_0|\psi_+(x)\overline{\psi}_-(y)|\psi_0\rangle_{x_0>y_0}=(2\pi)^{-3}\sum_k\int d\vec{p}\,\rho(E_p)u_k(p)\sum_l\int d\vec{q}\,\rho(E_q)\overline{u}_l(q)\langle\psi_0|a_k(p)(e^{iP(x-y)})a_l^\dagger(q)|\psi_0\rangle$$
$$=i(2\pi)^{-4}\sum_k\int d\vec{p}\,\rho(E_p)u_k(p)\overline{u}_k(p)\int_{\mathrm{e}}\frac{dm'\exp[i(m'/m)p(x-y)]}{m'-m-R(m')}, \quad (65)$$

$$\langle\psi_0|\overline{\psi}_+(y)\psi_-(x)|\psi_0\rangle_{y_0>x_0}=i(2\pi)^{-4}\sum_k\int d\vec{p}\,\rho(E_p)v_k(p)\overline{v}_k(p)\int_{\mathrm{e}}\frac{dm'\exp[i(m'/m)p(y-x)]}{m'-m-R(m')}. \quad (66)$$

For bosons (fermions) the spinors u_k, v_k satisfy

$$\sum_k u_k(p)\overline{u}_k(p)=[m^{2s}+(i)^{2s}\gamma_{[\mu]}p_{[\mu]}](2m^{2s})^{-1},$$
$$\sum_k v_k(p)\overline{v}_k(p)=+(-)[m^{2s}+(-)(i)^{2s}\gamma_{[\mu]}p_{[\mu]}](2m^{2s})^{-1}, \quad (67)$$

where

$$\gamma_{[\mu]}p_{[\mu]}\equiv\gamma_{\mu_1\mu_2\cdots\mu_{2s}}p_{\mu_1}p_{\mu_2}\cdots p_{\mu_{2s}}$$

and the γ's are the $2(2s+1)$ generalizations of the Dirac matrices.[8] For arbitrary spin the remainder of the derivation is straightforward but arduous. The significant steps are illustrated by the spin-zero case which is algebraically more simple. One begins in the usual way by defining the step functions

$$\Theta(x_0-y_0)=-(2\pi i)^{-1}\int_{-\infty}^{\infty}d\lambda\,\frac{e^{-i\lambda(x_0-y_0)}}{\lambda+i\epsilon},$$

and rewriting Eq. (65) as

$$\langle\psi_0|\psi_+(x)\overline{\psi}_-(y)|\psi_0\rangle_{x_0>y_0}=-(2\pi)^{-5}\int\frac{d\vec{p}}{2E_p}\int_{\mathrm{e}}\frac{dm'\exp[i(m'/m)p(x-y)]}{m'-m-R(m')}\int_{-\infty}^{\infty}\frac{d\lambda\exp[-i\lambda(x_0-y_0)]}{\lambda+i\epsilon}.$$

If the substitutions

$$\vec{q}=(m'/m)\vec{p},$$
$$q_0=\lambda+E_p(m'/m),$$
$$dq=d\vec{q}\,dq_0$$

are made,

$$\langle\psi_0|\psi_+(x)\overline{\psi}_-(y)|\psi_0\rangle_{x_0>y_0}=-(2\pi)^{-5}\int_{\mathrm{e}}\frac{dm'(m/m')^3}{m'-m-R(m')}\int\frac{dq}{2E_p}\,\frac{e^{iq(x-y)}}{q_0-E_p(m'/m)+i\epsilon}, \quad (68)$$

where m' can always be considered real and positive.

Similarly the step function

$$\Theta(y_0-x_0)=(2\pi i)^{-1}\int_{-\infty}^{\infty}d\lambda\,\frac{e^{i\lambda(x_0-y_0)}}{\lambda-i\epsilon}$$

can be defined and used to rewrite Eq. (66) as

$$\langle\psi_0|\psi_+(x)\overline{\psi}_-(y)|\psi_0\rangle_{y_0>x_0}=(2\pi)^{-5}\int_{\mathrm{e}}\frac{dm'(m/m')^3}{m'-m-R(m')}\int\frac{dq}{2E_p}\,\frac{e^{iq(x-y)}}{q_0+E_p(m'/m)-i\epsilon}. \quad (69)$$

These equations can now be summed to give

$$G(x-y)\equiv\tfrac{1}{2}\Delta_F(x-y),$$

$$\Delta_F(x-y)=2(2\pi)^{-5}\int_{\mathcal{C}}\frac{dm'\,(m/m')^2}{m'-m-R(m')}\int dq\,\frac{e^{iq(x-y)}}{q^2+m'^2-i\epsilon}\,, \tag{70}$$

where the poles at $m'^2=i\epsilon-q^2$ are outside of $\mathcal{C}$.

For stable particles this equation is

$$\Delta_F(x-y)=2(2\pi)^{-5}\int_{\mathcal{C}}\frac{dm'\,(m/m')^2 Z(m',m)}{m'-m}\int\frac{dq\,e^{iq(x-y)}}{q^2+m'^2-i\epsilon}\,. \tag{71}$$

The pole contribution to Eq. (71) is the renormalized free-particle spin-zero propagator

$$\begin{aligned}\Delta_F^p(x-y)&\equiv Z(m,m)\Delta_F^0(x-y;m^2)\\&=Z(m,m)(-2i)(2\pi)^{-4}\int dq\,\frac{e^{iq(x-y)}}{q^2+m^2-i\epsilon}\,.\end{aligned} \tag{72}$$

Substitution into Eq. (71) for Δ_F^0 then gives

$$\begin{aligned}\Delta_F(x-y)&=-(2\pi i)^{-1}\int_{\mathcal{C}}\frac{dm'\,(m/m')^2 Z(m',m)\Delta_F^0(x-y;m'^2)}{m'-m}\\&=-(2\pi i)^{-1}\int_{\mathcal{C}}\frac{dm'\,(m/m')^2\Delta_F^0(x-y;m'^2)}{m'-m-R(m')}\,.\end{aligned} \tag{73}$$

This form is especially useful for comparison to the Lehmann spectral representation[5,14] for the propagator,

$$\Delta_F(x-y)=\int_0^\infty dm^2\rho(m^2)\Delta_F^0(x-y;m^2). \tag{74}$$

The connection is made by noting first that Eq. (73) can be rewritten as

$$\Delta_F(x-y)=-(2\pi i)^{-1}\int_{\mathcal{C}} dm'\,(m/m')^2\left(\frac{1}{m'-m-R(m')}+\frac{1}{m'+m+R(-m')}\right)\Delta_F^0(x-y;m'^2) \tag{75}$$

since the term containing $R(-m')$ has singularities only along the negative m' axis whereas $\mathcal{C}$ is a closed contour along the positive m' axis. The integrand, including dm', is an even function of m' and consequently can be written as

$$\begin{aligned}\Delta_F(x-y)&=(-2\pi i)^{-1}\int_{\mathcal{C}}(dm'^2)\rho'(m'^2)\Delta_F^0(x-y;m'^2)\\&=Z(m,m)\Delta_F^0(x-y;m^2)-(2\pi i)^{-1}\int_\mu^\infty(dm'^2)\rho_+'(m'^2)\Delta_F^0(x-y;m'^2)-(2\pi i)^{-1}\int_\infty^\mu(dm'^2)\rho_-'(m'^2)\Delta_F^0(x-y;m'^2),\end{aligned}$$

where μ corresponds to the threshold and $\rho_\pm'(m'^2)$ is the value of $\rho'(m'^2)$ above (below) the cut. Both integrals can be extended to the origin since $\rho_+'=\rho_-'$ below threshold. Comparison to Eq. (74) then gives

$$(2\pi i)\rho(m'^2)=\rho_-'(m'^2)-\rho_+'(m'^2)+2\pi i\,Z(m,m)\delta(m'^2-m^2). \tag{76}$$

The unstable-particle propagator follows in a parallel way except Eq. (71) must be written as

$$\Delta_F(x-y)=2(2\pi)^{-5}\int_{\mathcal{C}}dm'\,\frac{(m/m')^2 Z(m',m_1)}{m'-m_1}\int\frac{dq\,e^{iq(x-y)}}{q^2+m'^2-i\epsilon}=-(2\pi i)^{-1}\int_{\mathcal{C}}\frac{dm'\,(m/m')^2 Z(m',m_1)}{m'-m_1}\Delta_F^0(x-y;m'^2). \tag{77}$$

The spectral function can also be defined for the unstable case. However, the pole term in Eq. (76) is missing, so that

$$(2\pi i)\rho(m'^2)=\rho_-'(m'^2)-\rho_+'(m'^2). \tag{78}$$

The propagators for the arbitrary spin case can be derived in a similar way. The result for both the stable and unstable cases is

$$\begin{aligned}G^{(s)}(x-y)&=(\tfrac{1}{2})\Delta_F^{(s)}(x-y),\\ \Delta_F^{(s)}(x-y)&=-(2\pi i m^{2s-1})^{-1}\rho(m)\int_{\mathcal{C}}\frac{dm'\,(m/m')^{2s+2}}{m'-m-R(m')}(i^{2s}\gamma_{[\mu]}p_{[\mu]}+m'^{2s})\Delta_F^0(x-y;m'^2),\end{aligned} \tag{79}$$

where $\rho(m)$ is $(2m)^{-1}$ for bosons and unity for fermions and p is the c-number operator $p\equiv(-i\vec{\nabla},-\partial/\partial t)$.

In parallel to the spin-zero case, the renormalized free-particle propagator for the stable case can be defined as the pole term or on-mass shell contribution of $\Delta_F^{(s)}(x-y)$. This gives

$$\Delta_F^{p(s)}(x-y)=Z(m,m)\Delta_F^{(s)}(x-y;m^2),$$
$$\Delta_F^{(s)}(x-y;m^2)=\rho(m)m^{1-2s}(i^{2s}\gamma_{[\mu]}p_{[\mu]}+m^{2s})\Delta_F^0(x-y;m^2) \tag{80}$$

and allows Eq. (79) to be rewritten as

$$\Delta_F^{(s)}(x-y)=-(2\pi i)^{-1}\int_{\mathfrak{C}}\frac{dm'\,(m/m')^{2s+2}}{m'-m-R(m')}\Delta_F^{(s)}(x-y;m'^2). \tag{81}$$

For spin $\frac{1}{2}$ the free-particle propagator defined in Eq. (80) reduces to the usual Feynman propagator for the Dirac case. The spin-one specialization agrees with the propagator defined by Tucker and Hammer [15] but disagrees with the result of Weinberg,[16] which contains unphysical, noncovariant terms. This is a point worthy of future investigation since in the work presented here and that of Tucker and Hammer,[15] the propagator is derived in the Heisenberg representation whereas Weinberg's results apply in the interaction representation where such unphysical terms are needed to cancel similar terms which appear in the interaction.

Schwinger's result[2] for scalar unstable particles is most easily obtained from the spin-zero specialization of Eq. (81), or, equivalently, Eqs. (65) and (66). The cut contribution to the integral is ignored and the evaluation is made assuming that $\mathfrak{C}$ encircles the pole in the unphysical sheet. The result is an exponential decay as in Eq. (60). This same approximation applied to either Eq. (77) or (81) gives the covariant form

$$\Delta_F^{(0)}(x-y)=Z(m_1,m_1)(m/m_1)^2\Delta_F^0(x-y;m_1^{\,2}), \tag{82}$$

where, as before, $m_1=m-i(\frac{1}{2}\Gamma)$.

Finally, if one defines the Lehmann spectral density for all spin as

$$\Delta_F^{(s)}(x-y)=\int_0^\infty(dm^2)\rho(m^2)\Delta_F^{(s)}(x-y;m^2),$$

then Eqs. (72) and (78) also apply for the arbitrary-spin case.

VII. IN AND OUT OPERATORS AND ALL THAT

While the derivations in the previous sections lead to propagators which represent single particles it is not clear how these propagators arise in a scattering calculation using the theory presented here. The calculational procedures become apparent if in- and out-states can be defined since then the standard S-matrix theory can be applied.

As is usually the case, the starting point is the definition of the in and out operators[17] for the stable particles,

$$a_{k\,\text{in,out}}=\underset{x_0\to-\infty,+\infty}{\text{Weak limit}}\int d\vec{x}\,j_0(\bar{f}_k(px),\psi_D(x)),$$
$$b_{k\,\text{in,out}}=\underset{x_0\to-\infty,+\infty}{\text{Weak limit}}\int d\vec{x}\,j_0(\bar{\psi}_D(x),g_k(px)), \tag{83}$$

where ψ_D are "dressed" field operators defined by Eqs. (48) and (62), f_k and g_k are any free-particle and antiparticle c-number solutions to the free-particle equations of motion, and $j_4=ij_0$ is the fourth component of the conserved current derived from the free-particle equations of motion. It is usually assumed that

$$\int d\vec{x}\,j_0(\bar{f}_k(px),f_l(qx))=\rho^{-1}(E_p)\delta_{kl}\delta(\vec{p}-\vec{q}),$$
$$\int d\vec{x}\,j_0(\bar{g}_k(px),g_l(qx))=\rho^{-1}(E_p)\delta_{kl}\delta(\vec{p}-\vec{q}), \tag{84}$$
$$\int d\vec{x}\,j_0(\bar{g}_k(px),f_l(qx))=0.$$

These solutions can be taken as the plane-wave solutions

$$f_k = (2\pi)^{-3/2} u_k e^{ipx}, \qquad (85)$$
$$g_k = (2\pi)^{-3/2} v_k e^{ipx}.$$

This assumption can be trivially avoided in the following if $\psi_D(x)$ is redefined in terms of f_k and g_k, requiring only the orthogonality relationships given in Eq. (84).

The definition in Eq. (83) can be simplified for the instantaneous hyperplane, since $\vec{P} = \vec{P}^0$ and

$$e^{-i\vec{P}\cdot\vec{x}} a_{Dl}(q) e^{i\vec{P}\cdot\vec{x}} = a_{Dl}(q) e^{i\vec{q}\cdot\vec{x}}. \qquad (86)$$

Then, for example $a_{k\,\text{in,out}}$ becomes

$$a_{k\,\text{in,out}}(p) = \underset{\tau\to-\infty,+\infty}{\text{Weak limit}} \sum_l \int d\vec{q}\,\rho(E_q) \left(\int d\vec{x}\, j_0(\bar{f}_k(px), f_l(qy))\; e^{iH\tau} a_{Dl}(q) e^{-iH\tau} e^{iE_q\tau} \right)$$
$$= \underset{\tau\to-\infty,+\infty}{\text{Weak limit}} [e^{iH\tau} a_{Dk}(p) e^{-iH\tau}] e^{iE_p\tau}, \qquad (87)$$

where as before τ is the time in the instantaneous hyperplane and it has been assumed that j_0 is linear in $\psi(x)$. Finally, if a_k is replaced by a_{Dk} in Eq. (32),

$$a_{k\,\text{in,out}}(p) = \underset{\tau\to-\infty,+\infty}{\text{Weak limit}}\, A_k(p,\tau) e^{iE_p\tau}. \qquad (88)$$

It is clear that $a_{k\,\text{in,out}}$, $A_k(p,\tau)$ and $a_{Dk}(p)$ all satisfy the same commutation rules.

An n-particle in- or out-state can also be defined in the instantaneous hyperplane as

$$(n!)^{-1/2}[a_{\text{in}}^\dagger(q_1)\cdots a_{\text{in}}^\dagger(q_n)]|\psi_0\rangle = |n, \text{in}\rangle$$
$$= \underset{\tau\to-\infty}{\text{Weak limit}} [A^\dagger(q_1,\tau)\cdots A^\dagger(q_n,\tau)] \exp[-i(E_{q_1}+\cdots+E_{q_n})\tau]|\psi_0\rangle$$
$$= \underset{\tau\to-\infty}{\text{Weak limit}}\, e^{iH\tau}|n\rangle_D e^{-iE\tau}, \qquad (89a)$$

$$(n!)^{-1/2}[a_{\text{out}}^\dagger(q_1)\cdots a_{\text{out}}^\dagger(q_n)]|\psi_0\rangle = |n, \text{out}\rangle$$
$$= \underset{\tau\to+\infty}{\text{Weak limit}} [A^\dagger(q_1,\tau)\cdots A^\dagger(q_n,\tau)] \exp[-i(E_{q_1}+\cdots+E_{q_n})\tau]|\psi_0\rangle$$
$$= \underset{\tau\to+\infty}{\text{Weak limit}}\, e^{iH\tau}|n\rangle_D e^{-iE\tau}, \qquad (89b)$$

where $|n\rangle_D$ and E are abbreviations for the free-particle states

$$|n\rangle_D \equiv (n!)^{-1/2}[a_D^\dagger(q_1)a_D^\dagger(q_2)\cdots a_D^\dagger(q_n)]|\psi_0\rangle, \qquad (90)$$
$$H_0|n\rangle_D = E|n\rangle_D, \quad E = \sum_i E_{q_i}.$$

If $|\alpha\rangle$ is any arbitrary state vector, then by Eq. (89)

$$\langle n, \text{in}|\alpha\rangle = \lim_{\tau\to-\infty} \langle n|e^{-iH\tau}|\alpha\rangle e^{iE\tau}$$
$$= \lim_{\tau\to-\infty} (-2\pi i)^{-1} \int_{\text{e}} dm'\, e^{-i(m'-E)\tau}\, {}_D\langle n|(m'-H)^{-1}|\alpha\rangle. \qquad (91a)$$

$$\langle n, \text{out}|\alpha\rangle = \lim_{\tau\to+\infty} {}_D\langle n|e^{-iH\tau}|\alpha\rangle e^{iE\tau}$$
$$= \lim_{\tau\to+\infty} (-2\pi i)^{-1} \int_{\text{e}} dm' e^{-i(m'-E)\tau}\, {}_D\langle n|(m'-H)^{-1}|\alpha\rangle. \qquad (91b)$$

The identity

$$(m'-H)^{-1} = (m'-H_0)^{-1} + (m'-H_0)^{-1} V (m'-H)^{-1} \qquad (92a)$$

can be used to rewrite this equation as

$$\langle n, \text{in}|\alpha\rangle = (-2\pi i)^{-1} \lim_{\tau\to-\infty} \int_{\text{e}} \frac{dm'\, e^{-i(m'-E)\tau}}{m'-E} \langle\psi_n(m')|\alpha\rangle, \qquad (92b)$$

$$\langle n, \text{out}|\alpha\rangle = (-2\pi i)^{-1} \lim_{\tau\to+\infty} \int_{\mathcal{C}} \frac{dm'\, e^{-i(m'-E)\tau}}{m'-E} \langle \psi_n(m')|\alpha\rangle,$$

where

$$|\psi_n(m')\rangle = |n\rangle_D + (m'-H)^{-1} V |n\rangle_D, \quad m' \neq E. \tag{93}$$

The restriction $m' \neq E$ must be put on this equation since the operator $(E-H)^{-1}$ is singular. The integral in Eq. (92b) is well defined for all m' and E so the pole at $m'=E$ can be moved through the contour to the position $m'=E \pm i\eta$, depending upon whether $\tau \to \pm\infty$. This gives

$$\langle n, \text{in}|\alpha\rangle = \langle \psi_n^+(E)|\alpha\rangle + \mathcal{J}(\tau),$$
$$\langle n, \text{out}|\alpha\rangle = \langle \psi_n^-(E)|\alpha\rangle + \mathcal{J}(\tau), \tag{94}$$

where

$$|\psi_n^\pm(E)\rangle = |n\rangle_D + (E-H \pm i\eta)^{-1} V |n\rangle_D \tag{95}$$

are the usual outgoing or incoming scattering states of the exact Hamiltonian and

$$\mathcal{J}(\tau) = -(2\pi i)^{-1} \lim_{\tau\to\mp\infty} \int_{\mathcal{C}'} \frac{dm' e^{-i(m'-E)\tau}}{m'-E \pm i\eta} \int \frac{dE' {}_D\langle n|V|E'\rangle \langle E'|\alpha\rangle}{m'-E'}. \tag{96}$$

Here $\mathcal{C}'$ is a contour which does not include the pole at $m'=E \mp i\eta$ and $\int dE' |E'\rangle\langle E'|$ represents a sum on the complete set of eigenstates of H with eigenvalue E'.

It can be shown that $\mathcal{J}(\tau)$ approaches zero in the asymptotic limit for large $|\tau|$. Addition and subtraction of H_0 to V allows Eq. (96) to be rewritten as

$$\mathcal{J}(\tau) = -(2\pi i)^{-1} \lim_{\tau\to\mp\infty} \int_{\mathcal{C}'} \frac{dm' e^{-i(m'-E)\tau}}{m'-E \pm i\eta} \int \frac{dE' (E'-E)}{m'-E'} {}_D\langle n|E'\rangle \langle E'|\alpha\rangle. \tag{97}$$

If the matrix elements lead to a δ function on the energies, $\delta(E'-E)$, $\mathcal{J}(\tau)$ vanishes since then the integrand vanishes ($m' \neq E$ anywhere on $\mathcal{C}'$ and the pole at $m'=E$ has been removed from within the contour). If the matrix elements do not lead to a δ function on the energies the sum on E' gives rise to branch lines in the m' plane with branch points at the various thresholds. For $\tau \to -\infty$ the pole is at $m'=E - i\eta$ in the lower half of the m' plane. The contours around the various branches can be rotated to the line of steepest descent in the upper half of the m' plane as in Fig. 1(c) so that in the limit only the branches contribute to give

$$\mathcal{J}(\tau) = O(\tau^{-\gamma}). \tag{98}$$

Similarly for $\tau \to \infty$, the pole is at $m' = E + i\eta$ in the upper-half m' plane and the contour can be rotated to the line of steepest descent in the lower half of the m' plane, again giving the contribution expressed by Eq. (98). Consequently, for any $|\alpha\rangle$,

$$\langle n, \text{in}|\alpha\rangle = \langle \psi_n^+(E)|\alpha\rangle,$$
$$\langle n, \text{out}|\alpha\rangle = \langle \psi_n^-(E)|\alpha\rangle, \tag{99}$$

so that the in- and out-states become equivalent to the exact scattering solutions in the asymptotic limit.

It then follows from Eq. (99) that the S matrix defined by

$$S_{nn'} = \langle n, \text{out}|n', \text{in}\rangle \tag{100}$$

is the usual result[18]

$$S_{nn'} = \langle \psi_n^- | \psi_{n'}^+ \rangle. \tag{101}$$

Although the derivation of Eq. (101) proceeds similarly to previous derivations[18] it differs in two essential respects:

(1) No adiabatic cutoff for the interaction is required. The parameter η enters into the calculations as a consequence of the contour integral which gives a precise definition for Eq. (91).

(2) The Cauchy integral representation of Eq. (13) guarantees that the field operators represent Heisenberg fields for all time t and that the states are solutions to the time-dependent Schrödinger equation for all t. This is in contrast to the integral representation used by Goldberg and Watson[19] where, for example,

$$\psi(t)=(2\pi i)^{-1}\int_{\mathcal{C}_2} dE\,\frac{e^{-iEt}}{E-H}X_a\,, \tag{102}$$

where $\mathcal{C}_2$ is a contour in the upper-half E plane extending from $-\infty+i\eta$ to $\infty+i\eta$. It is easy to verify that the $\psi(t)$ satisfies

$$H\psi=i(\partial/\partial t)\psi+i\delta(t)X_a \tag{103}$$

rather than the Schrödinger equation.

Finally, it is possible to write the S matrix as

$$S_{nn'}=\lim_{x_0\to\infty;\;y_0\to-\infty}\left[{}_D\langle n|e^{-iH(x_0-y_0)}|n'\rangle_D\,e^{iE_nx_0}e^{-iE_{n'}y_0}\right]. \tag{104}$$

This is particularly suitable for the evaluation of the S matrix since it can be shown that

$$e^{-iH(x_0-y_0)}=e^{-iH_0(x_0-y_0)}-i\int_{y_0}^{x_0}d\xi\,e^{-iH_0(x_0-\xi)}\,V\,e^{-iH_0(\xi-y_0)}-\int_{y_0}^{x_0}d\xi'\int_{y_0}^{x_0}d\xi\,e^{-H_0(x_0-\xi')}Ve^{-iH(\xi'-\xi)}Ve^{-iH_0(\xi-y_0)}\Theta(\xi',\xi),$$

$$x_0>y_0. \tag{105}$$

The derivation of this equation starts with the identities which are valid for $x_0>y_0$,

$$e^{-iH(x_0-y_0)}=e^{-iH_0(x_0-y_0)}-i\int_{y_0}^{x_0}d\xi\,e^{-iH(x_0-\xi)}\,Ve^{-iH_0(\xi-y_0)}, \tag{106a}$$

and

$$e^{-iH(x_0-y_0)}=e^{-iH_0(x_0-y_0)}-i\int_{y_0}^{x_0}d\xi\,e^{-iH_0(x_0-\xi)}Ve^{-iH(\xi-y_0)}. \tag{106b}$$

Both of these equations are readily verified by observing that for $x_0>y_0$, $x_0>\xi$, $\xi>y_0$ the Cauchy representations for the exponential operators can be changed to Eq. (102), that is, the contour $\mathcal{C}$ can be opened up to the contour $\mathcal{C}_2$. The ξ integration can then be done leaving identities similar to that expressed in Eq. (92a). Substitution for $e^{-iH(\xi-y_0)}$ from Eq. (106a) into Eq. (106b) then gives the desired result.

The S matrix can now be rewritten as

$$S_{n,n'}={}_D\langle n|n'\rangle_D-i\int_{-\infty}^{\infty}d\xi\,e^{-i(E_{n'}-E_n)\xi}{}_D\langle n|V|n'\rangle_D-\int_{-\infty}^{\infty}d\xi'\int_{-\infty}^{\infty}d\xi\,e^{-iE_{n'}\xi+iE_n\xi'}{}_D\langle n|V\,e^{-iH(\xi'-\xi)}V|n'\rangle_D\Theta(\xi',\xi). \tag{107}$$

This form is especially useful for the discussion of interactions that involve the exchange of a single particle.

It is clear that the usual perturbation expansion for $S_{nn'}$ can be obtained by successive iterations using Eq. (105) or (106).

VIII. SINGLE-PARTICLE INTERMEDIATE STATES

Many high-energy processes involve interactions of the form

$$V=\lambda\int d\vec{x}\,f(\bar{\psi}(\vec{x},0),\psi(\vec{x},0))A(\vec{x},0)+\text{H.c.}, \tag{108}$$

where f is a linear function of the ψ fields and where both H_0 and V can be evaluated at $t=0$ since H is time independent. As an illustrative example consider the special case of a scalar interaction where all the fields represent spin-zero, scalar bosons. This specialization, as will be seen, contains all the essentials of the S-matrix calculation without the purely algebraic complications of the higher-spin propagators.

The S matrix for the direct-channel process two particles in going to two particles out is

$$S_{2,2'}=-\int d\xi'\int d\xi\,e^{iE_2\xi'-iE_{2'}\xi}\Theta(\xi',\xi)\langle p_3,p_4|V\,e^{-H(\xi'-\xi)}V|p_1,p_2\rangle, \tag{109}$$

where

$$E_2=E_{p_3}+E_{p_4},\quad E_{2'}=E_{p_1}+E_{p_2}.$$

For fields defined by Eq. (26) the matrix element reduces to

$$\langle p_3, p_4|V e^{-iH(\xi'-\xi)}V|p_1, p_2\rangle = (2\pi)^{-6}|\lambda|^2\int d\vec{x}\int d\vec{y}\, e^{-i(\vec{p}_3+\vec{p}_4)\cdot\vec{x}}\, e^{i(\vec{p}_1+\vec{p}_2)\cdot\vec{y}}\,[\langle\psi_0|A(\vec{x},0)e^{-iH(\xi'-\xi)}A^\dagger(\vec{y},0)|\psi_0\rangle + \langle\psi_0|A^\dagger(\vec{x},0)e^{-iH(\xi'-\xi)}A(\vec{y},0)|\psi_0\rangle]$$

$$= (2\pi)^{-6}|\lambda|^2\int d\vec{x}\int d\vec{y}\, e^{-i(\vec{p}_3+\vec{p}_4)\cdot\vec{x}}\, e^{i(\vec{p}_1+\vec{p}_2)\cdot\vec{y}}[\langle\psi_0|A(\vec{x},\xi')A^\dagger(\vec{y},\xi)|\psi_0\rangle + \langle\psi_0|A^\dagger(-\vec{x},-\xi)A(-\vec{y},-\xi')|\psi_0\rangle], \tag{110}$$

where in the last term the sign of the spatial dependence may be changed since the vacuum expectation value depends only upon $|\vec{x}-\vec{y}|$. Substitution back into Eq. (109) then gives, after some minor algebraic manipulations,

$$S_{2,2'} = -(2\pi)^{-6}|\lambda|^2\int dx\int dy\, e^{-ip_2 x}e^{ip_{2'}y}\langle\psi_0|TA(x)A^\dagger(y)|\psi_0\rangle,$$

where

$$p_2 = (\vec{p}_3+\vec{p}_4, iE_2), \quad p_{2'} = (\vec{p}_1+\vec{p}_2, iE_{2'}). \tag{111}$$

It should be noted that the field $A(x)$ can represent either a stable or unstable particle and that for the simple process envisioned here, Eq. (111) is exact.

Finally, the single-particle propagator for the scalar field given by Eq. (77) can be used to further reduce Eq. (111) to

$$S_{2,2'} = -(2\pi)^{-3}|\lambda|^2\int_{\mathcal{C}}\frac{dm'\,(m/m')^2 Z(m',m_1)}{m'-m_1}\,\frac{\delta(p_1+p_2-p_3-p_4)}{q^2+m'^2-i\epsilon}, \tag{112}$$

where

$$q = p_1+p_2 = p_3+p_4, \quad m_1 = m - i(\tfrac{1}{2}\Gamma).$$

The Born approximation is obtained by encircling the pole at m_1 (or m for the stable case) and discarding the cut contributions which are higher order in λ. This gives

$$S_{2,2'} = i(2\pi)^{-2}\frac{|\lambda|^2(m/m_1)^2 Z(m_1,m_1)\delta(p_1+p_2-p_3-p_4)}{q^2+m_1^2}. \tag{113}$$

For small Γ (or for the stable case) $(m/m_1)=1$ so that the Born approximation is

$$(S_{2,2'})_B = \frac{i(2\pi)^{-2}\lambda'^2\delta(p_1+p_2-p_3-p_4)}{q^2+m_1^2}, \tag{114}$$

where

$$\lambda'^2 = |\lambda|^2 Z(m_1,m_1)$$

is the renormalized coupling constant. This gives rise to the usual Breit-Wigner shape for the cross section in the unstable case.

However, since Eq. (112) is exact, the analysis is not restricted to the Born approximation. Because of the m'^{-5} dependence of the integrand it is reasonable to assume that the contour can be taken to infinity everywhere in the physical sheet of the m' plane thereby picking up only the poles at $m'^2 = -q^2+i\epsilon$ and $m'=0$. If this assumption is valid, the S matrix can be evaluated exactly as

$$S_{2,2'} = \frac{i(2\pi)^{-2}m^2|\lambda|^2}{2(-q^2)^{3/2}}\left(\frac{[(-q^2)^{1/2}+m_1]Z((-q^2)^{1/2},m_1)+[(-q^2)^{1/2}-m_1]Z(-(-q^2)^{1/2},m_1)}{q^2+m_1^2} - \frac{2(-q^2)^{1/2}}{m_1^2}\frac{Z^2(0,m_1)}{Z(0,0)}\right) \times\delta(p_1+p_2-p_3-p_4). \tag{115}$$

Note that this reduces to the Born approximation, as it should, if Z is set equal to unity, that is, if the cut contribution is completely ignored. It follows from Eq. (45) that if Z^{-1} exists, then $Z^{-1}(\pm(-q^2)^{1/2},m_1)\to 1$ as $(-q^2)\to\infty$. Consequently in the high-energy limit

$$S_{2,2'} = (S_{2,2'})_B \frac{Z^2(0,m_1)}{Z(0,0)Z(m_1,m_1)}\left[1+\frac{m_1^2}{q^2}\left(1-\frac{Z(0,0)}{Z^2(0,m_1)}\right)\right]. \tag{116}$$

If crossing symmetry applies to this process, then the requirement that the cross section be finite as $t\to 0$ places the requirement on Eq. (115) that

$$Z(0,m_1)=0,$$
$$\lim_{t\to 0}\left[(t^{1/2}+m_1)Z(t^{1/2},m_1)+(t^{1/2}-m_1)Z(-t^{1/2},m_1)\right]=\frac{2c}{m_1^2}Z(m_1,m_1)t^{3/2}, \tag{117}$$

where c is some dimensionless constant. Since Z is itself dimensionless and symmetric with respect to the exchange of its arguments, Eq. (117) merely implies

$$Z(m',m_1)=\left[1+\frac{1}{2}\left(\frac{\mu}{m'}\right)^{\gamma}f\left(\frac{m'}{\mu},\frac{m_1}{\mu}\right)+\frac{1}{2}\left(\frac{\mu}{m_1}\right)^{\gamma}f\left(\frac{m_1}{\mu},\frac{m'}{\mu}\right)\right]^{-1},$$

where μ is the threshold energy, $\gamma>1$, and $f(m'/\mu, m_1/\mu)$ is any function such that

$$\lim_{m'\to 0}\left[\left(\frac{m'}{\mu}\right)^{\gamma}\frac{df}{dm'}\left(\frac{m_1}{\mu},\frac{m'}{\mu}\right)\right]=0.$$

Thus Eq. (117) is consistent with the requirement that $Z(m',m_1)$ be bounded for large m'.

With the restriction of Eq. (117), the cross-channel S-matrix element for low momentum transfer is

$$(S_t)_L = c(S_t)_B, \tag{118a}$$

whereas for high momentum transfer

$$(S_t)_H = Z^{-1}(m_1,m_1)t^{-1}(S_t)_B. \tag{118b}$$

The results expressed by Eqs. (118) can be generalized to more physical reality by including a form factor in the definition of the vertex specified by V. In this event the coupling constant $|\lambda|^2$ can be reinterpreted to be the form factor so that Eqs. (118) remain unaltered. This result is interesting since it predicts an extra factor t^{-1} in the amplitude of the scattering matrix which depends only on the boundedness of the integrals within Z and the assumption of crossing symmetry. Consequently one should expect this result to manifest itself in a variety of experiments.

This is especially interesting from the point of view of the vector-dominance model[20] for the form factors of the pion and nucleon which predict form factors of the type $(1-t/a^2)^{-1}$. This combined with Eqs. (118) gives an over-all behavior t^{-2} in the "high" momentum transfer limit which is in fact observed[20,21] and difficult to explain.

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Manifestly Conformal-Covariant Expansion on the Light Cone

S. Ferrara and A. F. Grillo
Laboratori Nazionali di Frascati del CNEN, Frascati, Italy

and

R. Gatto
Istituto di Fisica dell'Università, Roma, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Roma, Italy
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The isomorphism of the conformal algebra on space-time to the orthogonal O(4,2) algebra is exploited to derive in a manifestly covariant way an operator-product expansion on the light cone in terms of irreducible operator representations of the conformal algebra. The expansion provides for a solution of the causality problem for operator expansion on the light cone. Additional properties of the conformally covariant expansion, as well as its relation to the conformally invariant three-point function, are discussed.

I. INTRODUCTION

Wilson has advocated the relevance of scale invariance applied to an operator-product expansion.[1] The possible relevance of the stronger conformal invariance[2] for equal-time commutators[3] and operator-product expansions[4,5] has recently been proposed. In particular, in Ref. 5 the following "improved" light-cone expansion was derived:

$$A(x)B(0) \underset{x^2\to 0}{\sim} \sum_{n=0}^{\infty} c_n^{AB}\left(\frac{1}{x^2}\right)^{(l_A+l_B+n-l_n)/2} x^{\alpha_1}\cdots x^{\alpha_n}\; {}_1F_1\!\left(\tfrac{1}{2}(l_A-l_B+l_n+n);\, l_n+n;\, x\cdot\partial\right) O_{\alpha_1\cdots\alpha_n}(0)\,.$$

In the above equation $A(x)$ and $B(x)$ are local scalar (for simplicity) operators of dimensions l_A and l_B (in energy units), both annihilated by K_λ (the generator of special conformal transformations), i.e., satisfying $[K_\lambda, A(0)]=0$, $[K_\lambda, B(0)]=0$; $O_{\alpha_1\cdots\alpha_n}(0)$ (symmetric traceless tensors of dimension l_n) are those operators of the expansion basis which are annihilated by K_λ; c_n^{AB} are unknown constants; the hypergeometric function ${}_1F_1(a;c;z)$ arises from the structure of the conformal algebra.

In this paper we shall

(i) offer a manifestly conformal-covariant derivation of the improved expansion using the isomorphism of the conformal algebra to the orthogonal algebra O(4, 2). In a subsequent paper[6] the proof is extended (in view of later applications) to derive a conformally covariant operator-product expansion valid over the whole space-time;

(ii) present additional discussion on the properties of the improved expansion. Besides offering a solution of the important causality problem in operator expansions, and of translation invariance on a Hermitian basis, as extensively discussed in Ref. 5, the improved expansion is directly related to the conformally covariant expressions for the vacuum expectation value of a product of three local operators and for the vertex function.

II. CONFORMALLY COVARIANT FORMALISM

It is well known[2] that the conformal algebra on space-time is isomorphic to the orthogonal algebra O(4, 2), whose generators constitute an antisymmetric tensor J_{AB} $(A, B=0, 1, 2, 3, 5, 6)$ with