Hence,

$$|\chi_q^B(0)|^2 = \frac{2}{3\lambda}\left[\left(\lambda\,\frac{\partial Q(q^2)}{\partial q^2}\right)_{q^2=0}\right]^{-1}. \quad \text{(B9)}$$

From (2.50), we obtain

$$\left(\frac{g_B}{3}\right)^2 = \frac{2}{3}\left[\left(\frac{\partial Q(q^2)}{\partial q^2}\right)_{q^2=0}\right]^{-1} \quad \text{(B10)}$$

by virtue of

$$\langle 0|\,B^{\text{in}}(x)\,|\,B_q\rangle = \frac{1}{(2\pi)^{3/2}}\,\frac{1}{(2\,q_0)^{1/2}}\,e^{iqx}. \quad \text{(B11)}$$

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# High-Energy Delbrück Scattering from Nuclei*

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We study the elastic scattering of a high-energy photon from a heavy nucleus, considered to be a static Coulomb field due to the charge $Ze$. Exchange of an arbitrarily large number of photons with the nucleus is taken into account, and the effect of this multiphoton exchange is found to be very large, particularly for momentum transfers which are large compared with the rest mass of the electron. In addition, an interesting theoretical problem in this connection is formulated but unfortunately not solved.

## I. INTRODUCTION

Three years ago, we studied in detail all two-body elastic-scattering amplitudes in quantum electrodynamics at high energies.[1-5] Among these processes, the one with the most direct experimental interest is Delbrück scattering,[6] or the elastic scattering of a photon by a nucleus, considered to be a static Coulomb field. At the time when we carried out our theoretical analysis, the only relevant experimental data on Delbrück scattering were those of Moffatt and Stringfellow[7] at an energy of about 90 MeV, and a comparison of these data with our theoretical results is given in III. Recently, the experimental group F39 of DESY obtained data on Delbrück scattering and photon splitting at energies of several BeV and momentum transfer of a few MeV/$c$, although the data analysis is as yet incomplete. Motivated by this new information on copper, silver, gold, and uranium, we give in this paper the basic theoretical formulas for Delbrück scattering and some of the simple consequences.

The lowest-order diagrams for Delbrück scattering[6] are of the sixth order, and hence the matrix elements are of the order of $Z^2e^6$, where as usual $Z$ is the atomic number of the target nucleus, and $e^2/4\pi = \alpha \approx 1/137.04$ is the fine-structure constant. For these diagrams, two photons are exchanged with the nucleus. If, more generally, $2n$ photons are exchanged with the nucleus, then the matrix element is of the order of $(Ze^2)^{2n}e^2$ for $n = 1, 2, 3, \ldots$. Since for heavy nuclei $Ze^2$ is not small, the effects of multiphoton exchange must be taken into account. At high energies, the inclusion of multiphoton exchange does not complicate much the basic formula, which was first given in Ref. 1. Since the matrix elements from the lowest-order diagrams have been analyzed in great detail in III, we are here concerned with the effects of multiphoton exchange, or in other words, the effects of Coulomb correction.

In this paper we shall follow closely the development in the later part of III. We emphasize that, here as well as in III, no artificial nonzero photon mass is ever introduced. The generalization to include multiphoton exchange is, however, far from being trivial. In fact, as discussed in Sec. 7, there is an important problem that we do not know how to solve.

The effect of multiphoton exchange is quite large for heavy elements. For example, as seen in Sec. 6, for a momentum transfer of several MeV/$c$, the resulting reduction in differential cross section can be more than a factor of 10 in the case of uranium.

## 2. HIGH-ENERGY AMPLITUDES

Let $\omega$ be the energy of the incident photon in the laboratory system; then, when $\omega$ is much larger than the electron mass $m$, the Delbrück amplitude is given asymptotically for fixed nonzero momentum transfer $\vec{\Delta} = 2\vec{r}_1$ by[1]

$$\mathfrak{M}^{(D)} \sim i\omega Z^2 e^2 (2\pi)^{-2} \int d\vec{q}_\perp \, [(\vec{q}_\perp + \vec{r}_1)^2]^{-1+i\alpha Z} [(\vec{q}_\perp - \vec{r}_1)^2]^{-1-i\alpha Z} \mathcal{g}^{\gamma}_{ij}(\vec{r}_1, \vec{q}_\perp) , \tag{2.1}$$

where $i$ and $j$ are respectively the directions of polarization for the incident and scattered photons, and $\mathcal{g}^{\gamma}_{ij}$ is the photon impact factor given by[1,2,4,5]

$$\mathcal{g}^{\gamma}_{ij}(\vec{r}_1, \vec{q}_\perp) = 8\alpha^2 \int_0^1 d\beta\, d\beta'\, \delta(1-\beta-\beta') \int_0^1 dx \left( \frac{8\beta^3\beta' x(1-x) r_{1i} r_{1j} - \beta^2 \vec{r}_1^{\,2} \delta_{ij}[1 - 8\beta\beta'(x-\frac{1}{2})^2]}{4\vec{r}_1^{\,2}\beta^2 x(1-x) + m^2} - \frac{8\beta\beta' x(1-x) Q_i Q_j - \vec{Q}^2 \delta_{ij}[1 - 8\beta\beta'(x-\frac{1}{2})^2]}{4\vec{Q}^2 x(1-x) + m^2} \right) , \tag{2.2}$$

with

$$\vec{Q} = \tfrac{1}{2}(\vec{q}_\perp + \vec{r}_1) - \beta \vec{r}_1 . \tag{2.3}$$

We are interested in the properties of $\mathfrak{M}^{(D)}$ as given by (2.1).

## 3. FEYNMAN PARAMETERS

In order to make use of the methods and results of III, it is necessary to introduce Feynman parameters.[8] Since

$$[(\vec{q}_\perp + \vec{r}_1)^2]^{-1+i\alpha Z} [(\vec{q}_\perp - \vec{r}_1)^2]^{-1-i\alpha Z} = (\pi Z\alpha)^{-1} \sinh(\pi Z\alpha) \times \int_0^1 d\alpha_5\, d\alpha_6\, \delta(1-\alpha_5-\alpha_6) [(\vec{q}_\perp + \vec{r}_1)^2 \alpha_5 + (\vec{q}_\perp - \vec{r}_1)^2 \alpha_6]^{-2} (\alpha_6/\alpha_5)^{iZ\alpha} , \tag{3.1}$$

the effects of multiphoton exchange are contained entirely in the factor

$$(\alpha_6/\alpha_5)^{iZ\alpha} (\sinh \pi Z\alpha)/(\pi Z\alpha) , \tag{3.2}$$

which approaches 1 as $Z \to 0$. Accordingly, by (3.1)–(3.5) of III, we can write down immediately that

$$\mathfrak{M}^{(D)} \sim \tfrac{1}{2} i (2\pi)^{-3} e^6 Z^2 \omega |t|^{-1} G , \tag{3.3}$$

$$G = -G_1 \delta_{ij} + G_2 r_{1i} r_{1j} / |\vec{r}_1^{\,2}| , \tag{3.4}$$

$$G_1 = G_1(\tau)$$

$$= [(\sinh \pi Z\alpha)/(\pi Z\alpha)] \int_0^1 d\beta \int_0^1 d\beta' \int_0^1 dx \int_0^1 d\alpha_6 \int_0^{\alpha_6} d\alpha_5 \cos[Z\alpha \ln(\alpha_6/\alpha_5)] \delta(1-\beta-\beta'-\alpha_5-\alpha_6)(\beta+\beta')^{-3}$$
$$\times \Bigg[ (\beta+\beta')^2 \tau^2 \Bigg( \frac{\beta^2(\alpha_5+\alpha_6)}{\{\tau[(\beta+\beta')\alpha_5\alpha_6 + \beta^2 x(1-x)(\alpha_5+\alpha_6)] + (\beta+\beta')^2(\alpha_5+\alpha_6)\}^2}$$
$$- \frac{(\beta'\alpha_5 - \beta\alpha_6)^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1}}{\{\tau[(\beta+\beta')\alpha_5\alpha_6 + x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)] + (\beta+\beta')^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]\}^2}\Bigg)$$
$$+\tau \Bigg( \frac{4\beta\beta'}{\tau[(\beta+\beta')\alpha_5\alpha_6 + \beta^2 x(1-x)(\alpha_5+\alpha_6)] + (\beta+\beta')^2(\alpha_5+\alpha_6)}$$
$$- \frac{[4\beta\beta'(\alpha_5+\alpha_6)+(\beta+\beta')^3][(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1}}{\tau[(\beta+\beta')\alpha_5\alpha_6 + x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)] + (\beta+\beta')^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]}\Bigg)\Bigg] \tag{3.5}$$

and

$$G_2 = G_2(\tau)$$

$$= [(\sinh \pi Z\alpha)/(\pi Z\alpha)] \int_0^1 d\beta \int_0^1 d\beta' \int_0^1 dx \int_0^1 d\alpha_6 \int_0^{\alpha_6} d\alpha_5 \cos[Z\alpha \ln(\alpha_6/\alpha_5)]\, \delta(1-\beta-\beta'-\alpha_5-\alpha_6)(\beta+\beta')^{-3} 8\beta\beta'$$
$$\times\; x(1-x)\tau^2 \Bigg( \frac{\beta^2(\alpha_5+\alpha_6)}{\{\tau[(\beta+\beta')\alpha_5\alpha_6 + \beta^2 x(1-x)(\alpha_5+\alpha_6)] + (\beta+\beta')^2(\alpha_5+\alpha_6)\}^2}$$
$$- \frac{(\beta'\alpha_5 - \beta\alpha_6)^2}{[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]\{\tau[(\beta+\beta')\alpha_5\alpha_6 + x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)] + (\beta+\beta')^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]\}^2}\Bigg). \tag{3.6}$$

In (3.5) and (3.6), $\tau = |t|/m^2$ and $t = -4\vec{r}_1^{\,2}$.

Equations (3.5) and (3.6) can be slightly simplified by the change of variables

$$\sigma = \beta+\beta', \quad z = \beta/(\beta+\beta'), \quad \text{and} \quad z' = \alpha_5/(\alpha_5+\alpha_6). \tag{3.7}$$

In terms of the matrix elements for perpendicular and parallel polarizations, the results are

$$\begin{pmatrix}\mathfrak{M}_\perp^{(D)} \\ \mathfrak{M}_\parallel^{(D)}\end{pmatrix} \sim -\tfrac{1}{2} i (2\pi)^{-3} e^6 Z^2 \omega [(\sinh \pi Z\alpha)/(\pi Z\alpha)] \int_0^1 dx \int_0^1 d\sigma \int_0^1 dz \int_0^{1/2} dz' \cos\left(Z\alpha \ln \frac{1-z'}{z'}\right)$$
$$\times \Bigg[ \frac{4\sigma^{-1} z(1-z)}{\tau[(1-\sigma)z'(1-z') + \sigma z^2 x(1-x)] + \sigma}$$
$$- \frac{(1-\sigma)[4\sigma^{-1}(1-\sigma)z(1-z)+1]}{[1-\sigma+\sigma x(1-x)]\{\tau[(1-\sigma)^2 z'(1-z') + x(1-x)\sigma(1-\sigma)(z^2-2zz'+z')] + \sigma[1-\sigma+\sigma x(1-x)]\}}$$
$$+\tau \begin{pmatrix} 1 \\ 1 - 8z(1-z)x(1-x)\end{pmatrix} \Bigg( \frac{z^2}{\{\tau[(1-\sigma)z'(1-z') + \sigma z^2 x(1-x)] + \sigma\}^2}$$
$$- \frac{(1-\sigma)^3(z-z')^2}{[1-\sigma+\sigma x(1-x)]\{\tau[(1-\sigma)^2 z'(1-z') + x(1-x)\sigma(1-\sigma)(z^2-2zz'+z')] + \sigma[1-\sigma+\sigma x(1-x)]\}^2}\Bigg)\Bigg]. \tag{3.8}$$

Numerical calculation on the basis of (3.8) is being carried out by members of F39 at DESY, especially Willutzki.

Using a method similar to that in III, we apply Mellin transformation[9] to study the behavior of $G_1$ and $G_2$ as given by (3.5) and (3.6), respectively:

$$\overline{G}_n(\zeta) = \int_0^\infty G_n(\tau)\tau^{-1-\zeta} d\tau \tag{3.9}$$

for $n = 1, 2$. This task is greatly facilitated by merely including in addition the factor (3.2), or rather its real part, in numerous formulas given in III. For example, from (3.18) and (3.19) of III, we get

$$\overline{G}_1(\zeta)=\pi\csc\pi\zeta[\Gamma(\zeta)]^2[\Gamma(2\zeta)]^{-1}[(\sinh\pi Z\alpha)/(\pi Z\alpha)]$$
$$\times\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\alpha_5^{-1}\alpha_6^{-1}\cos[Z\alpha\ln(\alpha_6/\alpha_5)]$$
$$\times\{[1-2\beta\beta'(1-2\zeta)^{-1}][\beta^{2\zeta}-\pi\zeta(\csc\pi\zeta)(1-\zeta)(\alpha_5\alpha_6)^{-1+\zeta}(\beta^2\alpha_6+\beta'^2\alpha_5)F(2-\zeta,1-\zeta;2;z)]$$
$$-\pi\zeta^2(\csc\pi\zeta)(\alpha_5\alpha_6)^{\zeta}[1-4\beta\beta'(1+2\zeta)^{-1}]F(1-\zeta,1-\zeta;2;z)\} \quad (3.10)$$

and

$$\overline{G}_2(\zeta)=\pi\csc\pi\zeta[\Gamma(1+\zeta)]^2[\Gamma(2+2\zeta)]^{-1}[(\sinh\pi Z\alpha)/(\pi Z\alpha)]$$
$$\times\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)8\beta\beta'\alpha_5^{-1}\alpha_6^{-1}\cos[Z\alpha\ln(\alpha_6/\alpha_5)]$$
$$\times[\beta^{2\zeta}-\tfrac{1}{2}(1-\zeta^2)\pi\zeta(\csc\pi\zeta)(\alpha_5\alpha_6)^{-1+\zeta}(\beta'\alpha_5-\beta\alpha_6)^2F(2-\zeta,1-\zeta;3;z)], \quad (3.11)$$

where

$$z=-(\beta'\alpha_5-\beta\alpha_6)^2/(\alpha_5\alpha_6). \quad (3.12)$$

## 4. BEHAVIOR FOR $\omega\gg\Delta\gg m$

As in III, we want to study the behavior of the scattering amplitudes in the two extreme cases $\omega\gg m\gg\Delta$ and $\omega\gg\Delta\gg m$. Since the former case is completely understood,[10] we concentrate on the latter one.

We need to find, for this purpose, the properties of $\overline{G}_1(\zeta)$ and $\overline{G}_2(\zeta)$ in the neighborhood of $\zeta=0$. By (3.26), (3.27), and (3.31) of III, we know that, for $\zeta\approx 0$,

$$\overline{G}_1(\zeta)=\zeta^{-1}(G_3+G_4+G_5)+O(1) \quad (4.1)$$

and

$$\overline{G}_2(\zeta)=2\zeta^{-1}G_3+O(1), \quad (4.2)$$

where $G_3$, $G_4$, and $G_5$ are given by the integrals

$$G_3=4[(\sinh\pi Z\alpha)/(\pi Z\alpha)]\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)$$
$$\times\cos[Z\alpha\ln(\alpha_6/\alpha_5)]\beta\beta'(\beta'\alpha_5-\beta\alpha_6)^{-2}\ln[\alpha_5^{-1}\alpha_6^{-1}(\beta^2\alpha_6+\beta'^2\alpha_5)], \quad (4.3)$$

$$G_4=4[(\sinh\pi Z\alpha)/(\pi Z\alpha)]\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)$$
$$\times\cos[Z\alpha\ln(\alpha_6/\alpha_5)]\beta\beta'\alpha_5^{-1}\alpha_6^{-1}\ln[\beta^{-2}(\beta^2\alpha_6+\beta'^2\alpha_5)], \quad (4.4)$$

and

$$G_5=-2[(\sinh\pi Z\alpha)/(\pi Z\alpha)]\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)$$
$$\times\cos[Z\alpha\ln(\alpha_6/\alpha_5)]\alpha_5^{-1}\alpha_6^{-1}\ln[\beta^{-2}(\beta^2\alpha_6+\beta'^2\alpha_5)]. \quad (4.5)$$

We shall show that these integrals can all be carried out explicitly and the results are

$$G_3=2[1-Z\alpha\,\mathrm{Im}\psi'(1-iZ\alpha)], \quad (4.6)$$

$$G_4=(3Z\alpha)^{-1}[(Z\alpha)^{-1}-2\pi(1+4Z^2\alpha^2)\,\mathrm{csch}(2\pi Z\alpha)], \quad (4.7)$$

and

$$G_5=-(Z\alpha)^{-1}[(Z\alpha)^{-1}-2\pi\,\mathrm{csch}(2\pi Z\alpha)]. \quad (4.8)$$

In (4.6), $\psi'$ is the second derivative of the logarithm of the $\Gamma$ function, and is sometimes called the trigamma function. By (4.6)–(4.8), we get

$$\mathfrak{M}_{\perp}^{(D)} \sim -i(2\pi)^{-2}e^4 Z\omega\Delta^{-2}\left[-\tfrac{2}{3}(1-3Z^2\alpha^2)(Z\alpha)^{-1}+\tfrac{1}{3}(4\pi)(1-2Z^2\alpha^2)\operatorname{csch}(2\pi Z\alpha)-2Z^2\alpha^2\operatorname{Im}\psi'(1-iZ\alpha)\right] \tag{4.9}$$

and

$$\mathfrak{M}_{\parallel}^{(D)} \sim i(2\pi)^{-2}e^4 Z\omega\Delta^{-2}\left[-\tfrac{2}{3}(1+3Z^2\alpha^2)/(Z\alpha)+\tfrac{1}{3}(4\pi)(1-2Z^2\alpha^2)\operatorname{csch}(2\pi Z\alpha)+2Z^2\alpha^2\operatorname{Im}\psi'(1-iZ\alpha)\right] \tag{4.10}$$

for $\omega \gg \Delta \gg m$.

We give below a derivation of (4.6)–(4.8) from (4.3)–(4.5), but this derivation may not be the best. Let us consider $G_4$ and $G_5$ first; as a first step, we rewrite (4.4) and (4.5) in the form

$$G_4 = -2(\pi Z\alpha)^{-1}\operatorname{Re}\int_0^1 d\beta\int_C d\alpha_5\{[\alpha_5/(\alpha_5-1)]^{iZ\alpha}-1\}\,\beta(1-\beta)\alpha_5^{-1}\ln(\beta^{-2}[\beta^2+(1-2\beta)\alpha_5]) \tag{4.11}$$

and

$$G_5 = (\pi Z\alpha)^{-1}\operatorname{Re}\int_0^1 d\beta\int_C d\alpha_5\{[\alpha_5/(\alpha_5-1)]^{iZ\alpha}-1\}\,\alpha_5^{-1}\ln(\beta^{-2}[\beta^2+(1-2\beta)\alpha_5])\,. \tag{4.12}$$

In (4.11) and (4.12), the contour $C$ is around the branch cut from 0 to 1, as shown in Fig. 1. In both cases, this contour $C$ of integration can be shifted so that it wraps around the branch cut due to the logarithmic factor $\ln[\beta^2+(1-2\beta)\alpha_5]$. This logarithmic branch cut is from $-\infty$ to $-\beta^2/(1-2\beta)$ if $\beta<\frac{1}{2}$ and is from $\beta^2/(2\beta-1)$ to $+\infty$ if $\beta>\frac{1}{2}$. Therefore,

$$G_4 = -2(\pi Z\alpha)^{-1}\operatorname{Re}(2\pi i)\int_0^1 d\beta\int_{\mp\infty}^{-\beta^2/(1-2\beta)} d\alpha_5\{[\alpha_5/(\alpha_5-1)]^{iZ\alpha}-1\}\,\beta(1-\beta)\alpha_5^{-1} \tag{4.13}$$

and

$$G_5 = -2(\pi Z\alpha)^{-1}\operatorname{Re}(2\pi i)\int_0^1 d\beta\int_{\mp\infty}^{-\beta^2/(1-2\beta)} d\alpha_5\{[\alpha_5/(\alpha_5-1)]^{iZ\alpha}-1\}\,\alpha_5^{-1}\,, \tag{4.14}$$

where the $-$ sign in the range of integration for $\alpha_5$ applies for $\beta<\frac{1}{2}$ and the $+$ sign for $\beta>\frac{1}{2}$. These formulas can be slightly simplified with the change of variable

$$\alpha' = \alpha_5/(\alpha_5-1)\,, \tag{4.15}$$

and the results are

$$G_4 = 2(\pi Z\alpha)^{-1}\operatorname{Re}(2\pi i)\int_0^1 d\beta\int_{\beta^2/(1-\beta)^2}^1 d\alpha'\,\alpha'^{-1}(1-\alpha')^{-1}\beta(1-\beta)(\alpha'^{iZ\alpha}-1) \tag{4.16}$$

and

$$G_5 = -(\pi Z\alpha)^{-1}\operatorname{Re}(2\pi i)\int_0^1 d\beta\int_{\beta^2/(1-\beta)^2}^1 d\alpha'\,\alpha'^{-1}(1-\alpha')^{-1}(\alpha'^{iZ\alpha}-1)\,. \tag{4.17}$$

The remainder of the calculation for these two cases is completely straightforward: We merely integrate by parts with respect to $\beta$ and carry out the necessary elementary integrals. Equations (4.7) and (4.8) result.

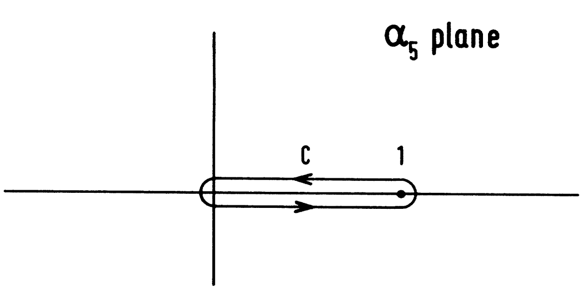


FIG. 1. The contour $C$ of integration around the branch cut from 0 to 1.

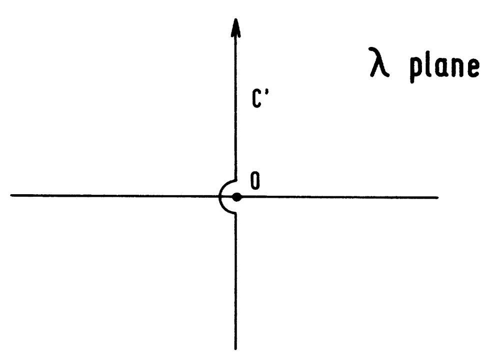


FIG. 2. The contour $C'$ of integration for the inverse Mellin transform of the step function.

We choose not to apply this procedure directly to $G_3$, because the structure of branch cuts in the $\alpha_5$ plane is somewhat more complicated. Instead we find it expedient to write, with a change of variable,

$$G_3 = G_{31} + G_{32}, \tag{4.18}$$

where

$$G_{31} = -4[(\sinh\pi Z\alpha)/(\pi Z\alpha)]\,\mathrm{Re}\int_0^1 d\alpha_5\,[\alpha_5/(1-\alpha_5)]^{iZ\alpha}[\alpha_5(1-\alpha_5)]^{1/2}\int_0^{[\alpha_5/(1-\alpha_5)]^{1/2}} dx\,(1-x^{-2})\ln(1+x^2) \tag{4.19}$$

and

$$G_{32} = -4[(\sinh\pi Z\alpha)/(\pi Z\alpha)]\,\mathrm{Re}\int_0^1 d\alpha_5\,[\alpha_5/(1-\alpha_5)]^{iZ\alpha}(1-2\alpha_5)\int_0^{[\alpha_5/(1-\alpha_5)]^{1/2}} dx\,x^{-1}\ln(1+x^2). \tag{4.20}$$

The $x$ integration in (4.19) is elementary, and thus

$$G_{31} = -4[\psi(1) - \mathrm{Re}\,\psi(1-iZ\alpha)]. \tag{4.21}$$

The evaluation of $G_{32}$ is more complicated. One way is to use the integral representation of the step function

$$(-2\pi i)^{-1}\int_{C'} x^{\lambda}\,d\lambda/\lambda = \begin{cases} 0 & \text{if } x>1 \\ 1 & \text{if } x<1 \end{cases} \tag{4.22}$$

where the contour $C'$ of integration is shown in Fig. 2. Thus

$$G_{32} = -4[(\sinh\pi Z\alpha)/(\pi Z\alpha)]\,\mathrm{Re}\int_{C'} d\lambda\,\lambda^{-1}\int_0^1 d\alpha_5\,[\alpha_5/(1-\alpha_5)]^{iZ\alpha}(1-2\alpha_5)\int_0^{\infty} dx\,x^{-1}\ln(1+x^2)[x(1-\alpha_5)^{1/2}\alpha_5^{-1/2}]^{\lambda}. \tag{4.23}$$

In this form, all the integrations can be evaluated, and we get

$$G_{32} = 2 + 4[\psi(1) - \mathrm{Re}\,\psi(1-iZ\alpha)] - 2Z\alpha\,\mathrm{Im}\,\psi'(1-iZ\alpha). \tag{4.24}$$

Equation (4.6) follows from (4.20), (4.21), and (4.24).

## 5. SERIES EXPANSION IN $Z\alpha$

As an indication of the importance of multiphoton exchange, we expand the right-hand sides of (4.9) and (4.10) into power series in $Z\alpha$. Since[11]

$$\psi(1+z) = -\gamma + \sum_{n=1}^{\infty}(-1)^n\zeta(n)z^{n-1}, \tag{5.1}$$

we get

$$\mathrm{Im}\,\psi'(1-iZ\alpha) = 2\sum_{n=1}^{\infty}(-1)^{n-1}n\zeta(2n+1)(Z\alpha)^{2n-1}. \tag{5.2}$$

Here $\zeta$ is the Riemann $\zeta$ function and $\gamma$ is Euler's constant. Furthermore

$$\mathrm{csch}\,z = -\sum_{n=0}^{\infty} 2(2^{2n-1}-1)\mathrm{B}_{2n}z^{2n-1}/(2n)!, \tag{5.3}$$

where $\mathrm{B}_{2n}$ are the Bernoulli numbers. Because of the appearance of $\zeta$ in (5.2), it is desirable to use the relation[12]

$$\mathrm{B}_{2n} = (-1)^{n+1}(2\pi)^{-2n}2(2n)!\,\zeta(2n) \tag{5.4}$$

to get the expansion

$$\mathrm{csch}(2\pi Z\alpha) = (\pi Z\alpha)^{-1}\sum_{n=0}^{\infty}(-1)^n 2(2^{2n-1}-1)\zeta(2n)(Z\alpha)^{2n}. \tag{5.5}$$

Write

$$\mathfrak{M}_{\perp}^{(D)} = \sum_{n=0}^{\infty}\mathfrak{M}_{\perp n} \tag{5.6}$$

and

$$\mathfrak{M}_{\parallel}^{(D)}=\sum_{n=0}^{\infty}\mathfrak{M}_{\parallel n}\,, \tag{5.7}$$

where $\mathfrak{M}_{\perp n}$ and $\mathfrak{M}_{\parallel n}$ are both proportional to $Z^2\alpha^3(Z\alpha)^{2n}$. Then it follows from (4.9) and (4.10) that, for $\omega \gg \Delta \gg m$,

$$\mathfrak{M}_{\perp 0}\sim \tfrac{1}{3}i(2\pi)^{-3}e^6Z^2\omega\Delta^{-2}(\tfrac{2}{3}\pi^2-1)\,, \tag{5.8}$$

$$\mathfrak{M}_{\parallel 0}\sim \tfrac{1}{3}i(2\pi)^{-3}e^6Z^2\omega\Delta^{-2}(\tfrac{2}{3}\pi^2+5)\,, \tag{5.9}$$

$$\mathfrak{M}_{\perp n}\sim -\tfrac{1}{3}i(2\pi)^{-3}e^6Z^2\omega\Delta^{-2}(-1)^n(Z\alpha)^{2n}[4(2^{2n+1}-1)\zeta(2n+2)-6n\zeta(2n+1)+8(2^{2n-1}-1)\zeta(2n)]\,, \tag{5.10}$$

and

$$\mathfrak{M}_{\parallel n}\sim \tfrac{1}{3}i(2\pi)^{-3}e^6Z^2\omega\Delta^{-2}(-1)^n(Z\alpha)^{2n}[4(2^{2n+1}-1)\zeta(2n+2)+6n\zeta(2n+1)+8(2^{2n-1}-1)\zeta(2n)]\,, \tag{5.11}$$

for $n \geq 1$. Equations (5.8) and (5.9) agree with (4.4) and (4.5) of III.

## 6. SOME NUMERICAL RESULTS

To show the importance of multiphoton exchange, we plot in Fig. 3 the ratio $\mathfrak{M}_{\perp}^{(D)}/\mathfrak{M}_{\perp 0}$ and $\mathfrak{M}_{\parallel}^{(D)}/\mathfrak{M}_{\parallel 0}$ as given by (4.9), (4.10), (5.8), and (5.9) for $\omega \gg \Delta \gg m$. It is seen that the matrix elements are *decreased* in each case. In particular, for uranium

$$\mathfrak{M}_{\perp}^{(D)}/\mathfrak{M}_{\perp 0}\approx 0.1693$$

and (6.1)

$$\mathfrak{M}_{\parallel}^{(D)}/\mathfrak{M}_{\parallel 0}\approx 0.2969\,.$$

Furthermore, the ratio is always smaller for perpendicular polarization. Therefore, the polarization $P$ of the scattered beam, as given by

$$P=(|\mathfrak{M}_{\parallel}^{(D)}|^2-|\mathfrak{M}_{\perp}^{(D)}|^2)/(|\mathfrak{M}_{\parallel}^{(D)}|^2+|\mathfrak{M}_{\perp}^{(D)}|^2)\,, \tag{6.2}$$

is *increased*. In Fig. 4, we show this polarization for $\omega \gg \Delta \gg m$ as a function of $Z$. In particular, for uranium

$$P\approx 86.96\%\,, \tag{6.3}$$

compared with $P\approx 62.32\%$ for hydrogen [see Eq. (4.6) of III].

The effect is more drastic for differential cross sections. In Fig. 5 we show the reduction of dif-

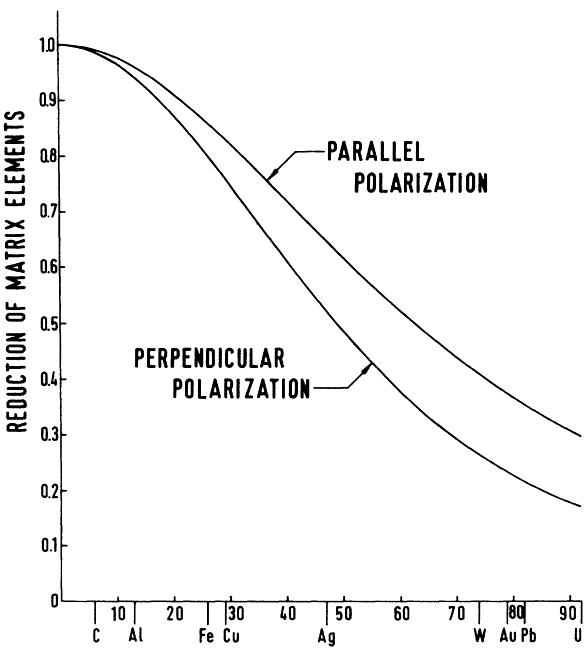


FIG. 3. Reduction of the matrix elements for Delbrück scattering due to multiphoton exchange. Note that neither matrix element changes sign.

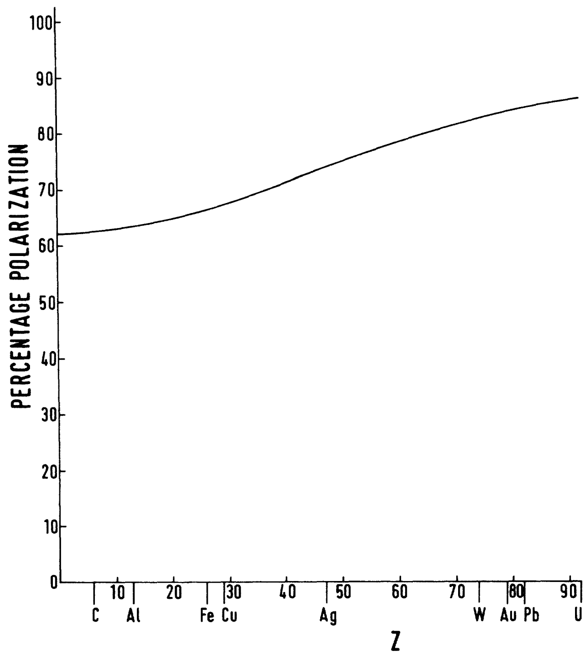


FIG. 4. Polarization of scattered photon at high energies when the incident photon is unpolarized and the momentum transfer is much larger than $mc$.

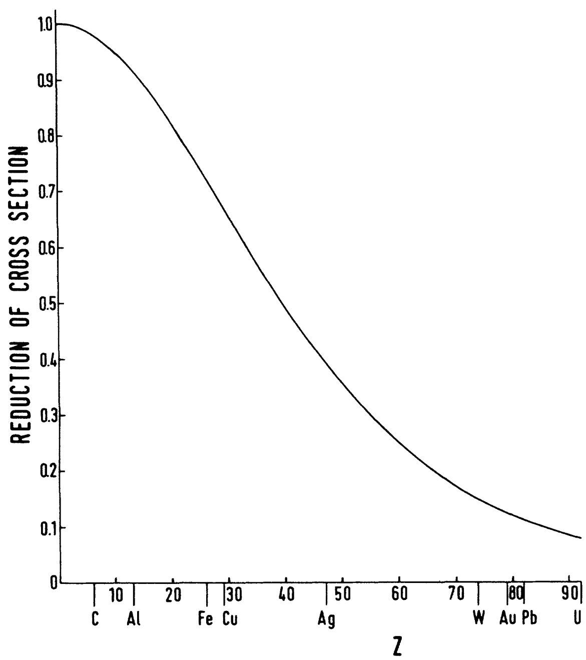


FIG. 5. Reduction of the differential cross section for Delbrück scattering due to multiphoton exchange.

ferential cross sections given by

$$(|\mathfrak{M}_{\parallel}^{(D)}|^2 + |\mathfrak{M}_{\perp}^{(D)}|^2)/(|\mathfrak{M}_{\parallel 0}|^2 + |\mathfrak{M}_{\perp 0}|^2) \qquad (6.4)$$

for $\omega \gg \Delta \gg m$. For uranium, this ratio is 0.076 96, i.e., a reduction by a factor of 13.

## 7. DISCUSSION

We have shown that the effect of multiphoton exchange is quite large for Delbrück scattering. In order to use either the simple results (4.9) and (4.10) for $\omega \gg \Delta \gg m$ or the more complete (3.8) for the purpose of comparison with experiments, the following points must be kept in mind.

(i) The size of the nucleus has not been taken into account. Theoretically, it is not difficult to include this effect[13] but the resulting formulas are more complicated to evaluate numerically.

(ii) The matrix elements discussed here refer only to elastic scattering. For example, if the target is lead, we have studied the process

$$\gamma + \mathrm{Pb} \to \gamma + \mathrm{Pb}. \qquad (7.1)$$

In particular, in (7.1) the lead nucleus remains in its ground state. Experimentally, it is rather difficult to ascertain that the nucleus is neither excited nor broken up when the momentum transfer is more than a few MeV/$c$. One possibility, as suggested by Ting,[14] is to use different isotopes of the nucleus as the target. With possibly a small correction due to nucleus size, different lead isotopes, for example, give the same Delbrück cross section, although the levels of the nucleus can be quite different.

A possible *practical application* of Delbrück scattering is to produce a beam of linearly polarized photons. The value given by (6.3) seems quite encouraging for this purpose.

We conclude with a few more theoretical remarks.

(i) The series expansion given in Sec. 5 converges if and only if

$$Z\alpha < \tfrac{1}{2}. \qquad (7.2)$$

In particular, this means that for heavy elements the convergence of the perturbation expansion in $Z\alpha$ for Delbrück scattering is non-uniform. Thus Delbrück scattering at a momentum transfer of a few MeV/$c$ provides an excellent way of testing perturbation calculations of very high orders. Just how high the order is must be determined by numerical calculation from (3.8).

(ii) Strictly speaking, the condition $\omega \gg \Delta \gg m$ for (4.9), (4.10), and many other equations is not precise enough. These results are valid when $\omega/\Delta$ is very large and $\Delta/m$ is also large. That is, (4.9) for example should be written as

$$\lim_{\Delta\to\infty}\lim_{\omega\to\infty} \mathfrak{M}_{\perp}^{(D)}/[-i(2\pi)^{-2}e^4 Z\omega\Delta^{-2}] = -\tfrac{2}{3}(1-3Z^2\alpha^2)(Z\alpha)^{-1} + \tfrac{1}{3}(4\pi)(1-2Z^2\alpha^2)\,\mathrm{csch}(2\pi Z\alpha) - 2Z^2\alpha^2\,\mathrm{Im}\,\psi'(1-iZ\alpha). \qquad (7.3)$$

(iii) There is a most interesting unsolved theoretical problem. Since the asymptotic expressions (4.9) and (4.10) are analytic at $Z\alpha=0$, we ask whether this is also the case for the next order terms, i.e., terms down by a factor $\Delta^{-2}$ with possibly additional logarithmic factors. In the Appendix we study these terms assuming that $Z\alpha$ is fixed and positive. A comparison with the results of III then shows that these terms are different depending on whether $Z\alpha$ is zero or positive. This means that we do not know what the correction terms to (4.9) and (4.10) are. What is needed is the next term in the asymptotic expansion of (3.8) for large $\Delta$ but uniform in $Z\alpha$. Even though we are most anxious to get this term, we are so far unable to make any progress because Mellin transformation cannot be used for this purpose.

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## APPENDIX

In this Appendix, we study briefly one of the leading correction terms to (4.9) and (4.10), assuming that $Z\alpha$ is of the order of unity. For this purpose, we need to find the behavior of $\overline{G}_1(\zeta)$ and $\overline{G}_2(\zeta)$ for $\zeta$ in the neighborhood of $-1$.

Let

$$\xi = \zeta + 1\,, \tag{A1}$$

then by (D1), (D2), (C1), and (C2) of III, we have

$$\overline{G}_1(\zeta) = -\pi \csc\pi\xi[\Gamma(-1+\xi)]^2[\Gamma(-2+2\xi)]^{-1}[(\sinh\pi Z\alpha)/(\pi Z\alpha)]H_1(\xi) \tag{A2}$$

and

$$\overline{G}_2(\zeta) = -\pi \csc\pi\xi[\Gamma(\xi)]^2[\Gamma(2\xi)]^{-1}[(\sinh\pi Z\alpha)/(\pi Z\alpha)]H_2(\xi)\,, \tag{A3}$$

where

$$\begin{aligned} H_1(\xi) = \int_0^1 d\beta \int_0^1 d\beta' \int_0^1 d\alpha_6 \int_0^{1/2} d\alpha_5 \delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\, \mathrm{Re}(\alpha_6/\alpha_5)^{iZ\alpha}\alpha_5{}^{-3+\xi}\alpha_6{}^{-3+\xi} \\ \times \{[1+2\beta\beta'(1-2\xi)^{-1}][\beta^{-2+2\xi}(\alpha_5\alpha_6)^{2-\xi}\pi\csc\pi\xi(1-\xi)(2-\xi)(\beta^2\alpha_6+\beta'^2\alpha_5)F(3-\xi,\ 2-\xi;2;z)] \\ +\pi\csc\pi\xi(1-\xi)^2\alpha_5\alpha_6[1+4\beta\beta'(1-2\xi)^{-1}]F(2-\xi,2-\xi;2;z)\} \end{aligned} \tag{A4}$$

and

$$\begin{aligned} H_2(\xi) = \int_0^1 d\beta \int_0^1 d\beta' \int_0^1 d\alpha_6 \int_0^{1/2} d\alpha_5 \delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\, \mathrm{Re}(\alpha_6/\alpha_5)^{iZ\alpha} 8\beta\beta'\alpha_5{}^{-1}\alpha_6{}^{-1} \\ \times[\beta^{-2+2\xi} - \tfrac{1}{2}\Gamma(1+\xi)\Gamma(3+\xi)(\beta'\alpha_5-\beta\alpha_6)^2(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi}F(\xi,2-\xi;3;\overline{z})]. \end{aligned} \tag{A5}$$

In (A4), $z$ is given by (3.12); in (A5),

$$\overline{z} = (\beta'\alpha_5 - \beta\alpha_6)^2/(\beta^2\alpha_6 + \beta'^2\alpha_5)\,. \tag{A6}$$

In the previous case of III where the factor $\mathrm{Re}(\alpha_6/\alpha_5)^{iZ\alpha}$ is absent, $H_2(\xi)$, for example, is of the order of $\xi^{-2}$ for small $\xi$: one power comes from $\beta \approx 0$ and the other power from $\alpha_5 \approx \beta^2 \approx 0$. With this additional factor $\mathrm{Re}(\alpha_6/\alpha_5)^{iZ\alpha}$, the $H_2(\xi)$ of (A5) is instead of the order of $\xi^{-1}$ at most. This is the basic reason for the complication discussed at the end of Sec. 7.

We shall study only the $H_2(\xi)$ of (A5). By (C.8)–(C.11) of III, we have

$$H_2(\xi) \sim H_{21}(\xi) + \xi H_{22}(\xi)\,, \tag{A7}$$

where

$$\begin{aligned} H_{21}(\xi) = \int_0^1 d\beta \int_0^1 d\beta' \int_0^1 d\alpha_6 \int_0^{1/2} d\alpha_5\, \delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\, \mathrm{Re}(\alpha_6/\alpha_5)^{iZ\alpha} \\ \times 8\beta\beta'\alpha_5{}^{-1}\alpha_6{}^{-1}[\beta^{-2+2\xi} - (\beta'\alpha_5-\beta\alpha_6)^2(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi}] \end{aligned} \tag{A8}$$

and

$$\begin{aligned} H_{22}(\xi) = \int_0^1 d\beta \int_0^1 d\beta' \int_0^1 d\alpha_6 \int_0^{1/2} d\alpha_5\, \delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\, \mathrm{Re}(\alpha_6/\alpha_5)^{iZ\alpha} 8\beta\beta'(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi} \\ \times\{-1+[1+(\beta'\alpha_5-\beta\alpha_6)^{-2}(\beta^2\alpha_6+\beta'^2\alpha_5)]\ln[\alpha_5{}^{-1}\alpha_6{}^{-1}(\beta^2\alpha_6+\beta'^2\alpha_5)]\}. \end{aligned} \tag{A9}$$

We need to obtain terms of the order of $\xi^{-1}$, 1, and $\xi$ for $H_{21}(\xi)$, and the term of the order of 1 for $H_{22}(\xi)$. In particular, we can just set $\xi = 0$ on the right-hand side of (A9).

We shall not calculate all these terms; instead, we shall get only the leading term. Hence it is sufficient to consider $H_{21}(\xi)$. Unlike the case previously treated in III, the two terms on the right-hand side of (A8) are separately meaningful. In fact, the term $\beta^{-2+2\xi}$ gives no contribution because, for $Z\alpha\neq 0$,

$$\int_0^1 d\alpha_5\, \alpha_5^{-1}\alpha_6^{-1}(\alpha_5/\alpha_6)^{iZ\alpha}=0\,. \tag{A10}$$

Therefore, $H_{21}(\xi)$ does not contain any term of order $\xi^{-1}$, and we concentrate on the term of order 1:

$$H_{21}(\xi)\sim H_{21}(0)=-\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\,\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\,\mathrm{Re}(\alpha_6/\alpha_5)^{iZ\alpha}$$
$$\times 8\beta\beta'\alpha_5^{-1}\alpha_6^{-1}(\beta'\alpha_5-\beta\alpha_6)^2(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2}\,. \tag{A11}$$

If the contour $C$ of Fig. 1 is again used, we get

$$H_{21}(0)=4\,\mathrm{csch}(\pi Z\alpha)\int_0^1 d\beta\,\beta(1-\beta)\,\mathrm{Re}\int_C d\alpha_5\{[\alpha_5/(\alpha_5-1)]^{iZ\alpha}-1\}\,\alpha_5^{-1}\{(\beta-\alpha_5)^2[\beta^2+(1-2\beta)\alpha_5]^{-2}-\beta^{-2}\}$$
$$=4\,\mathrm{csch}(\pi Z\alpha)2\pi\int_0^1 d\beta\,\beta(1-\beta)\,\mathrm{Im}[\beta/(1-\beta)]^{2iZ\alpha}[(1-2\beta)^{-2}-\beta^{-2}+iZ\alpha\beta^{-2}(1-2\beta)^{-1}]$$
$$=8\pi\,\mathrm{csch}(\pi Z\alpha)\int_0^1 d\beta\{-[\beta/(1-\beta)]^{2iZ\alpha-1}+\tfrac{1}{2}(\partial/\partial\beta)(1-\beta)^2(1-2\beta)^{-1}[\beta/(1-\beta)]^{2iZ\alpha}\}$$
$$=8\pi^2\,\mathrm{csch}(\pi Z\alpha)\,\mathrm{csch}(2\pi Z\alpha)\,. \tag{A12}$$

In particular, it is interesting to note that

$$H_{21}(0)=4(Z\alpha)^{-2}+O(1)\quad\text{as}\quad Z\alpha\to 0\,. \tag{A13}$$

Accordingly, by (A2), for $\zeta$ near $-1$

$$\overline{G}_2(\zeta)=-16\pi(Z\alpha)^{-1}\,\mathrm{csch}(2\pi Z\alpha)(\zeta+1)^{-2}+O((\zeta+1)^{-1})\,. \tag{A14}$$

When (A14) is combined with (4.2) and (4.6), we get

$$G_2(\tau)=4[1-Z\alpha\,\mathrm{Im}\psi'(1-iZ\alpha)]-16\pi(Z\alpha)^{-1}\,\mathrm{csch}(2\pi Z\alpha)\tau^{-1}\ln\tau+O(\tau^{-1}) \tag{A15}$$

for large $\tau$ with fixed $Z\alpha\neq 0$. Numerically, for uranium, (A15) is approximately

$$G_2(\tau)\sim 2.493-41.97\tau^{-1}\ln\tau+O(\tau^{-1})\,. \tag{A16}$$

Thus, in this case, the second term gives a contribution of 26.1%, 13.2%, 8.1%, and 4.0%, respectively, for $\Delta=10$, 15, 20, and 30 MeV/$c$.

We are now in a position to state more explicitly the unsolved problem of (iii) near the end of Sec. 7. Define, by (4.1), (4.2), and (4.6)–(4.8),

$$G_1(\ln\tau, Z\alpha)=\tau\{G_1(\tau)-(Z\alpha)^{-1}[-\tfrac{2}{3}(1-3Z^2\alpha^2)(Z\alpha)^{-1}$$
$$+\tfrac{1}{3}(4\pi)(1-2Z^2\alpha^2)\,\mathrm{csch}(2\pi Z\alpha)-2Z^2\alpha^2\,\mathrm{Im}\psi'(1-iZ\alpha)]\} \tag{A17}$$

and

$$G_2(\ln\tau, Z\alpha)=\tau\{G_2(\tau)-4[1-Z\alpha\,\mathrm{Im}\psi'(1-iZ\alpha)]\}\,. \tag{A18}$$

What are the behaviors of $G_1(x,y)$ and $G_2(x,y)$ as $x\to\infty$? In particular, by (A13), what are the behaviors of $G_1(x,y)$ and $G_2(x,y)$ for fixed $xy$ as $x\to\infty$?

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# Field Theory for Stable and Unstable Particles*

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A completely covariant field theory is developed which includes both stable and unstable particle fields. Exact single-particle propagators for both the unstable and stable cases are derived for arbitrary spin in terms of matrix elements of the basic interaction. The free-particle approximation to these propagators does not contain the unphysical terms which are usually present in the propagators derived in the interaction picture. The relationship to the Lehmann spectral representation is established and general equations for the various renormalization constants are given. Based upon general considerations it is shown that in the limit of high momentum transfer an extra factor $t^{-2}$ occurs in the cross section for 2-particle-to-2-particle scattering, more in line with experimental observations.

## I. INTRODUCTION

In a previous paper,[1] in an investigation of the Lee model, the authors showed that a $V$-particle state is well defined even though no stable $V$-particle state exists as an in- or out-state. This state, which can be simply described as

$$|V(p,t)\rangle = e^{-iHt}|p\rangle, \quad (1)$$

where $H$ is the exact Hamiltonian and $|p\rangle$ is the renormalized "mathematical" $V$-particle state, is shown to be the scattered-wave part of the exact $N, \theta$ scattering solution thereby relating the unstable state to the production process and, therefore, to the stable in-states of the model. In the large-time limit, corresponding to an out-state for the stable case, $|V(p,t)\rangle_{t\to\infty}$ approaches the exact $V$-particle eigenstate of $H$. In the unstable case, for large mean life $\Gamma$ and $\Gamma t \ll 1$,

$$\begin{aligned}&\lim_{t\to\infty} |\langle p|V(p,t)\rangle|^2 \sim e^{-\Gamma t} + O(t^{-3/2}),\\ &\lim_{t\to\infty}\int \frac{d\vec{k}}{2\omega}|\langle k, p-k|V(p,t)\rangle|^2 \sim (1-e^{-\Gamma t}) + O(t^{-3/2}),\end{aligned} \quad (2)$$

where $|k, p-k\rangle$ is the $N, \theta$ in-state, precisely what one expects for the time dependence for an unstable state and its decay products. Thus, for the Lee model there appears to be no difficulty in extending the usual field-theoretical approach to include a discussion of "asymptotic" states, rather than just in- or out-states, thereby including the possibility for a description of an unstable particle within the framework of the theory.

In this paper the authors extend the analysis used on the Lee model to a general relativistically invariant field theory that includes stable as well as unstable particles. Covariant Heisenberg field operators are defined which create single-particle states with the properties of the $V$-particle state given in Eqs. (1) and (2). Expressions are derived, in terms of the basic matrix elements of the interaction, for exact propagators for particles of arbitrary spin for both the stable and unstable cases. Since the calculations are carried out in the Heisenberg representation rather than the interaction representation, no unphysical contact terms arise in these expressions. Along the way expressions are also obtained for the various renormalization constants.

A sample calculation of the $S$ matrix for the process 2 particles in and 2 particles out is done as an illustration. In the limit of high momentum transfer, for both stable- and unstable-particle exchange, extra factors of the momentum transfer appear which depress the cross section over the usual Born-approximation results.

The approach presented differs from that of other authors[2,3] who describe a field theory of un-