Spontaneous Violation of Dilatation Invariance in Relativistic Field Theories*

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Adopting the point of view that the mass of observed particles is dynamical in origin, the effect of spontaneous breakdown of dilatation invariance is investigated. Neither extra fields nor anomalous dimensions are introduced from the outset so that the Lagrangians in our formalism are strictly invariant against dilatation transformation. The Bethe-Salpeter equation is employed throughout in order to demonstrate that, in the pair approximation, the appearance of massive fields does not contradict the formal invariance of the theory. The result is interpreted in terms of the dimensional transformation previously introduced by the authors. It is shown that the dilatation transformation for the Heisenberg operators turns into the dimensional transformation for the massive asymptotic fields and the inhomogeneous dilatation transformation for the Goldstone boson, and that the very *appearance of the inhomogeneous term cancels the extra dimension brought in by the mass*. A Goldberger-Treiman-like relation is derived. Finally, it is pointed out that the composite nature of the Goldstone boson in one model (Sec. II) seems to be the distinctive feature leading to a nontrivial scattering matrix.

I. INTRODUCTION

It is an attractive speculation that the apparent finite mass of observed particles may be due to the spontaneous breakdown of the dilatation invariance, and the dilaton which emerges as the Goldstone boson may be identified with the graviton. The objectionable aspects of this conjecture may be (1) the lack of satisfactory theory of the graviton from the quantum-field-theoretical viewpoint, (2) the apparent discrepancy of spin between graviton and the Goldstone particle,[1] and (3) the difficulty in accounting for the nonzero mass within the scheme of dilatation invariance. It is not our purpose to propose a solution of these basic problems, but rather to make a modest attempt at clarifying the third problem, the least fundamental one of all.

It has been suggested[2] from a purely kinematical viewpoint that the tracelessness of the energy-momentum tensor does not necessarily contradict the nonvanishing mass of material particles if the massless Goldstone particle is introduced. We shall take particular models and confirm the above point by an explicit dynamical calculation.[3] Section II, which will be devoted to the $\lambda\phi^4$ theory, is divided into six subsections. In the first two subsections, we define the model and consider the spontaneous breakdown of the dilatation invariance. In discussing the breakdown of dilatation symmetry, it is not appropriate to consider

$$\langle 0|\phi|0\rangle \neq 0 \tag{1.1}$$

since this condition implies the breakdown of the dilatation symmetry as well as that of the symmetry under

$$\phi \to -\phi , \tag{1.2}$$

which unnecessarily complicates the issue. In order to focus our attention to the violation of the dilatation symmetry, we seek a solution under the condition that the vacuum expectation value of ϕ^2, with some numerical factor, yields the nonvanishing mass of the asymptotic field. The following two subsections are devoted to the explicit dynamical calculation to exhibit the massive asymptotic field and the Goldstone boson. For this purpose, the Bethe-Salpeter technique with the pair approximation is employed.[4] The energy-momentum tensor is rewritten in terms of the asymptotic field and the Goldstone boson; thereby it is seen explicitly that the tracelessness of the energy-momentum tensor, namely, the dilatation invariance, is maintained in spite of the appearance of the nonvanishing mass. It is pointed out that the dilatation transformation of the Heisenberg fields now turns into the dimensional transformation which was introduced by us in a separate paper.[5] In actual calculations, the diverging quantities will inevitably appear. The usual cutoff procedure is evidently irrelevant to the consideration of the dilatation invariance. Our remedy for this difficulty is that taking advantage of the fact that the coupling constant λ appears with the logarithmic divergence, we put λ to zero, as the cutoff momentum approaches infinity in such a manner that the net effect stays finite. It should be pointed out in this connection that the coupling constant of the asymptotic field and the Goldstone particle is independent of the value of the constant λ due to the compositeness of the Goldstone particle. Hence, even at the limit $\lambda \to 0$, the interaction does not disappear.

It will be shown in Sec. II E that our solution is self-consistent so that the Goldstone commutator of the Heisenberg operators can be reproduced at the level of the asymptotic fields. The final subsection consists of a few remarks on our model considered in this section.

Another model[2,6] will be discussed briefly in Sec. III. This model is of an essentially different type from the previous one in that the Goldstone particle is not the composite but the asymptotic field associated with the Heisenberg scalar field assumed at the outset. The pattern of the argument is similar to that given in the preceding section. Again the appearance of the nonvanishing mass of the asymptotic field induces the spontaneous violation of the dilatation invariance. Nevertheless, the trace of the energy-momentum tensor stays zero throughout, and the dilatation transformation becomes the dimensional transformation. However, it is pointed out that in this model, the renormalized coupling constant between asymptotic fields vanishes when the bare coupling constant tends to zero.

II. MODEL I: SELF-INTERACTING MASSLESS SCALAR FIELD

A. Energy-Momentum Tensor and Dilatation Transformation

We consider a self-interacting scalar field characterized by the dilatation-invariant Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi(x)\,\partial_\mu\phi(x) - \tfrac{1}{4}\lambda\phi^4(x)\,, \tag{2.1}$$

from which the field equation follows:

$$\Box\phi(x) = \lambda\phi^3(x)\,. \tag{2.2}$$

The canonical energy-momentum tensor

$$T_{\mu\nu} = -\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{2}\delta_{\mu\nu}(\partial_\lambda\phi\partial_\lambda\phi + \tfrac{1}{2}\phi\Box\phi) \tag{2.3}$$

obeys

$$\partial_\mu T_{\mu\nu} = 0\,, \tag{2.4}$$

$$T_{\lambda\lambda} = \tfrac{1}{2}\Box\phi^2\,. \tag{2.5}$$

Following Callan, Coleman, and Jackiw[7] or the prescription given in I,[5] we define

$$\Theta_{\mu\nu} = T'_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{6}(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\phi^2\,. \tag{2.6}$$

This tensor satisfies the conditions

$$\partial_\mu\Theta_{\mu\nu} = 0\,, \tag{2.7}$$

$$\Theta_{\lambda\lambda} = 0\,. \tag{2.8}$$

The generator of the dilatation transformation

$$D = \int d\sigma_\mu \Theta_{\mu\nu} x_\nu \tag{2.9}$$

is conserved on account of (2.8), i.e.,

$$\frac{\delta D}{\delta\sigma(x)} = \Theta_{\lambda\lambda} = 0\,. \tag{2.10}$$

It is easy to prove that

$$[\phi(x), D] = i(x_\mu\partial_\mu + 1)\phi(x)\,. \tag{2.11}$$

B. Spontaneous Breakdown of Dilatation Invariance and the Goldstone Commutator

We shall look for the solution of Eq. (2.2) under the condition that the asymptotic field has a nonvanishing mass m, i.e.,

$$(\Box - m^2)\phi^{\text{in}}(x) = 0\,, \tag{2.12}$$

where the mass m is determined by

$$m^2 = 3\lambda\langle 0|\phi^2(x)|0\rangle\,. \tag{2.13}$$

It is this mass that causes the spontaneous breakdown of the dilatation invariance, as is seen below. We first calculate

$$\langle 0|[\lambda\phi(x)\phi(y), D]|0\rangle = i\left(2 + x_\mu\frac{\partial}{\partial x_\mu} + y_\mu\frac{\partial}{\partial y_\mu}\right)\langle 0|\lambda\phi(x)\phi(y)|0\rangle\,. \tag{2.14}$$

The quantity appearing on the right-hand side is highly ambiguous at $y_\mu \to x_\mu$. However, the dimensional relation

$$\left(2 + x_\mu\frac{\partial}{\partial x_\mu} + y_\mu\frac{\partial}{\partial y_\mu}\right)\langle 0|\lambda\phi(x)\phi(y)|0\rangle = m\frac{\partial}{\partial m}\langle 0|\lambda\phi(x)\phi(y)|0\rangle \tag{2.15}$$

enables us to cast (2.14) into the form

$$\langle 0|[\lambda\phi^2(x), D]|0\rangle = im\frac{\partial}{\partial m}\langle 0|\lambda\phi^2(x)|0\rangle \tag{2.16}$$

from which it follows that

$$D|0\rangle \neq 0 \tag{2.17}$$

if $m \neq 0$.

Further information is provided by the spectral representation. Following Goldstone *et al.*,[8] we put

$$\langle 0|[\lambda\phi^2(x),\Theta_{\mu\nu}(y)]|0\rangle = \int_0^\infty d\kappa^2[\rho_1(\kappa^2)\delta_{\mu\nu}+\rho_2(\kappa^2)\partial_\mu\partial_\nu]\Delta(x-y;\kappa^2). \tag{2.18}$$

The divergenceless and traceless conditions (2.7) and (2.8) restrict the spectral functions to

$$\rho_1(\kappa^2)=0, \quad \kappa^2\rho_2(\kappa^2)=0. \tag{2.19}$$

Hence, we may put

$$\rho_2(\kappa^2)=\tfrac{1}{3}c\delta(\kappa^2). \tag{2.20}$$

Consequently,

$$\langle 0|[\lambda\phi^2(x),\Theta_{\mu\nu}(y)]|0\rangle=\tfrac{1}{3}c\,\partial_\mu\partial_\nu D(x-y), \tag{2.21}$$

which implies the existence of a massless boson, as long as $c\neq 0$. Multiplying by y_ν and integrating over $d\sigma_\mu(y)$, we obtain from (2.21)

$$\langle 0|[\lambda\phi^2(x),D]|0\rangle=c, \tag{2.22}$$

and comparison of Eqs. (2.22) and (2.16) leads to the immediate identification

$$c=im\frac{\partial}{\partial m}\langle 0|\lambda\phi^2(x)|0\rangle. \tag{2.23}$$

This quantity does not vanish when $m\neq 0$. We shall refer to (2.22) as the Goldstone commutator.

The quantity c is divergent if the right-hand side of (2.23) is calculated explicitly. In the discussion of the dilatation invariance, the cutoff procedure cannot be employed for the obvious reason. This difficulty can be avoided however by adopting the prescription that the coupling constant λ is to be put equal to zero as the cutoff momentum goes to infinity in such a manner that the total effect stays finite. This is the reason why the combination $\lambda\phi^2(x)$ was considered in (2.14). We shall come back to this point later.

C. The Bethe-Salpeter Equation and the Goldstone Boson

To see explicitly the intimate connection between the nonvanishing mass m and the appearance of the Goldstone boson designated by $B^{\text{in}}(x)$, we shall set up the Bethe-Salpeter (B-S) equation for the two-body state denoted by $|q\rangle$.[4] The B-S wave function

$$\chi_q(x,y)\equiv\langle 0|T(\phi(x)\phi(y))|q\rangle \tag{2.24}$$

satisfies

$$(\Box_x-m^2)\chi_q(x,y)(\overleftarrow{\Box}_y-m^2) = 3i\lambda\delta^{(4)}(x-y)\langle 0|\phi^2(x)|q\rangle+\langle 0|T(J(x)J(y))|q\rangle, \tag{2.25}$$

with

$$J(x)=\lambda\phi^3(x)-m^2\phi(x). \tag{2.26}$$

We shall be content with the pair approximation which regards the contribution of the last term negligible. On introducing the internal wave function by

$$\chi_q(x,y)=\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iq(x+y)/2}\chi_q(z), \tag{2.27}$$

with

$$z=x-y, \tag{2.28}$$

Eq. (2.25) becomes

$$\chi_q(z)=\chi_q^{(0)}(z)+\lambda Q(z;q)\chi_q(0), \tag{2.29}$$

where

$$Q(z;q)=3i\int d^4\eta\,\Delta_c(\tfrac{1}{2}z-\eta)\Delta_c(\tfrac{1}{2}z+\eta)\,e^{iq\eta} \tag{2.30}$$

and

$$\chi_q^{(0)}(z)=(2\pi)^{3/2}(2q_0)^{1/2}\langle 0|T(\phi^{\text{in}}(\tfrac{1}{2}z)\phi^{\text{in}}(-\tfrac{1}{2}z))|q\rangle. \tag{2.31}$$

The causal function $\Delta_c(x)$ satisfies

$$(\Box-m^2)\Delta_c(x-x')=\delta^{(4)}(x-x'). \tag{2.32}$$

The inhomogeneous term $\chi_q^{(0)}(z)$ is present if $|q\rangle$ is the two-single-particle state $|s,q\rangle$, say. Hence, the B-S wave function becomes

$$\chi_q^s(z)=\chi_q^{(0)}(z)+\frac{\lambda Q(z;q)}{1-\lambda Q(q^2)}\chi_q^{(0)}(0), \tag{2.33}$$

where

$$Q(q^2)\equiv Q(0;q)=3i\int d^4\eta\,\Delta_c(-\eta)\Delta_c(\eta)\,e^{iq\eta}. \tag{2.34}$$

On the other hand, if $|q\rangle$ is the bound state denoted by $|B_q\rangle$, the inhomogeneous term vanishes and the B-S equation reads

$$\chi_q^B(z)=\lambda Q(z;q)\chi_q^B(0). \tag{2.35}$$

By putting z equal to zero, we obtain

$$[1-\lambda Q(q^2)]\chi_q^B(0)=0. \tag{2.36}$$

Since we have (Appendix A)

$$1-\lambda Q(0)=0, \tag{2.37}$$

the bound state has no mass. Thus, we see explicitly that the Goldstone particle emerges.

D. The Generator of the Dilatation Transformation

Our next task is to investigate how the generator (2.9) is expressed in terms of the asymptotic field $\phi^{\text{in}}(x)$ and the Goldstone particle $B^{\text{in}}(x)$ obeying

$$\Box B^{\text{in}}(x)=0\,. \tag{2.38}$$

For this purpose, we employ the formulas

$$\langle 0|\partial_\mu\phi(x)\partial_\nu\phi(x)|q\rangle=\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}\times\lim_{z\to 0}\left(-\tfrac{1}{4}q_\mu q_\nu-\frac{\partial}{\partial z_\mu}\frac{\partial}{\partial z_\nu}\right)\chi_q(z) \tag{2.39}$$

and

$$\langle 0|\phi(x)\Box\phi(x)|q\rangle=\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}\times\lim_{z\to 0}\left(-\tfrac{1}{4}q^2+\frac{\partial}{\partial z_\lambda}\frac{\partial}{\partial z_\lambda}\right)\chi_q(z)\,, \tag{2.40}$$

and rewrite the energy-momentum tensor (2.6). If we introduce

$$Q_{\mu\nu}(q)\equiv\lim_{z\to 0}\frac{\partial}{\partial z_\mu}\frac{\partial}{\partial z_\nu}Q(z;q)\,, \tag{2.41}$$

the tensor (2.6) becomes

$$\langle 0|\Theta_{\mu\nu}(x)|s,q\rangle=\langle 0|\Theta^{\text{in}}_{\mu\nu}(x)|s,q\rangle+\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}(-\tfrac{1}{4}m^2)\delta_{\mu\nu}\chi^{(0)}_q(0)+\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}t_{\mu\nu}(q)\frac{1}{1-\lambda Q(q^2)}\chi^{(0)}_q(0)\,, \tag{2.42}$$

where

$$\Theta^{\text{in}}_{\mu\nu}(x)=-\partial_\mu\phi^{\text{in}}\partial_\nu\phi^{\text{in}}+\tfrac{1}{2}\delta_{\mu\nu}(\partial_\lambda\phi^{\text{in}}\partial_\lambda\phi^{\text{in}}+m^2\phi^{\text{in}}\phi^{\text{in}})-\tfrac{1}{6}(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)(\phi^{\text{in}})^2 \tag{2.43}$$

and

$$t_{\mu\nu}(q)=\lambda Q_{\mu\nu}(q)+\tfrac{1}{12}\lambda q_\mu q_\nu Q(q^2)-\delta_{\mu\nu}[\tfrac{1}{48}\lambda q^2Q(q^2)+\tfrac{1}{4}\lambda Q_{\lambda\lambda}(q)]\,. \tag{2.44}$$

The form of $t_{\mu\nu}(q)$ can be simplified if we observe the identity following from (2.44):

$$t_{\mu\mu}(q)=0\,. \tag{2.45}$$

Since we want to preserve Lorentz invariance, $\Theta_{\mu\nu}$ must be divergenceless, i.e.,

$$q_\mu t_{\mu\nu}(q)=\tfrac{1}{4}m^2q_\nu[1-\lambda Q(q^2)]\,. \tag{2.46}$$

This condition, which has nothing to do with scale invariance, removes the ambiguity coming from the singular nature of Q. The only tensor satisfying (2.45) and (2.46) is given by

$$t_{\mu\nu}(q)\frac{1}{1-\lambda Q(q^2)}=\tfrac{1}{4}m^2\delta_{\mu\nu}-\tfrac{1}{3}m^2(\delta_{\mu\nu}-q_\mu q_\nu q^{-2})\,. \tag{2.47}$$

Substituting (2.47) into (2.42) and using (2.31), we arrive at

$$\langle 0|\Theta_{\mu\nu}(x)|s,q\rangle=\langle 0|\Theta^{\text{in}}_{\mu\nu}(x)|s,q\rangle-\tfrac{1}{3}m^2\left(\delta_{\mu\nu}-\partial_\mu\partial_\nu\frac{1}{\Box}\right)\langle 0|[\phi^{\text{in}}(x)]^2|s,q\rangle\,. \tag{2.48}$$

The energy-momentum tensor expressed in terms of $\phi^{\text{in}}(x)$ is nothing but $\hat{\Theta}^{\text{in}}_{\mu\nu}(x)$ introduced in I in connection with the dimensional transformation, namely,

$$\hat{\Theta}^{\text{in}}_{\mu\nu}(x)=-\partial_\mu\phi^{\text{in}}\partial_\nu\phi^{\text{in}}+\tfrac{1}{2}\delta_{\mu\nu}(\partial_\lambda\phi^{\text{in}}\partial_\lambda\phi^{\text{in}}+m^2\phi^{\text{in}}\phi^{\text{in}})-\tfrac{1}{6}(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)(\phi^{\text{in}})^2-\tfrac{1}{3}m^2\left(\delta_{\mu\nu}-\partial_\mu\partial_\nu\frac{1}{\Box}\right)(\phi^{\text{in}})^2\,; \tag{2.49}$$

as was shown in I, the generator

$$\hat{D}^{\text{in}}=\int d\sigma_\mu\hat{\Theta}^{\text{in}}_{\mu\nu}(x)\,x_\nu \tag{2.50}$$

induces

$$[\phi^{\text{in}}(x),\hat{D}^{\text{in}}]=i\left(x_\mu\partial_\mu+1-m\frac{\partial}{\partial m}\right)\phi^{\text{in}}(x)\,. \tag{2.51}$$

Note that *the mass derivative acts only on the wave function but not on the creation and annihilation operators.*

Let us next consider the bound state $|B_q\rangle$. From (2.6) and (2.35), we have

$$\langle 0|\Theta_{\mu\nu}(x)|B_q\rangle=\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}t_{\mu\nu}(q)\chi_q^B(0) \tag{2.52}$$

by virtue of (2.39) and (2.40). Substituting (2.47) into (2.52), we obtain

$$\begin{aligned}\langle 0|\Theta_{\mu\nu}(x)|B_q\rangle&=\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}\{\tfrac{1}{4}m^2\delta_{\mu\nu}[1-\lambda Q(q^2)]\chi_q^B(0)-\tfrac{1}{3}m^2(\delta_{\mu\nu}-q_\mu q_\nu q^{-2})[1-\lambda Q(q^2)]\chi_q^B(0)\}\\&=\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}(-\tfrac{1}{3}m^2)\lambda q_\mu q_\nu\frac{\partial Q(q^2)}{\partial q^2}\chi_q^B(0),\end{aligned} \tag{2.53}$$

where the relations (2.36) and (2.37) have been taken into account. We now put

$$\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}\chi_q^B(0)=\frac{g_B}{3\lambda}\langle 0|B^{\rm in}(x)|B_q\rangle \tag{2.54}$$

and determine g_B by the normalization condition of the B-S wave function (Appendix B) as

$$\left(\frac{g_B}{3}\right)^2=\frac{2}{3}\left[\left(\frac{\partial Q(q^2)}{\partial q^2}\right)_{q^2=0}\right]^{-1}, \tag{2.55}$$

whence

$$\langle 0|\Theta_{\mu\nu}(x)|B_q\rangle=\tfrac{1}{3}m^2\left(\frac{\partial Q(q^2)}{\partial q^2}\right)_{q^2=0}\tfrac{1}{3}g_B\,\partial_\mu\partial_\nu\langle 0|B^{\rm in}(x)|B_q\rangle=\frac{2}{3}\frac{m^2}{g_B}\partial_\mu\partial_\nu\langle 0|B^{\rm in}(x)|B_q\rangle. \tag{2.56}$$

The two-Goldstone-particle contribution to the energy-momentum tensor cannot be calculated unfortunately within the pair approximation. However, we may reasonably assume

$$\langle 0|\Theta_{\mu\nu}(x)|B_q,B_p\rangle=\langle 0|\Theta^B_{\mu\nu}(x)|B_q,B_p\rangle, \tag{2.57}$$

with

$$\Theta^B_{\mu\nu}(x)=-\partial_\mu B^{\rm in}\partial_\nu B^{\rm in}+\tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda B^{\rm in}\partial_\lambda B^{\rm in}-\tfrac{1}{6}(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)(B^{\rm in})^2. \tag{2.58}$$

Since such a term must appear in the expression of the Hamiltonian, it is reasonable to assume that it is present in the full $\Theta_{\mu\nu}$. Collecting all the results obtained above, we have

$$\Theta_{\mu\nu}(x)=\hat{\Theta}^{\rm in}_{\mu\nu}(x)+\Theta^B_{\mu\nu}(x)+\frac{2}{3}\frac{m^2}{g_B}\partial_\mu\partial_\nu B^{\rm in}(x), \tag{2.59}$$

where $\hat{\Theta}^{\rm in}_{\mu\nu}(x)$ and $\Theta^B_{\mu\nu}(x)$ are given by (2.49) and (2.58), respectively. The generator D thus consists of two parts,

$$D=\hat{D}^{\rm in}+D^B, \tag{2.60}$$

with

$$\hat{D}^{\rm in}=\int d\sigma_\mu\hat{\Theta}^{\rm in}_{\mu\nu}(x)x_\nu, \tag{2.61}$$

$$D^B=\int d\sigma_\mu\Theta^B_{\mu\nu}(x)x_\nu-2\frac{m^2}{g_B}\int d\sigma_\mu\partial_\mu B^{\rm in}(x). \tag{2.62}$$

The generator D^B induces the change on $B^{\rm in}$ as

$$[B^{\rm in}(x),D^B]=i(x_\mu\partial_\mu+1)B^{\rm in}(x)+2i\frac{m^2}{g_B}. \tag{2.63}$$

E. Commutators

It is now a simple matter to reproduce the Goldstone commutator (2.22) and thereby to justify the consistency of our solution. The foregoing calculation indicates that

$$\lambda\phi^2(x)=\frac{\lambda}{1-\lambda Q(-\Box)}[\phi^{\rm in}(x)]^2+\tfrac{1}{3}g_B B^{\rm in}(x)+\cdots. \tag{2.64}$$

Hence,

$$\langle 0|[\lambda\phi^2(x),D]|0\rangle = i\frac{\lambda}{1-\lambda Q(-\Box)}\left(x_\mu\partial_\mu + 2 - m\frac{\partial}{\partial m}\right)\langle 0|[\phi^{\text{in}}(x)]^2|0\rangle + \tfrac{1}{3}ig_B(x_\mu\partial_\mu+1)\langle 0|B^{\text{in}}(x)|0\rangle + i\tfrac{2}{3}m^2 . \tag{2.65}$$

The first and the second terms vanish on dimensional ground, whence

$$\langle 0|[\lambda\phi^2(x),D]|0\rangle = i\tfrac{2}{3}m^2 . \tag{2.66}$$

This agrees with (2.22), since

$$c = im\frac{\partial}{\partial m}\langle 0|\lambda\phi^2(x)|0\rangle = -m\frac{\partial}{\partial m}\lambda\triangle_c(0) = -2m^2\lambda\int d^4\eta\,\triangle_c(-\eta)\triangle_c(\eta) = i\tfrac{2}{3}m^2\lambda Q(0) = i\tfrac{2}{3}m^2 . \tag{2.67}$$

Our final task is to construct the energy-momentum vector and the angular momentum tensor as a functional of the asymptotic fields $\phi^{\text{in}}(x)$ and $B^{\text{in}}(x)$. As was proved in Ref. 9, the terms proportional to $\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu$ and $\delta_{\mu\nu}-\partial_\mu\partial_\nu(1/\Box)$ will not contribute to the integral and

$$P_\mu = -\int d\sigma_\nu[T^{\text{in}}_{\nu\mu}(x)+T^B_{\nu\mu}(x)] , \tag{2.68}$$

$$M_{\mu\nu} = -\int d\sigma_\lambda[x_\mu(T^{\text{in}}_{\lambda\nu}+T^B_{\lambda\nu}) - x_\nu(T^{\text{in}}_{\lambda\mu}+T^B_{\lambda\mu})] , \tag{2.69}$$

where

$$T^{\text{in}}_{\mu\nu}(x) = -\partial_\mu\phi^{\text{in}}\partial_\nu\phi^{\text{in}} + \tfrac{1}{2}\delta_{\mu\nu}(\partial_\lambda\phi^{\text{in}}\partial_\lambda\phi^{\text{in}} + m^2\phi^{\text{in}}\phi^{\text{in}}) , \tag{2.70}$$

$$T^B_{\mu\nu}(x) = -\partial_\mu B^{\text{in}}\partial_\nu B^{\text{in}} + \tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda B^{\text{in}}\partial_\lambda B^{\text{in}} . \tag{2.71}$$

Notice that the linear term in B^{in} also has no contribution. The commutators with D become[5]

$$[P_\mu, D] = iP_\mu - im\frac{\partial}{\partial m}P_\mu , \tag{2.72}$$

$$[M_{\mu\nu}, D] = -im\frac{\partial}{\partial m}M_{\mu\nu} . \tag{2.73}$$

The apparent difference from those of the Heisenberg operators is due to the asymptotic limit taken before the calculation of the commutators.[5]

F. Remarks

As a final comment of this section, we discuss the physical meaning of the constant g_B, and derive the Goldberger-Treiman relation. To see the meaning of the constant g_B, we refer to Eq. (2.25), which reads in the pair approximation

$$\begin{aligned}(\Box_x - m^2)\langle 0|T(\phi(x)\phi(y))|B_q\rangle(\overleftarrow{\Box}_y - m^2) &= 3i\lambda\delta^{(4)}(x-y)\langle 0|\phi^2(x)|B_q\rangle \\ &= ig_B\delta^{(4)}(x-y)\langle 0|B^{\text{in}}(x)|B_q\rangle\end{aligned} \tag{2.74}$$

with the aid of the relation (2.54). By using the reduction formula, we find that the constant g_B is the coupling constant of the process

$$B_q \to \phi_p + \phi_{p'} .$$

The relation (2.55) implies

$$g_B{}^2 = 6\left[\left(\frac{\partial Q(q^2)}{\partial q^2}\right)_{q^2=0}\right]^{-1} , \tag{2.75}$$

which has a remarkable feature that it is independent of the original coupling constant λ. As we remarked earlier, the constant λ is taken to be zero to make the quantity $\lambda Q(q^2)$ meaningful. Nevertheless, the coupling constant g_B stays finite. In fact if we express $Q(q^2)$, given in (2.31), in spectral form and calculate $\partial Q(q^2)/\partial q^2$ at $q^2=0$, we obtain

$$\left(\frac{\partial Q(q^2)}{\partial q^2}\right)_{q^2=0} = \frac{1}{32\pi^2 m^2} , \tag{2.76}$$

whence

$$g_B{}^2 = 192\pi^2 m^2 . \tag{2.77}$$

The Goldberger-Treiman relation can easily be derived: Define the coupling constant f_B between the vacuum and the one B^{in} state by

$$\langle 0|\Theta_{\mu\nu}(0)|B_q\rangle \equiv -\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}f_B q_\mu q_\nu . \tag{2.78}$$

The comparison of this equation and (2.56) leads at once to

$$f_B = \frac{2}{3}\frac{m^2}{g_B}\,. \tag{2.79}$$

III. MODEL II: SCALAR FIELD WITH THE YUKAWA COUPLING

A. Lagrangian and the Field Equations

As an alternative example, we consider a model in which a massless fermion field interacts with a massless boson characterized by the Lagrangian

$$\mathcal{L} = -\bar{\psi}(x)\gamma_\lambda\partial_\lambda\psi(x) - g\,\bar{\psi}(x)\psi(x)\Phi(x) - \tfrac{1}{2}\partial_\lambda\Phi(x)\partial_\lambda\Phi(x)\,. \tag{3.1}$$

The equations of motion and the Callan-Coleman-Jackiw tensor can readily be derived:

$$\gamma\cdot\partial\psi(x) = -g\,\psi(x)\Phi(x)\,, \tag{3.2}$$

$$\Box\Phi(x) = g:\bar{\psi}(x)\psi(x): \tag{3.3}$$

and

$$\Theta_{\mu\nu}(x) = -\tfrac{1}{4}\bar{\psi}(x)\gamma_\mu(\partial_\nu - \overleftarrow{\partial}_\nu)\psi(x) - \tfrac{1}{4}\bar{\psi}(x)\gamma_\nu(\partial_\mu - \overleftarrow{\partial}_\mu)\psi(x) - \partial_\mu\Phi(x)\partial_\nu\Phi(x) + \tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda\Phi(x)\partial_\lambda\Phi(x) - \tfrac{1}{6}(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\Phi^2(x)\,. \tag{3.4}$$

As the Lagrangian (3.1) is dilatation invariant, the generator

$$D = \int d\sigma_\mu(x)\Theta_{\mu\nu}(x)\,x_\nu \tag{3.5}$$

is time independent. The use of canonical equal-time commutator gives

$$[\Phi(x), D] = i(x_\mu\partial_\mu + 1)\Phi(x)\,, \tag{3.6}$$

$$[\psi(x), D] = i(x_\mu\partial_\mu + \tfrac{3}{2})\psi(x)\,. \tag{3.7}$$

B. Spontaneous Breakdown of Dilatation Invariance

We now seek a solution under the condition that

$$\langle 0|\,g\Phi(x)|0\rangle = m\,. \tag{3.8}$$

Observe that the right-hand side of this equation has been purposely designated to generate a finite physical fermion mass. In this sense, the relation (3.8) merely defines the quantity m. The nonvanishing vacuum expectation value of Φ implies a spontaneous breakdown of dilatation invariance. Indeed, from (3.6), it follows at once that

$$\langle 0|[\,g\Phi(x), D]|0\rangle = im \tag{3.9}$$

so that

$$D|0\rangle \neq 0 \tag{3.10}$$

if $m \neq 0$.

We can again ascertain the existence of a massless particle in this model by considering the spectral representation of $\langle 0|[\,g\Phi(x),\,\Theta_{\mu\nu}(y)]|0\rangle$. We have

$$\langle 0|[\,g\Phi(x), \Theta_{\mu\nu}(y)]|0\rangle = -\tfrac{1}{3}\,a\,\partial_\mu\partial_\nu D(x-y)\,, \tag{3.11}$$

and consequently, the Goldstone commutator

$$\langle 0|[\,g\Phi(x), D]|0\rangle = a = im \neq 0 \tag{3.12}$$

is obtained.

C. Bethe-Salpeter Equation and Physical States

In order to take into account the condition (3.8), it is convenient to introduce a new field $\phi(x)$ by

$$g\Phi(x) = g\,\phi(x) + m \tag{3.13}$$

so that

$$\langle 0|\,g\,\phi(x)|0\rangle = 0\,. \tag{3.14}$$

The equations of motion and the $\Theta_{\mu\nu}$ then read

$$(\gamma\partial + m)\psi(x) = -g\,\psi(x)\phi(x)\,, \tag{3.15}$$

$$\bar{\psi}(x)(-\gamma\overleftarrow{\partial} + m) = -g\,\bar{\psi}(x)\phi(x)\,, \tag{3.16}$$

$$\Box\,\phi(x) = g:\bar{\psi}(x)\psi(x):\,, \tag{3.17}$$

and

$$\Theta_{\mu\nu}(x) = -\tfrac{1}{4}\bar{\psi}(x)\gamma_\mu(\partial_\nu - \overleftarrow{\partial}_\nu)\psi(x) - \tfrac{1}{4}\bar{\psi}(x)\gamma_\nu(\partial_\mu - \overleftarrow{\partial}_\mu)\psi(x) - \partial_\mu\phi(x)\partial_\nu\phi(x) + \tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda\phi(x)\partial_\lambda\phi(x) - \tfrac{1}{6}(\Box\,\delta_{\mu\nu} - \partial_\mu\partial_\nu)\phi^2(x) - \frac{m}{3g}(\Box\,\delta_{\mu\nu} - \partial_\mu\partial_\nu)\phi(x)\,. \tag{3.18}$$

As before, the B-S wave function is defined by

$$\chi_q(x,y) \equiv \langle 0|T(\psi(x)\bar{\psi}(y))|\,q\rangle \equiv \frac{1}{(2\pi)^{3/2}}\frac{1}{(2\,q_0)^{1/2}}\,e^{iq(x+y)/2}\chi_q(z)\,, \tag{3.19}$$

with

$$z = x - y\,.$$

In the pair approximation, $\chi_q(z)$ satisfies

$$\chi_q(z) = \chi_q^{(0)}(z) + g\,R(z\,;\,q)\,u_q\,. \tag{3.20}$$

Equation (3.17) reads

$$q^2 u_q = g\,\mathrm{Tr}\chi_q(0)\,, \tag{3.21}$$

where

$$\langle 0|\phi(x)|\,q\rangle = \frac{1}{(2\pi)^{3/2}}\frac{1}{(2\,q_0)^{1/2}}\,e^{iqx}u_q\,, \tag{3.22}$$

$$R(z\,;\,q) = \int d^4\eta\, S_c(\tfrac{1}{2}z - \eta)S_c(\tfrac{1}{2}z + \eta)\,e^{iq\eta} \tag{3.23}$$

(see Ref. 10),

$$\chi_q^{(0)}(z) = (2\pi)^{3/2}\ (2q_0)^{1/2} \langle 0|T(\psi^{\rm in}(\tfrac{1}{2}z)\bar{\psi}^{\rm in}(-\tfrac{1}{2}z))|q\rangle . \tag{3.24}$$

We further introduce

$$R_{\mu\nu}(q) \equiv \lim_{z\to 0} \frac{\partial}{\partial z_\nu} \mathrm{Tr}[R(z;q)\gamma_\mu], \tag{3.25}$$

$$R(q^2) \equiv \lim_{z\to 0} \mathrm{Tr}[R(z;q)]. \tag{3.26}$$

If we take the two-fermion scattering state denoted by $|F, q\rangle$, we obtain from (3.20) and (3.21)

$$\mathrm{Tr}[\chi_q^F(0)] = \frac{\mathrm{Tr}[\chi_q^{(0)}(0)]}{1-(g^2/q^2)R(q^2)} . \tag{3.27}$$

On the other hand, to obtain the one-particle state, we first drop $\chi_q^{(0)}$ and then eliminate χ_q from (3.20) and (3.21). Thus

$$[q^2 - g^2R(q^2)]u_q = 0 . \tag{3.28}$$

Following the same pattern as in the model in the preceding section, we can see that Eq. (3.28) admits a zero-mass solution, i.e.,

$$\lim_{q^2\to 0} [q^2 - g^2R(q^2)] = 0 , \tag{3.29}$$

so that u_q represents the Goldstone boson and, in fact, is the wave function of the incoming field operator associated with the Heisenberg field $\phi(x)$. This is in contrast to the fact that in the previous example the Goldstone boson was a composite particle.

D. Generator of Dilatation Transformation

The contribution of the physical states to the energy-momentum tensor $\Theta_{\mu\nu}$ is obtained by using the same technique as in the previous model. We find

$$\begin{aligned}\langle 0|\Theta_{\mu\nu}(x)|F,q\rangle = &-\tfrac{1}{4}\langle 0|\bar\psi^{\rm in}\gamma_\mu(\partial_\nu - \overleftarrow{\partial}_\nu)\psi^{\rm in}|F,q\rangle - \tfrac{1}{4}\langle 0|\bar\psi^{\rm in}\gamma_\nu(\partial_\mu - \overleftarrow{\partial}_\mu)\psi^{\rm in}|F,q\rangle \\ &-\tfrac{1}{3}m(q^2\delta_{\mu\nu} - q_\mu q_\nu)\frac{1}{q^2-g^2R(q^2)}\langle 0|\bar\psi^{\rm in}\psi^{\rm in}|F,q\rangle \\ &-\tfrac{1}{2}[R_{\mu\nu}(q)+R_{\nu\mu}(q)]\frac{g^2}{q^2-g^2R(q^2)}\langle 0|\bar\psi^{\rm in}\psi^{\rm in}|F,q\rangle .\end{aligned} \tag{3.30}$$

The conservation of $\Theta_{\mu\nu}$ then implies that

$$q_\mu[R_{\mu\nu}(q)+R_{\nu\mu}(q)] = 0 . \tag{3.31}$$

In order to determine $R_{\mu\nu}(q)+R_{\nu\mu}(q)$, we make use of the field equation (3.15):

$$\langle 0|\bar\psi(x)\gamma_\lambda(\partial_\lambda - \overleftarrow{\partial}_\lambda)\psi(x)|q\rangle + m\langle 0|\bar\psi(x)\psi(x)|q\rangle = -2g\langle 0|\bar\psi(x)\psi(x)\phi(x)|q\rangle . \tag{3.32}$$

However, in the pair approximation, the right-hand side can be neglected. Substituting (3.20), we obtain

$$R_{\lambda\lambda}(q) = -mR(q^2) . \tag{3.33}$$

The only symmetric tensor satisfying (3.31) and (3.33) is

$$\tfrac{1}{2}[R_{\mu\nu}(q)+R_{\nu\mu}(q)] = -\tfrac{1}{3}m(\delta_{\mu\nu} - q^{-2}q_\mu q_\nu)R(q^2) , \tag{3.34}$$

which leads to

$$\begin{aligned}\langle 0|\Theta_{\mu\nu}(x)|F,q\rangle = &-\tfrac{1}{4}\langle 0|\bar\psi^{\rm in}(x)\gamma_\mu(\partial_\nu - \overleftarrow{\partial}_\nu)\psi^{\rm in}(x)|F,q\rangle - \tfrac{1}{4}\langle 0|\bar\psi^{\rm in}(x)\gamma_\nu(\partial_\mu - \overleftarrow{\partial}_\mu)\psi^{\rm in}(x)|F,q\rangle \\ &-\tfrac{1}{3}m\left(\delta_{\mu\nu} - \partial_\mu\partial_\nu\frac{1}{\Box}\right)\langle 0|\bar\psi^{\rm in}(x)\psi^{\rm in}(x)|F,q\rangle .\end{aligned} \tag{3.35}$$

Here, again we meet the traceless tensor $\hat\Theta_{\mu\nu}$ introduced in I. Analogously, the contribution of the single-particle state is found to be

$$\begin{aligned}\langle 0|\Theta_{\mu\nu}(x)|q\rangle &= +\frac{1}{2}\frac{g}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}[R_{\mu\nu}(q)+R_{\nu\mu}(q)]u_q + \frac{m}{3g}\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}(q^2\delta_{\mu\nu} - q_\mu q_\nu)u_q \\ &= \frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}\frac{m}{3g}\left(\delta_{\mu\nu} - \frac{1}{q^2}q_\mu q_\nu\right)[q^2 - g^2R(q^2)]u_q \\ &= -\frac{1}{(2\pi)^{3/2}}\frac{1}{(2q_0)^{1/2}}e^{iqx}q_\mu q_\nu\frac{m}{3g}\left(1 - g^2\frac{\partial R(q^2)}{\partial q^2}\right)u_q \\ &= +\frac{m}{3g}Z_3^{-1}\partial_\mu\partial_\nu\langle 0|\phi(x)|q\rangle .\end{aligned} \tag{3.36}$$

Noting that

$$\langle 0|\phi(x)|q\rangle = Z_3^{1/2}\langle 0|\phi^{\text{in}}(x)|q\rangle , \tag{3.37}$$

$$gZ_3^{1/2} = g_{\text{ren}} , \tag{3.38}$$

we arrive at

$$\langle 0|\Theta_{\mu\nu}(x)|q\rangle = \frac{m}{3g_{\text{ren}}}\partial_\mu\partial_\nu\langle 0|\phi^{\text{in}}(x)|q\rangle . \tag{3.39}$$

The total energy-momentum tensor now becomes

$$\Theta_{\mu\nu}(x) = \hat{\Theta}^F_{\mu\nu}(x) + \Theta^B_{\mu\nu}(x), \tag{3.40}$$

with

$$\hat{\Theta}^F_{\mu\nu}(x) = -\tfrac{1}{4}\bar{\psi}^{\text{in}}(x)\gamma_\mu(\partial_\nu - \overleftarrow{\partial}_\nu)\psi^{\text{in}}(x) - \tfrac{1}{4}\bar{\psi}^{\text{in}}(x)\gamma_\nu(\partial_\mu - \overleftarrow{\partial}_\mu)\psi^{\text{in}}(x) - \tfrac{1}{3}m\left(\delta_{\mu\nu} - \partial_\mu\partial_\nu\frac{1}{\Box}\right)\bar{\psi}^{\text{in}}(x)\psi^{\text{in}}(x), \tag{3.41}$$

$$\Theta^B_{\mu\nu} = \frac{m}{3g_{\text{ren}}}\partial_\mu\partial_\nu\phi^{\text{in}}(x) - \partial_\mu\phi^{\text{in}}(x)\partial_\nu\phi^{\text{in}}(x) + \tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda\phi^{\text{in}}(x)\partial_\lambda\phi^{\text{in}}(x) - \tfrac{1}{6}(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)[\phi^{\text{in}}(x)]^2 \tag{3.42}$$

The dilatation generator is

$$D \equiv \int d\sigma_\mu \Theta_{\mu\nu}(x)\, x_\nu = \hat{D}^F + D^B , \tag{3.43}$$

where

$$\hat{D}^F = \int d\sigma_\mu \hat{\Theta}^F_{\mu\nu}(x)\, x_\nu , \tag{3.44}$$

$$D^B = \int d\sigma_\mu \Theta^B_{\mu\nu}(x)\, x_\nu . \tag{3.45}$$

They satisfy

$$i[\psi^{\text{in}}(x), \hat{D}^F] = -\left(x_\mu\partial_\mu + \tfrac{3}{2} - m\frac{\partial}{\partial m}\right)\psi^{\text{in}}(x) \tag{3.46}$$

$$i[\phi^{\text{in}}(x), D^B] = -(x_\mu\partial_\mu + 1)\phi^{\text{in}}(x) - \frac{m}{g_{\text{ren}}} . \tag{3.47}$$

The vacuum expectation value of (3.47) multiplied by g_{ren} reproduces the Goldstone commutator (3.12).

IV. DISCUSSION

Our investigation on the two models in Secs. II and III shows how it is possible that the appearance of nonvanishing physical masses due to the spontaneous breakdown of the dilatation invariance does not contradict the original invariance. Namely, the invariance of the theory is preserved by the rearrangement of the dilatation to the dimensional transformation, which was introduced in I. It is seen that the generator of the dilatation transformation of the Heisenberg operators turns into that of the dimensional transformation of asymptotic fields. However, the rearrangement of the invariance is accompanied by the Goldstone boson. Since the Goldstone boson undergoes the inhomogeneous dilatation transformation [(2.63) and (3.47)], it plays a particular role in our theory and restricts physical results. For example, in the former model, the dilatation transformation generates

$$[\phi(x), D] = i(x\cdot\partial + 1)\phi(x) . \tag{4.1}$$

On the other hand, the dimensional transformation generates

$$[\phi^{\text{in}}(x, m), \hat{D}^{\text{in}}] = i\left(x\cdot\partial + 1 - m\frac{\partial}{\partial m}\right)\phi^{\text{in}}(x, m), \tag{4.2}$$

$$[B^{\text{in}}(x, m), D^B] = i(x\cdot\partial + 1)B^{\text{in}} + i\frac{2m^2}{g_B} . \tag{4.3}$$

Now, we can construct two quantities out of $B^{\text{in}}(x)$ which have normal homogeneous dimension. They are

$$\partial_\mu B^{\text{in}}(x) \quad \text{and} \quad m + \frac{g_B}{2m}B^{\text{in}}(x) ,$$

i.e.,

$$[\partial_\mu B^{\text{in}}(x), D^B] = i(x\cdot\partial + 2)\partial_\mu B^{\text{in}}(x) , \tag{4.4}$$

$$\left[m+\frac{g_B}{2m}B^{\text{in}}(x), D^B\right]=i(x\cdot\partial+1)\left(m+\frac{g_B}{2m}B^{\text{in}}(x)\right). \tag{4.5}$$

We further note that

$$\left[\exp\left(\frac{g_B}{2m}B^{\text{in}}(x)\frac{\partial}{\partial m}\right)\phi^{\text{in}}(x, m), \hat{D}^{\text{in}}+D^B\right] =i(x\cdot\partial+1)\left[\exp\left(\frac{g_B}{2m}B^{\text{in}}(x)\frac{\partial}{\partial m}\right)\phi^{\text{in}}(x, m)\right]. \tag{4.6}$$

In other words, the exponential factor just cancels the mass derivative term appearing in (4.2). We infer therefore that the Heisenberg operator $\phi(x)$ is of the form

$$\phi(x)=F\left[m+\frac{g_B}{2m}B^{\text{in}}(x), \exp\left(\frac{g_B}{2m}B^{\text{in}}(x)\frac{\partial}{\partial m}\right)\phi^{\text{in}}(x,m), \text{ their derivatives}\right], \tag{4.7}$$

where F is functional with the dimension L^{-1}. Indeed, it is not difficult to show that

$$[F, \hat{D}^{\text{in}}+D^B]=i(x\cdot\partial+1)F, \tag{4.8}$$

which is nothing but the relation (4.1). The function of the exponential factor can best be understood if we note the relation

$$\exp\left(\frac{g_B}{2m}B^{\text{in}}(x)\frac{\partial}{\partial m}\right)\phi^{\text{in}}(x, m) =\phi^{\text{in}}\left(x, m+\frac{g_B}{2m}B^{\text{in}}(x)\right). \tag{4.9}$$

This implies that in the right-hand side of (4.7), the mass and the Goldstone boson can appear only as a combination $m+(g_B/2m)B^{\text{in}}(x)$. Hence, the appearance of the mass does not bring in abnormal dimension and the original transformation property of the Heisenberg operator is maintained at every step.

Finally, we emphasize that the coupling constant g_B does not vanish even at the limit $\lambda\to 0$, as is seen in (2.75). This is due to the composite nature of $B^{\text{in}}(x)$. It is interesting to speculate that all the divergences may be eliminated, in a nontrivial manner, by the procedure $\lambda\to 0$ as the cutoff goes to infinity when particles are composite.

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APPENDIX A: A DERIVATION OF $1-\lambda Q(0)=0$ AND THE MASS EQUATION (2.13)

Let us put

$$Q_{\mu\nu}(q)=F(q^2)q_\mu q_\nu+G(q^2)\delta_{\mu\nu}. \tag{A1}$$

It follows from (2.44) and (2.46) that

$$1-\lambda Q(q^2)=q^2\,\frac{1+12\lambda F(q^2)}{q^2+4m^2}. \tag{A2}$$

If $\lambda F(q^2)$ has no singularity of the form $1/q^2$, then we have

$$1-\lambda Q(0)=0. \tag{A3}$$

To ensure that $\lambda F(q^2)$ has no singularity, we employ the spectrum representation

$$\lambda F(q^2)=\int_{4m^2}^{\infty}d\kappa^2\frac{\rho(\kappa^2)}{q^2+\kappa^2-i\epsilon}, \tag{A4}$$

where

$$\rho(\kappa^2)=\frac{\lambda}{64\pi^2}\left(1-\frac{4m^2}{\kappa^2}\right)^{3/2}, \tag{A5}$$

and calculate

$$\lambda F(0)=\int_{4m^2}^{\infty}d\kappa^2\,\frac{1}{\kappa^2}\,\rho(\kappa^2). \tag{A6}$$

If we put $\lambda\to 0$ as the cutoff momentum goes to infinity, we obtain a finite result. Thus, we arrive at (A3).

We now prove that the mass equation (2.13) can be written as

$$m^2=3i\lambda\Delta_c(0) \tag{A7}$$

and discuss the consistency with (A3). From the field equation, it follows that

$$(\Box_x-m^2)\langle 0|T(\phi(x)\phi(y))|0\rangle(\overleftarrow{\Box}_y-m^2) =i(\Box-m^2)\delta^{(4)}(x-y)-im^2\delta^{(4)}(x-y) +3i\lambda\delta^{(4)}(x-y)\langle 0|\phi^2(x)|0\rangle, \tag{A8}$$

where the pair approximation is employed. Equation (A8) implies that

$$\Delta_c'(z)=\Delta_c(z)+\lambda Q(z;0)\Delta_c'(0)+i\tfrac{1}{3}m^2Q(z;0). \tag{A9}$$

If we put $z=0$ and use (A3), we have

$$m^2=3i\lambda\Delta_c(0). \tag{A10}$$

The quantity $\langle 0|\lambda\phi^2(x)|0\rangle$ appearing in the mass equation (2.13) is highly ambiguous and was the subject of intensive study around 1949 in connection with the photon self-energy and the equivalence theorem of the pseudoscalar and pseudovector theories, etc. This quantity is in general stated as

$$\langle 0|\lambda\phi^2(x)|0\rangle = \lim_{\eta\to 0}\langle 0|\lambda\phi(\eta)\phi(0)|0\rangle$$
$$= \lim_{\eta\to 0}\left(c_1\frac{\lambda}{\eta^2} + c_2\lambda m^2 \ln m^2\eta^2 + \lambda c_3 m^2\right),$$

where the constants c_1, c_2, and c_3 depend on how the integration is carried out. In particular, the value of c_1 depends crucially on the method of integration. We shall take in this paper the view point that the constant c_1 must be determined consistently. If we require that the two relations (A4) and (A7) are consistent, the constant c_1 must be zero. The logarithmic divergent term causes no trouble if we take $\lambda\to 0$ limit in such a way that the expression $Q(0)$ if finite as the cutoff goes to infinity. The relation

$$\lim_{q^2\to 0}[q^2 - g^2 R(q^2)] = 0 \tag{A11}$$

can also be shown in exactly the same way so that it is not necessary to repeat it here.

APPENDIX B: NORMALIZATION OF $\chi_q^B(0)$

In the pair approximation in which the B-S equation is written in the form

$$\chi_q^B(z) = \lambda Q(z;q)\chi_q^B(0), \tag{B1}$$

it proves convenient to write the normalization condition in z space.

Let us first express the normalization condition in z space[11]:

$$\int d^4z'd^4z\,\bar{\chi}_q^B(z')\frac{\partial}{\partial q_0}[I(z',z;q) + G(z',z;q)]\chi_q^B(z) = 2iq_0\frac{1}{(2\pi)^4}, \tag{B2}$$

where

$$I(z',z;q) = \frac{1}{(2\pi)^8}\int d^4p\,d^4p'I(p',p;q)\,e^{ip'z'}e^{-ipz}$$
$$= \frac{1}{(2\pi)^8}\int d^4p[S'_{FA}(\tfrac{1}{2}q+p)]^{-1}[S'_{FB}(\tfrac{1}{2}q-p)]^{-1}e^{ip(z'-z)}, \tag{B3}$$

$$G(z',z;q) = \frac{1}{(2\pi)^8}\int d^4p\,d^4p'G(p',p;q)\,e^{ip'z'}e^{-ipz}. \tag{B4}$$

To apply the above formula to our model I, we note the following:

(i) In the pair approximation, $G(z',z;q)$ is independent of q, so that the second term in (B2) does not contribute.

(ii) The one-particle propagator in our case is

$$S'_{FA}(\tfrac{1}{2}q+p) = S'_{FB}(\tfrac{1}{2}q+p) = i\Delta_c(\tfrac{1}{2}q+p). \tag{B5}$$

(iii) Particles appearing in our model are indistinguishable. Hence the right-hand side of (B2) must be multiplied by two.

Considering the above all, we arrive at

$$\int d^4z\int d^4z'\bar{\chi}_q^B(z')\frac{\partial}{\partial q_0}I(z',z;q)\chi_q^B(z) = 4iq_0\frac{1}{(2\pi)^4}. \tag{B6}$$

Recall, however,

$$I(z',z;q) = \frac{-1}{(2\pi)^8}\int d^4p\,\Delta_c^{-1}(\tfrac{1}{2}q+p)\,\Delta_c^{-1}(\tfrac{1}{2}q-p)\,e^{ip(z'-z)}$$
$$= -\frac{3i}{(2\pi)^4}Q^{-1}(z'-z;q). \tag{B7}$$

Substituting (B7) and (B1) into (B6) we have

$$\lambda^2\int d^4z\int d^4z'Q(z';q)\frac{\partial Q^{-1}(z'-z;q)}{\partial q_0}Q(z;q)|\chi_q^B(0)|^2 = \lambda^2\frac{\partial}{\partial q_0}Q(0;q)|\chi_q^B(0)|^2 = -\tfrac{4}{3}q_0. \tag{B8}$$

Hence,

$$|\chi_q^B(0)|^2 = \frac{2}{3\lambda}\left[\left(\lambda\,\frac{\partial Q(q^2)}{\partial q^2}\right)_{q^2=0}\right]^{-1}. \tag{B9}$$

From (2.50), we obtain

$$\left(\frac{g_B}{3}\right)^2 = \frac{2}{3}\left[\left(\frac{\partial Q(q^2)}{\partial q^2}\right)_{q^2=0}\right]^{-1} \tag{B10}$$

by virtue of

$$\langle 0|\,B^{\rm in}(x)\,|\,B_q\rangle = \frac{1}{(2\pi)^{3/2}}\,\frac{1}{(2\,q_0)^{1/2}}\,e^{iqx}. \tag{B11}$$

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[1]D. Maison and H. Reeh, München report (unpublished).

[2]G. Mack, Nucl. Phys. B5, 499 (1968). G. Mack and A. Salam, Ann. Phys. (N.Y.) 53, 174 (1969). M. Gell-Mann, in *Proceedings of the Third Hawaii Topical Conference on Particle Physics*, edited by S. F. Tuan (Western Periodicals, North Hollywood, Calif., 1969); C. G. Callan, Jr. and P. Carruthers, Phys. Rev. D 4, 3214 (1971). The last paper contains several useful references on this subject.

[3]In this context see also M. Kugler and S. Nussinov, Nucl. Phys. B28, 97 (1971).

[4]A. Aurilia, Y. Takahashi, and H. Umezawa, Progr. Theoret. Phys. (Kyoto) (to be published).

[5]A. Aurilia, Y. Takahashi, and H. Umezawa, Phys. Rev. D 5, 851 (1972). This paper will be referred to as I.

[6]Y. Fujii, Lett. Nuovo Cimento 1, 384 (1971).

[7]C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).

[8]J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[9]Y. Takahashi, Proc. Roy. Irish Acad. 71A, 1 (1971).

[10]We define $(\gamma\cdot\partial + m)S_c(x-y) = \delta^{(4)}(x-y)$.

[11]D. Lurié, A. J. McFarlane, and Y. Takahashi, Phys. Rev. 140, B1091 (1965).

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High-Energy Delbrück Scattering from Nuclei*

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We study the elastic scattering of a high-energy photon from a heavy nucleus, considered to be a static Coulomb field due to the charge Ze. Exchange of an arbitrarily large number of photons with the nucleus is taken into account, and the effect of this multiphoton exchange is found to be very large, particularly for momentum transfers which are large compared with the rest mass of the electron. In addition, an interesting theoretical problem in this connection is formulated but unfortunately not solved.

I. INTRODUCTION

Three years ago, we studied in detail all two-body elastic-scattering amplitudes in quantum electrodynamics at high energies.[1-5] Among these processes, the one with the most direct experimental interest is Delbrück scattering,[6] or the elastic scattering of a photon by a nucleus, considered to be a static Coulomb field. At the time when we carried out our theoretical analysis, the only relevant experimental data on Delbrück scattering were those of Moffatt and Stringfellow[7] at an energy of about 90 MeV, and a comparison of these data with our theoretical results is given in III. Recently, the experimental group F39 of DESY obtained data on Delbrück scattering and photon splitting at energies of several BeV and momentum transfer of a few MeV/c, although the data analysis is as yet incomplete. Motivated by this new information on copper, silver, gold, and uranium, we give in this paper the basic theoretical formulas for Delbrück scattering and some of the simple consequences.

The lowest-order diagrams for Delbrück scat-