[23]During the writing of this paper, a study of the rainbow diagrams by Y. Shimizu, Phys. Rev. D (to be published), was received. By Mellin transform techniques he obtains results consistent with Eq. (6.1) but including additional terms which are here seen to be vanishing by the cluster decomposition method.

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[25]S.-J. Chang and P. M. Fishbane, Phys. Rev. D 2, 1084 (1970).

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Short-Distance Behavior of Quantum Electrodynamics and an Eigenvalue Condition for α

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We review and extend earlier work dealing with the short-distance behavior of quantum electrodynamics. We show that if the renormalized photon propagator is asymptotically finite, then in the limit of zero fermion mass all of the single-fermion-loop $2n$-point functions, regarded as functions of the coupling constant, must have a common infinite-order zero. In the usual class of asymptotically finite solutions introduced by Gell-Mann and Low, the asymptotic coupling α_0 is fixed to be this infinite-order zero and the physical coupling $\alpha < \alpha_0$ is a free parameter. We show that if the single-fermion-loop diagrams actually possess the required infinite-order zero, there is a unique, additional solution in which the physical coupling α is fixed to be the infinite-order zero. We conjecture that this is the solution chosen by nature. According to our conjecture, the fine-structure constant is determined by the eigenvalue condition $F^{[1]}(\alpha)=0$, where $F^{[1]}$ is a function related to the single-fermion-loop vacuum-polarization diagrams. The eigenvalue condition is independent of the number of fundamental fermion species which are assumed to be present.

I. INTRODUCTION AND SUMMARY

The fundamental constant regulating all microscopic electronic phenomena, from atomic physics to quantum electrodynamics, is the fine-structure constant α. Experimentally, the current value[1] $\alpha = 1/(137.03602 \pm 0.00021)$ is one of the best determined numbers in physics. Theoretically, the reason why nature selects this particular numerical value has remained a mystery, and has provoked much interesting speculation. The speculations may be divided roughly into three general types: (a) those in which α is cosmologically determined, either as a cosmological boundary condition (which makes α undeterminable) or as a function of time-varying cosmological parameters (which makes α a function of time)[2]; (b) theories in which α is a constant which is determined microscopically through the interplay of the electromagnetic interaction with interactions of other types, either strong, weak, or gravitational.[3] Since these interactions are currently even less well understood than is the electromagnetic interaction, such theories seem at present to offer little promise of an actual computation of α; (c) finally, theories in which α is microscopically determined through properties of the electromagnetic interaction alone, considered in isolation from other interactions. It is this restricted class of theories to which we will address ourselves in the present paper.

The idea that α may be determined electromagnetically is an old one. In the early days of renormalization theory there were hopes that α could be fixed by requiring the logarithmic divergences appearing in higher orders of perturbation theory to cancel or "compensate" the second-order divergence in the photon wave-function renormalization Z_3,[4] so that the renormalized photon propagator would be asymptotically finite. These hopes received a setback, however, when Jost and Luttinger[5] calculated the order-α^2 logarithmically divergent contribution to Z_3 and found that it has the same sign as the order-α divergence. Of course, it was obvious that the question could not be settled by calculations to any finite order of perturbation theory. A systematic nonperturbative attack on the problem was made by Gell-Mann and Low[6] in their classic 1954 paper on renormalization-group methods. They showed that there is indeed an eigenvalue condition imposed by requiring that the renormalized photon propagator behave as

$$\alpha d_c(-q^2/m^2, \alpha) = \alpha_0 + h(-q^2/m^2, \alpha), \qquad (1)$$

with α_0 finite and with h vanishing as $-q^2/m^2 \to \infty$. However, the condition takes the form $\psi(\alpha_0) = 0$, and determines the *asymptotic coupling* α_0 rather than the physical coupling α. Their analysis leaves α a free parameter of the theory, restricted only by the condition $\alpha < \alpha_0$ coming from spectral-function positivity. This essential conclusion was retained in the subsequent important work of Johnson, Baker, and Willey,[7] who showed that if the eigenvalue condition is satisfied *all* the renormalization constants of electrodynamics (m_0 and Z_2 as well as Z_3) can be finite, and who applied a simple argument based on the Federbush-Johnson theorem[8] to obtain a greatly simplified form of the eigenvalue equation for α_0. Thus, the prevailing view since 1954 has been that it is not possible to determine α within a purely electrodynamical context.

Our aim in the present paper is to give a reexamination and extension of the work of Gell-Mann and Low and of Johnson, Baker, and Willey, which, we believe, reopens the possibility of an electrodynamic determination of α. We continue to work within the same basic framework as these previous authors in that we assume, as they do, that asymptotically vanishing terms encountered in each order of perturbation theory do not sum to give an asymptotically dominant result. Our basic observation is that the work of Johnson *et al.*, assumes that α_0 is both a simple zero, and a point of regularity, of the Gell-Mann–Low function $\psi(y)$. In actual fact, we find that an extension of the argument given by Baker and Johnson to obtain their simplified eigenvalue condition indicates that neither of these assumptions is correct. We show that if ψ has a zero at all it must be a zero of infinite order – i.e., an essential singularity. This infinite-order zero in the coupling constant must also appear in all of the single-fermion-loop $2n$-point functions calculated in electrodynamics with zero fermion mass. The presence of an essential singularity has the important consequence that different orders of summing perturbation theory can lead to *inequivalent* forms of the eigenvalue condition. One natural method of summing perturbation theory is to proceed "vacuum-polarization-insertion-wise". One first sums all internal-photon self-energy parts, and then inserts the resulting full photon propagators in the vacuum-polarization skeleton graphs. This order of summation is the one used by Johnson, Baker, and Willey, and leads naturally to the class of asymptotically finite solutions introduced by Gell-Mann and Low, in which α_0 is fixed to be the infinite-order zero and $\alpha < \alpha_0$ is undetermined. An alternative summation method is to proceed "loopwise": One first sums all single-fermion-loop vacuum-polarization graphs, then one sums all two-fermion-loop vacuum-polarization graphs, and so forth. If we assume that the single-fermion-loop $2n$-point functions do actually have the infinite-order zero in the coupling constant as described above, then by using loopwise summation we show that there is a unique additional asymptotically finite solution, in which the physical coupling α is fixed to be the infinite-order zero. We conjecture that this is the solution actually chosen by nature. According to our con*jecture, the fine-structure constant α may be computed as follows. Let $F^{[1]}(y)$ be the coefficient of the logarithmically divergent part of the sum of single-fermion-loop vacuum-polarization diagrams illustrated in Fig. 1. We conjecture that $F^{[1]}(y)$ is analytic in an interval extending from $y=0$ to $y=\alpha$, where it has an infinite-order zero as y approaches α from below along the real axis.* If the function $F^{[1]}(y)$ has no infinite-order zero, then the renormalized photon propagator cannot be asymptotically finite. Our conjecture has the obvious virtue that it stands or falls according to the outcome of the mathematical problem of calculating the function $F^{[1]}(y)$. This problem will be well posed in perturbation theory if $F^{[1]}(y)$ is a function of the class which is uniquely defined by the coefficients of its formal power-series development in y.[9]

The paper is organized as follows. In Sec. II we give a review of previous work on the short-distance behavior of electrodynamics. We derive the Gell-Mann–Low equation for the asymptotic behavior of the photon propagator, discuss its properties, and establish its relation to the recent work of Callan and Symanzik.[10] We then review the program of Johnson, Baker, and Willey for the removal of infinities from electrodynamics. In Sec. III we show that the zero of the Gell-Mann–Low function must be an essential singularity and discuss the implications of this for the conventional

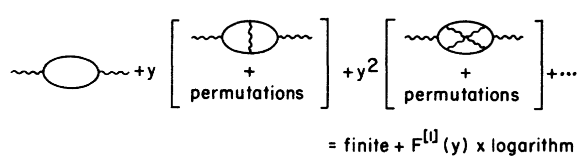

FIG. 1. Sum of single-fermion-loop vacuum-polarization diagrams which determines the function $F^{[1]}(y)$, with the dependence on the coupling constant y indicated explicitly. Throughout the paper we will adhere to the convention of using solid lines to denote fermions, wavy lines to denote photons.

eigenvalue condition on α_0 and for the asymptotic behavior of the renormalized electron propagator. In Sec. IV we introduce the idea of "loopwise" summation and show that, assuming the presence of the essential singularity, there is an asymptotically finite solution of electrodynamics in which α, rather than α_0, is fixed to be the infinite order zero. In Sec. V we motivate our conjecture that nature picks the solution in which α is fixed, and we suggest a possible connection of our work with a conjecture of Dyson[9] concerning singularities in electrodynamics at $\alpha = 0$. We also point out that our conjecture leads to a determination of α which is independent of the number of fundamental fermion species, and based on this fact, give a speculative argument justifying the neglect of the strong interactions in formulating our eigenvalue condition for α. In Appendix A we give a summary of notation, while in Appendix B we derive the Callan-Symanzik equations for massive photon (i.e., infrared-cutoff) spinor electrodynamics in an arbitrary covariant gauge, and briefly sketch the application of these equations to deriving the Johnson-Baker-Willey asymptotic form for the electron propagator.

II. REVIEW OF PREVIOUS WORK

We begin with a survey of the papers of Gell-Mann and Low, of Callan and Symanzik and of Johnson, Baker and Willey dealing with the short distance behavior of electrodynamics. Our particular aim will be to examine the underlying assumptions which these authors make and to discuss the connections between their approaches.

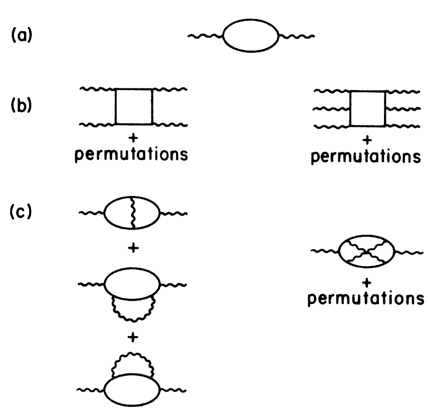

FIG. 2. (a) Lowest-order vacuum-polarization contribution to $\pi(q^2)_{\mu\nu}$. (b) Vacuum-polarization loops with four or more vertices. (c) Vacuum-polarization contributions to $\pi(q^2)$ which involve the loops with four or more vertices illustrated in (b).

A. Cutoff (Unrenormalized) and Renormalized Quantum Electrodynamics

In order for the renormalization constants and the unrenormalized propagators and vertex parts to be well-defined, it is necessary to introduce cutoffs. In addition to an ultraviolet cutoff Λ, we will eliminate infrared divergences by giving the photon a nonzero mass μ. The infrared cutoff will be needed for the derivation of the Callan-Symanzik equations for the electron propagator given in Appendix B. Where no infrared divergences are encountered, such as in the discussion of the asymptotic photon propagator which occupies the bulk of the paper, the photon mass μ will be set to zero. Specifically, we introduce the cutoffs as follows:

(i) The propagator for a bare photon of four-momentum q is given by

$$D_F^0(q)_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2-\mu_0{}^2}\,\frac{-\Lambda^2}{q^2-\Lambda^2} + Z_3(\xi-1)\frac{q_\mu q_\nu}{q^2}\,\frac{1}{q^2-\mu^2}\,\frac{-\Lambda^2}{q^2-\Lambda^2}\,, \qquad (2)$$

with μ_0 the bare photon mass and ξ a guage parameter. The reason for the peculiar choice of the longitudinal term in Eq. (2) will become apparent very soon.

(ii) The lowest-order vacuum-polarization contribution to the photon proper self-energy $\pi(q^2)_{\mu\nu}$ comes from the fermion loop diagram illustrated in Fig. 2(a). We calculate this contribution in the following manner: First we impose gauge invariance to remove the quadratic divergence, and then we regulate the fermion loop, with fermion regulator mass Λ, to remove the logarithmic divergence.

(iii) All vacuum polarization loops with four or more vertices [see Fig. 2(b)] are calculated by imposing gauge invariance, which makes them finite. The requirement of gauge-invariant calculation of fermion loops, together with the photon propagator cutoff specified in (i), renders convergent the vacuum polarization contribution to $\pi(q^2)_{\mu\nu}$ of the type illustrated in Fig. 2(c). The photon propagator cutoff also makes finite all electron self-energy parts and vertex parts, so our cutoff procedure is sufficient to make the unrenormalized theory well defined.

We can now proceed to define renormalization constants and renormalized (i.e., Λ-independent in the limit $\Lambda \to \infty$) n-point functions. The renormalized electron propagator and electron-photon vertex are introduced in the standard[11] manner; we review in detail only the construction of the renormalized photon propagator. Since rules (ii) and (iii) guarantee the gauge invariance of the photon proper self-energy part, we may write

$$\pi(q^2)_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) q^2 \pi(q^2) . \tag{3}$$

Letting α_b denote the canonical (bare) coupling and summing the series illustrated in Fig. 3 to get the full unrenormalized photon propagator $D_F'(q)_{\mu\nu}$, we get

$$D_F'(q)_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \times \frac{1}{q^2 - \mu_0^2 + \alpha_b q^2 \pi(q^2)[1 + O(q^2/\Lambda^2)]} \frac{-\Lambda^2}{q^2 - \Lambda^2} + Z_3(\xi - 1) \frac{q_\mu q_\nu}{q^2} \frac{1}{q^2 - \mu^2} \frac{-\Lambda^2}{q^2 - \Lambda^2} . \tag{4}$$

We fix the unrenormalized photon mass μ_0^2 by requiring Eq. (4) to have a pole at $q^2 = \mu^2$, i.e.,

$$\mu^2 - \mu_0^2 + \alpha_b \mu^2 \pi(\mu^2) = 0 . \tag{5}$$

We then make the algebraic rearrangement

$$\begin{aligned} q^2 - \mu_0^2 + \alpha_b q^2 \pi(q^2) &= q^2 - \mu^2 + \alpha_b[q^2 \pi(q^2) - \mu^2 \pi(\mu^2)] \\ &= (q^2 - \mu^2)[1 + \alpha_b \pi(\mu^2)] + \alpha_b q^2[\pi(q^2) - \pi(\mu^2)] \\ &= Z_3^{-1}\{q^2 - \mu^2 + \alpha q^2[\pi(q^2) - \pi(\mu^2)]\} , \end{aligned} \tag{6}$$

which introduces the photon wave-function renormalization constant Z_3,

$$Z_3^{-1} = 1 + \alpha_b \pi(\mu^2) , \tag{7a}$$

and the renormalized coupling constant α,

$$\alpha = \alpha_b Z_3 = \frac{\alpha_b}{1 + \alpha_b \pi(\mu^2)} . \tag{7b}$$

Comparing Eq. (7) with Eq. (5), we note that the photon bare mass can be reexpressed as

$$\mu_0^2 = Z_3^{-1} \mu^2 , \tag{8}$$

indicating that it is not an independent renormalization constant. To get the full renormalized photon propagator, we multiply Eq. (4) by α_b and let the cutoff Λ become infinite, giving

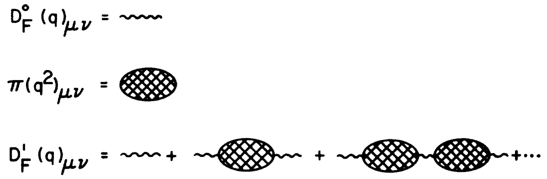

FIG. 3. Series which defines the full unrenormalized photon propagator $D_F'(q)_{\mu\nu}$.

$$\begin{aligned} \alpha \tilde{D}_F'(q)_{\mu\nu} &= \lim_{\Lambda\to\infty} \alpha_b D_F'(q)_{\mu\nu} \\ &= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \frac{\alpha}{q^2 - \mu^2 + \alpha q^2 \pi_c(q^2)} \\ &\quad + \alpha(\xi - 1) \frac{q_\mu q_\nu}{q^2} \frac{1}{q^2 - \mu^2} , \end{aligned} \tag{9}$$

with

$$\pi_c(q^2) = \lim_{\Lambda\to\infty} [\pi(q^2) - \pi(\mu^2)] . \tag{10}$$

We can now see why the longitudinal part of the bare propagator had to be chosen as in Eq. (2): Because of the transversality of $\pi(q^2)_{\mu\nu}$, the longitudinal part of the full propagator [Eq. (4)] is the same as the longitudinal part of the bare propagator. Therefore, in order for the longitudinal part of the renormalized propagator to be finite, the longitudinal part of the bare propagator must become finite when multiplied by α_b. This dictates the overall factor of Z_3, and the use of μ^2 rather than μ_0^2 in the denominator. The fact that the gauge parameter ξ always occurs in the combination $(\xi - 1)Z_3$ will be of importance in the derivation of the Callan-Symanzik equations for the electron propagator given in Appendix B. On the other hand, the form of the longitudinal terms is irrelevant to the subsequent discussion of the Gell-Mann–Low and Callan-Symanzik equations for the photon propagator, because the photon proper self-energy is strictly gauge-invariant (rather than being merely gauge-covariant, as is the case for the electron propagator and the electron-photon vertex) and hence receives no contribution from the longitudinal terms.

To conclude this section, we state the specialization of Eqs. (7)–(10) to the case of massless-photon electrodynamics ($\mu^2 = \mu_0^2 = 0$). We have

$$\alpha \tilde{D}_F'(q)_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \frac{\alpha d_c(-q^2/m^2, \alpha)}{q^2} + \alpha(\xi - 1) \frac{q_\mu q_\nu}{(q^2)^2} , \tag{11}$$

with

$$d_c(-q^2/m^2, \alpha) = [1 + \alpha \pi_c(q^2)]^{-1} \tag{12}$$

a dimensionless function which contains all the dynamical effects of vacuum polarization, and with $\pi_c(q^2)$ now given by

$$\pi_c(q^2) = \lim_{\Lambda\to\infty} [\pi(q^2) - \pi(0)] . \tag{13}$$

B. The Gell-Mann-Low Equation

We turn next to a review of the Gell-Mann–Low equation, which describes the asymptotic properties of the photon propagator. It will be useful to define an "asymptotic part" of the renormalized photon propagator, which we denote by $\alpha d_c^\infty(-q^2/m^2, \alpha)$, in the following manner: We develop $\alpha d_c(-q^2/m^2, \alpha)$, in a perturbation expansion in powers of α and *in each order of perturbation theory* drop terms which vanish as $-q^2/m^2 \to \infty$, while retaining terms which are constant or which increase logarithmically.[12] The resulting sum of constant and logarithmic terms is the "asymptotic part" and clearly has the form

$$\alpha d_c^\infty(-q^2/m^2, \alpha) = q(\alpha) + p(\alpha)\ln(-q^2/m^2) + r(\alpha)\ln^2(-q^2/m^2) + \cdots . \tag{14}$$

Throughout the analysis which follows we will make the assumption that the nonasymptotic terms which we have neglected in each order of perturbation theory do not sum to give a result which dominates asymptotically over the logarithmic series in Eq. (14). That is, we assume that the asymptotic behavior of the "asymptotic part" αd_c^∞ correctly describes the asymptotic behavior of the exact propagator αd_c.[13]

To facilitate the derivation of the Gell-Mann–Low equation we introduce a notation which explicitly indicates the point where the subtraction in the photon proper self-energy is made. Thus, letting $x = -q^2/m^2$, we write

$$\begin{aligned} \alpha d_c[x, w, \alpha] &\equiv \alpha\{1 + \alpha(\pi[x] - \pi[w])\}^{-1}, \\ \pi[x] &\equiv \pi(-m^2 x) = \pi(q^2) . \end{aligned} \tag{15}$$

In terms of the new notation, the usual renormalized photon propagator is

$$d_c(x, \alpha) = d_c[x, 0, \alpha] , \tag{16}$$

with the renormalized charge α given by

$$\alpha = \alpha d_c[0, 0, \alpha] . \tag{17}$$

Let us now imagine that, instead of making expansions in powers of the usual fine-structure constant α, we use as expansion parameter a new fine-structure constant α_w defined by

$$\alpha_w = \alpha d_c[w, 0, \alpha] = \alpha\{1 + \alpha(\pi[w] - \pi[0])\}^{-1} . \tag{18}$$

From the definition of Eq. (15), we may write

$$\begin{aligned} \alpha_w d_c[x, w, \alpha_w] &= \alpha_w\{1 + \alpha_w(\pi[x] - \pi[w])\}^{-1} \\ &= (\alpha_w^{-1} + \pi[x] - \pi[w])^{-1} , \end{aligned} \tag{19}$$

which on substitution of Eq. (18) becomes

$$\begin{aligned} \alpha_w d_c[x, w, \alpha_w] &= \{\alpha^{-1} + \pi[y] - \pi[0] + \pi[x] - \pi[w]\}^{-1} \\ &= \alpha d_c[x, 0, \alpha] = \alpha d_c(x, \alpha) . \end{aligned} \tag{20}$$

On the right-hand side we have the usual photon propagator, which involves a subtraction at the nonasymptotic point zero; Eq. (20) states that this can be reexpressed in terms of the new charge α_w and the photon propagator subtracted at w, with no further reference to the point zero.

Let us now let x and w both become large. According to our earlier discussion, the right-hand side of Eq. (20) becomes the logarithmic series $\alpha d_c^\infty(x, \alpha)$. For the left-hand side, we introduce the asymptotic assumption that the only dependence on x and w, when both are large, is through the ratio x/w. An equivalent statement of the asymptotic assumption is that when $x = -q^2/m^2$ and $w = -q'^2/m^2$ are both large, the quantity $\alpha_w d_c[x, y, \alpha_w]$, regarded as a power series in α_w, becomes independent of the electron mass m.[14] This assumption can actually be justified order-by-order in perturbation theory, either by using the analysis of Callan and Symanzik (see below) or by invoking the theorem on cancellation of infrared singularities of Kinoshita[15] and Lee and Nauenberg.[15] Equation (20) now becomes

$$\alpha_w D[x/w, \alpha_w] = \alpha d_c^\infty(x, \alpha) , \tag{21}$$

where, since w is large, we may rewrite Eq. (18) for α_w as

$$\begin{aligned} \alpha_w &= \alpha d_c^\infty(w, \alpha) \\ &= q(\alpha) + p(\alpha)\ln w + r(\alpha)\ln^2 w + \cdots . \end{aligned} \tag{22}$$

Equation (21) gives a functional relation for d_c^∞, which may be rewritten in a more useful form as follows: We introduce the function $\psi(z)$ by the definition

$$\psi(z) = \frac{\partial}{\partial v} z D[v, z]\bigg|_{v=1} . \tag{23}$$

Differentiating Eq. (21) with respect to x, and then setting $x = w$, we get the differential equation

$$\frac{1}{w}\psi(\alpha_w) = \frac{d\alpha_w}{dw} . \tag{24a}$$

Rewriting this as

$$\frac{dw}{w} = \frac{d\alpha_w}{\psi(\alpha_w)} , \tag{24b}$$

integrating with respect to w from 1 to x and using the boundary condition

$$\alpha_w|_{w=1} = q(\alpha) = \alpha d_c^\infty(1, \alpha) \tag{25}$$

we get finally the Gell-Mann–Low equation,

$$\ln x = \int_{q(\alpha)}^{\alpha d_c^\infty(x, \alpha)} \frac{dz}{\psi(z)} . \tag{26}$$

It is also useful to have the inversion formula relating the coefficients $q(\alpha), p(\alpha), \ldots,$ in the logarithmic expansion of Eq. (22) to the Gell-Mann–Low function $\psi(z)$. To get this, we write $z = \alpha_x = \alpha d_c^\infty(x, \alpha)$ and make a Taylor expansion of z with respect to $\ln x$,

$$z=\sum_{n=0}^{\infty}\frac{(\ln x)^n}{n!}\frac{d^n}{d(\ln x)^n}z\bigg|_{\ln x=0} \tag{27a}$$

According to Eq. (24), the derivative $d/d(\ln x)$ may be rewritten as

$$\frac{d}{d(\ln x)}=\frac{dz}{d(\ln x)}\frac{d}{dz}$$
$$=\psi(z)\frac{d}{dz}, \tag{27b}$$

giving the desired formula[6]

$$\alpha d_c^\infty(x,\alpha)=\sum_{n=0}^{\infty}\frac{(\ln x)^n}{n!}\left\{\left[\psi(z)\frac{d}{dz}\right]^n z\right\}\bigg|_{z=q(\alpha)}. \tag{28}$$

The function $\psi(z)$ appearing in these formulas is not explicitly known beyond its expansion to sixth order of perturbation theory, which is[16]

$$\psi(z)=z\left(\frac{z}{3\pi}+\frac{z^2}{4\pi^2}+\frac{z^3}{8\pi^3}\left[\tfrac{8}{3}\zeta(3)-\tfrac{101}{36}\right]+\cdots\right), \tag{29}$$

with $\zeta(3)$ the Riemann ζ function.

As Gell-Mann and Low have shown, Eq. (26) provides a powerful tool for analyzing the asymptotic behavior of the photon propagator, and leads one naturally to distinguish the following two possibilities: (a) The integral $\int dz/\psi(z)$ in Eq. (26) does not diverge until the upper limit becomes infinite. In this case $\alpha d_c^\infty(x,\alpha)\to\infty$ as $x\to\infty$, and so the photon propagator is asymptotically divergent. (b) For some finite value $z=\alpha_0$, the function $\psi(z)$ develops a sufficiently strong zero for $\int^{\alpha_0}dz/\psi(z)$ to diverge. In this case $\alpha d_c^\infty(x,\alpha)\to\alpha_0$ as $x\to\infty$ and the photon propagator is asymptotically finite. We will restrict our attention from here on exclusively to case (b), for which, as noted in the Introduction, we may write

$$\alpha d_c^\infty(x,\alpha)=\alpha_0+h(x,\alpha), \tag{30}$$
$$\lim_{x\to\infty}h(x,\alpha)=0.$$

Within the category of case (b), we wish to further distinguish between two different types of possible asymptotic behavior of the theory:

Type 1. The physical fine-structure constant α is equal to the particular value α_1 which satisfies

$$q(\alpha_1)=\alpha_0. \tag{31a}$$

According to Eq. (21), the coefficient of $(\ln x)^n$ with $n\geq 1$ is then

$$\left\{\left[\psi(z)\frac{d}{dz}\right]^n z\right\}\bigg|_{z=q(\alpha)}=\psi(q(\alpha))\left\{\frac{d}{dz}\left[\psi(z)\frac{d}{dz}\right]^{n-1}z\right\}\bigg|_{z=q(\alpha)}$$
$$=\psi(\alpha_0)\{\cdots\}=0; \tag{31b}$$

and the logarithmic series reduces to its constant term alone,

$$\alpha d_c^\infty(x,\alpha)=\alpha_0. \tag{31c}$$

In this case the Gell-Mann–Low equation degenerates to an integral over an interval of vanishing size located at the point where ψ^{-1} is infinite.

Type 2. The physical fine-structure constant α differs from α_1. The coefficients of the logarithmic terms in Eq. (28) then do not vanish and $\alpha d_c^\infty(x,\alpha)$ is a nontrivial function of x which approaches α_0 in the limit as $x\to\infty$. In this case the Gell-Mann–Low equation is nondegenerate, with the integral extending over an interval of finite size, and α is an undetermined parameter.

Clearly, as far as behavior of the photon propagator is concerned, the more general class of asymptotically finite solutions with type-2 behavior is just as satisfactory physically as the solution with type-1 behavior. (We will find additional evidence for this statement when we study the asymptotic electron propagator below.) Hence following Gell-Mann and Low, we conclude that requiring asymptotic finiteness of the renormalized photon propagator fixes α_0, but does not determine the fine-structure constant α.

To conclude this discussion of the Gell-Mann–Low equation we give a simple, concrete illustration of type-2 asymptotic behavior. Let us make the customary assumption that $\psi(z)$ is regular and vanishes with a simple zero and negative slope at $z=\alpha_0$. We ignore the fact that $\psi(z)$ also vanishes for small z and replace ψ by a linear approximation over the integration interval in the Gell-Mann–Low equation,

$$\psi(z)\approx\psi'(\alpha_0)(\alpha_0-z),\quad \psi'(\alpha_0)<0. \tag{32}$$

Then Eq. (26) can be immediately integrated to give

$$\ln x=\frac{1}{\psi'(\alpha_0)}\ln\left[\frac{\alpha d_c^\infty(x,\alpha)-\alpha_0}{q(\alpha)-\alpha_0}\right], \tag{33}$$

which can be rewritten as

$$\alpha d_c^\infty(x,\alpha)=\alpha_0+[q(\alpha)-\alpha_0]x^{\psi'(\alpha_0)}$$
$$=q(\alpha)+[q(\alpha)-\alpha_0]\sum_{n=1}^{\infty}\frac{\psi'(\alpha_0)^n(\ln x)^n}{n!}. \tag{34}$$

We see that the logarithmic series for $\alpha d_c^\infty(x,\alpha)$ is nontrivial, with all powers of $\ln x$ present, but that it sums to a function which approaches α_0 asymptotically. The nonasymptotic piece h, which is given in our example by

$$h(x,\alpha)=[q(\alpha)-\alpha_0]x^{\psi'(\alpha_0)}, \tag{35}$$

vanishes asymptotically as a power of x independently of the value of α.

C. The Callan-Symanzik Equation

A very powerful and elegant method for studying the asymptotic behavior of renormalized perturbation theory has recently been developed by Callan and Symanzik.[10] We briefly review here the derivation of the Callan-Symanzik equation for the renormalized photon propagator,[17] and indicate its connection with the Gell-Mann–Low equation discussed above. Our starting point is the formula relating the renormalized and unrenormalized photon propagators,

$$d_c(-q^2/m^2,\alpha)^{-1}=Z_3(\Lambda^2/m^2,\alpha)[1+\alpha_b\pi(q^2)]\,, \quad (36)$$

where we have explicitly indicated the cutoff dependence of Z_3. Since the photon propagator is gauge invariant, the quantities appearing in Eq. (36) have no dependence on the gauge parameter ξ. Let us now vary the physical electron mass m, with the canonical (bare) coupling α_b and the cutoff Λ held fixed. Under this variation the bare electron mass m_0 and the physical coupling α both change, since the renormalization conditions give both of them an implicit dependence on m. Thus, insofar as d_c and Z_3 are concerned, variation of m is described by

$$m\frac{d}{dm}=m\frac{\partial}{\partial m}+m\frac{d\alpha}{\partial m}\frac{\partial}{\partial\alpha}\,, \quad (37)$$

while for the bare propagator $1+\alpha_b\pi(q^2)$, which depends on m only through m_0, the mass variation is described by

$$m\frac{d}{dm}=m\frac{dm_0}{dm}\frac{\partial}{\partial m_0}\,. \quad (38)$$

Equating the mass variations of the left- and right-hand sides of Eq. (36) gives

$$\begin{aligned}\left(m\frac{\partial}{\partial m}+m\frac{d\alpha}{dm}\frac{\partial}{\partial\alpha}\right)d_c^{-1}&=m\frac{d}{dm}d_c^{-1}\\&=m\frac{d}{dm}Z_3[1+\alpha_b\pi]+Z_3m\frac{d}{dm}[1+\alpha_b\pi]\\&=Z_3^{-1}m\frac{d}{dm}Z_3d_c^{-1}+\alpha m\frac{dm_0}{dm}\frac{\partial}{\partial m_0}\pi\,.\end{aligned} \quad (39)$$

The term $\partial\pi/\partial m_0$ on the right-hand side of Eq. (39) is simply interpreted as a photon-photon-scalar vertex part, with the scalar current carrying zero four-momentum. It can be shown[18] that this vertex part is made finite by multiplication by the renormalization constant m_0, and so we can write

$$m_0\frac{\partial}{\partial m_0}\pi=\tilde{\Gamma}_{\gamma\gamma S}(q^2/m^2,\alpha)\,. \quad (40)$$

It can be further shown[10,18] that the quantities $\beta(\alpha)$ and $\delta(\alpha)$ defined by

$$\begin{aligned}\beta(\alpha)&=Z_3^{-1}m\frac{d}{dm}Z_3\,,\\1+\delta(\alpha)&=m_0^{-1}m\frac{d}{dm}m_0\end{aligned} \quad (41)$$

are cutoff-independent and therefore, as indicated, are functions of α alone. Finally, we can relate the quantity $m\,d\alpha/dm$ appearing on the left-hand side of Eq. (39) to $\beta(\alpha)$, as follows:

$$\begin{aligned}m\frac{d\alpha}{dm}&=m\frac{d}{dm}(\alpha_bZ_3)=\alpha_bm\frac{d}{dm}Z_3\\&=\alpha Z_3^{-1}m\frac{d}{dm}Z_3=\alpha\beta(\alpha)\,.\end{aligned} \quad (42)$$

Putting everything together we get the Callan-Symanzik equation for the photon propagator,

$$\left[m\frac{\partial}{\partial m}+\beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha}-1\right)\right]d_c(-q^2/m^2,\alpha)^{-1}=\alpha[1+\delta(\alpha)]\tilde{\Gamma}_{\gamma\gamma S}(q^2/m^2,\alpha)\,. \quad (43)$$

Let us now let $-q^2/m^2$ become infinite. Order by order in perturbation theory, the left-hand side of Eq. (43) becomes

$$\left[m\frac{\partial}{\partial m}+\beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha}-1\right)\right]d_c^\infty(-q^2/m^2,\alpha)^{-1}, \quad (44)$$

while a simple application of Weinberg's theorem[19] shows that, again order by order in perturbation theory, the right-hand side of Eq. (43) vanishes. So we learn that d_c^∞ satisfies the differential equation

$$\left[m\frac{\partial}{\partial m}+\beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha}-1\right)\right]d_c^\infty(-q^2/m^2,\alpha)^{-1}=0\,. \quad (45a)$$

Interestingly, when we substitute Eq. (37) for md/dm into Eq. (42), we learn[17] that $Z_3(\Lambda^2/m^2,\alpha)$ satisfies a differential equation identical in form to Eq. (45a),

$$\left[m\frac{\partial}{\partial m}+\beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha}-1\right)\right]Z_3(\Lambda^2/m^2,\alpha)=0\,. \quad (45b)$$

For the subsequent discussion, it will be useful to reexpress Eq. (45a) as a differential equation for d_c^∞,

$$\left[m\frac{\partial}{\partial m}+\beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha}+1\right)\right]d_c^\infty(-q^2/m^2,\alpha)=0\,. \tag{46}$$

We will now show that Eq. (46) is completely equivalent to Eq. (26), the Gell-Mann–Low equation. Letting x, as before denote $-q^2/m^2$, we rewrite Eq. (46) in the form

$$\left[-2x\frac{\partial}{\partial x}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}\right]\alpha d_c^\infty(x,\alpha)=0\,. \tag{47}$$

This equation has the integral

$$\alpha d_c^\infty(x,\alpha)=\Phi^{-1}\left[\ln x+\int_c^\alpha\frac{2dz}{z\beta(z)}\right], \tag{48}$$

with the function Φ determined by the $x=1$ boundary condition

$$\Phi[\alpha d_c^\infty(1,\alpha)]=\Phi[q(\alpha)]=\int_c^\alpha\frac{2dz}{z\beta(z)} \tag{49}$$

and with c an arbitrary constant of integration. (The presence of c merely reflects the freedom of changing Φ by an arbitrary additive constant.) Inverting Eq. (48), we thus can write

$$\ln x=\Phi[\alpha d_c^\infty(x,\alpha)]-\Phi[q(\alpha)] \tag{50}$$

which, if we write $\Phi[u]$ in the integral form

$$\Phi[u]=\int_{c'}^u\frac{dz}{\psi(z)}\,,\qquad \psi(z)=[\Phi'(z)]^{-1}, \tag{51}$$

can be further recast as

$$\ln x=\int_{q(\alpha)}^{\alpha d_c^\infty(x,\alpha)}\frac{dz}{\psi(z)}\,, \tag{52}$$

which is just the Gell-Mann–Low equation. Clearly, the derivation which we have just given does not involve the asymptotic assumption made in the discussion immediately following Eq. (20); in effect, the Callan-Symanzik route to the Gell-Mann–Low equation replaces a statement about infrared behavior (m independence of $\alpha_w d_c[x,w,\alpha_w]$ as $m\to 0$) with a statement about ultraviolet behavior (asymptotic vanishing of $\tilde{\Gamma}_{\gamma\gamma S}$) which can be justified in perturbation theory by the use of Weinberg's theorem.

Comparing Eq. (51) with Eq. (49), we can read off the following functional relationship between the Callan-Symanzik function $\beta(\alpha)$ and the Gell-Mann–Low function $\psi(z)$,

$$\beta(\alpha)=\frac{2\psi(q(\alpha))}{\alpha q'(\alpha)}\,. \tag{53}$$

Thus, in the case of type-1 asymptotic behavior, where $\alpha=\alpha_1$, we have $\beta(\alpha)=0$. Equation (47) then reduces to

$$x\frac{\partial}{\partial x}\alpha d_c^\infty(x,\alpha)=0\,, \tag{54}$$

which has, as expected, the integral

$$\alpha d_c^\infty(x,\alpha)=\alpha_0\,. \tag{55}$$

Similarly, Eq. (45b) tells us that when $\beta(\alpha)=0$, the photon wave-function renormalization Z_3 is cutoff-independent. As shown in Appendix B, the Callan-Symanzik equation for the renormalized electron propagator also has the function $\beta(\alpha)$ as coefficient of the $\partial/\partial\alpha$ term. Consequently, in the case of type-1 asymptotic behavior this equation also simplifies, and leads, by a simple argument,[18] to an elementary scaling form for the asymptotic electron propagator. Clearly, in the case of type-2 asymptotic behavior we have $\beta(\alpha)\neq 0$ and must deal with the Callan-Symanzik equations in their full complexity. Even so, we find in Appendix B that the scaling form for the asymptotic electron propagator still holds, again indicating, as asserted above, that there is no reason for favoring the type-1 solution over the more general class of asymptotically finite solutions with type-2 asymptotic behavior.

D. The Johnson-Baker-Willey Program

We conclude our review by surveying the recent work of Johnson, Baker, and Willey (JBW) dealing with the asymptotic properties of electrodynamics. As noted in Sec. I, this work has led to two principal results: a simplified form of the Gell-Mann–Low eigenvalue condition for α_0, and a demonstration that if the eigenvalue condition is satisfied, then the electron bare mass m_0 and wave-function renormalization constant Z_2 can be finite. We discuss these two aspects in turn.

1. *Simplified Eigenvalue Condition*

A key ingredient in the JBW formulation of the eigenvalue condition is the use of "vacuum-polarization-insertion-wise" summation of the photon proper self-energy part π. The basic idea is to first write down a modified skeleton expansion for the photon proper self-energy in which all diagrams with internal vacuum polarization insertions are omitted. Some typical diagrams which appear in this expansion are illustrated in Fig. 4; note that they still contain internal electron self-energy and electron-photon vertex parts. The next step is to replace all internal photon lines appearing in the expansion by full renormalized photon propagators $\alpha\tilde{D}_F'(q)_{\mu\nu}$ (we indicate explicitly the coupling constant α associated with the ends of the photon line).

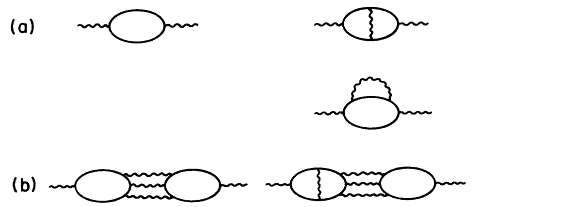

FIG. 4. Typical diagrams which appear in the modified skeleton expansion for the photon proper self-energy part π. All proper diagrams are included which do not have internal photon self-energy insertions. Diagrams (a) have a single fermion loop, while those labeled (b) contain two or more fermion loops.

This recipe leads to a "vacuum-polarization-insertion-wise" summed expression for the photon proper self-energy π, which, it is easy to see, correctly includes all of the relevant Feynman diagrams.

We next introduce the assumption that the renormalized photon propagator is asymptotically finite, which allows us to write it in the form

$$\alpha\tilde{D}_F'(q)_{\mu\nu}=\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2}[\alpha_0+h(-q^2/m^2,\alpha)] + \alpha(\xi-1)\frac{q_\mu q_\nu}{(q^2)^2}, \tag{56}$$

with h vanishing asymptotically. In order for this assumption to be self-consistent, we require that the renormalized photon proper self-energy part π_c, obtained by inserting Eq. (56) in the skeleton expansion as outlined above, must itself be asymptotically finite. That is, no powers of $\ln(-q^2/m^2)$ may be present in the asymptotic behavior of π_c. To determine the asymptotic properties of π_c, we must consider each contributing graph (or more exactly, each gauge-invariant set of graphs related by permutation of photon vertices) and examine the convergence properties of both the over-all integration, involving all lines in the graph, and the subintegrations involving subsets of these lines. In doing this we maintain our "vacuum-polarization-insertion-wise" summation by treating internal photon propagators as complete entities, described by Eq. (56), rather than also breaking these up into contributing graphs. By our assumption of Eq. (56), the internal photon propagators cannot give rise to any logarithmic terms. Subintegrations associated with electron-self-energy and electron-photon vertex parts also lead to no logarithms, because as shown in Sec. II D 2, there is a gauge (the Landau gauge) which makes these asymptotically finite. It can be shown[7,18] that there are no other troublesome subintegrations; hence logarithmic behavior of a graph contributing to π_c can only be associated with the over-all integration involving all lines in the graph. Referring to Eq. (56), we see that each internal photon line in the over-all integration contributes two parts, a part proportional to α_0 and a part proportional to h. As we have seen in Eqs. (32)–(35) above, the conventional assumptions about the nature of the zero of $\psi(z)$ imply that h decreases as a power of $-q^2/m^2$ as $-q^2/m^2\to\infty$. As a result, any contribution to the over-all integration involving one or more factors h converges, and leads to an asymptotically finite contribution to π_c. Hence the asymptotically logarithmic part of π_c is correctly obtained by neglecting h in each internal photon propagator, so that Eq. (56) becomes

$$\alpha\tilde{D}_F'(q)_{\mu\nu}=-g_{\mu\nu}\frac{\alpha_0}{q^2}+\text{gauge term}. \tag{57}$$

Thus, we are led to a simplified model for π_c (the so-called JBW model) in which no internal photon self-energy insertions appear; all internal photons are described by free propagators coupling with the asymptotic coupling strength α_0. An analysis[7,18] of the asymptotic behavior of π_c in this model shows that a single logarithm is present (corresponding to the fact that a single subtraction suffices to make the over-all integration converge), so we get finally

$$\pi_c \underset{-q^2/m^2\to\infty}{\sim} g(\alpha_0)+f(\alpha_0)\ln(-q^2/m^2). \tag{58}$$

Self-consistency of the assumption of asymptotic finiteness of π_c now requires

$$f(\alpha_0)=0, \tag{59}$$

which is the JBW form of the eigenvalue condition. Let us reiterate that Eq. (59) does not involve all vacuum polarization graphs [as does the Gell-Mann–Low eigenvalue condition $\psi(\alpha_0)=0$] but rather only the restricted class, illustrated in Fig. 4, which have no internal photon self-energy insertions.

Implicit in the derivation of Eq. (59) are rather stringent convergence assumptions. These arise because the argument leading to Eq. (59) involves replacing the limit of an infinite sum [the exact $\pi_c(q^2)$ is an infinite sum of skeleton graphs with photon self-energy insertions] by the sum of the limits of the individual terms. [Eq. (58) is the sum of skeletons with the photon self-energy insertions replaced by their asymptotic limits.] A necessary, but by no means sufficient, condition for the interchange of limit with sum to be valid is that the resulting series $f(\alpha_0)$ be convergent. This fact will be of importance in the discussion of "loopwise" summation given in Sec. IV below.

In their recent papers,[7] Baker and Johnson have extended in two respects the treatment of the eigenvalue equation sketched above: First, they have shown that $f(\alpha_0)=0$ implies that α_0 is also a zero of the Gell-Mann–Low function $\psi(y)$, and secondly, they have shown (again assuming "vacuum-polarization-insertion-wise" summation) that Eq. (59) can be replaced by the much simpler eigenvalue condition

$$F^{[1]}(\alpha_0)=0\,, \tag{60}$$

where $F^{[1]}(y)$ is the single fermion loop part of $f(y)$ introduced in Sec. I. The first assertion is proved by an argument (which we omit) based on properties of the modified skeleton expansion, showing that $\psi(y)$ and $f(y)$ are functionally related,

$$\psi(y)=\sum_{n=1}^{\infty}[f(y)]^n c_n(y) = f(y)c_1(y)+f^2(y)c_2(y)+\cdots\,. \tag{61}$$

Hence a zero of f is necessarily a zero of ψ. The second assertion follows from a simple argument based on the Federbush-Johnson theorem; we give details in the case, since the results are central to the discussion of Sec. III below. We assume that the Gell-Mann–Low eigenvalue equation $\psi(y)=0$ has a solution $y=\alpha_0$, so that the renormalized photon propagator takes the form of Eq. (1). If we now let the electron mass m approach zero, we learn from Eq. (1) that the renormalized photon propagator d_c approaches its asymptotic value α_0 for any $q^2\neq 0$. This means that *in a theory of massless spin-$\frac{1}{2}$ electrodynamics satisfying the eigenvalue condition, the full renormalized photon propagator is exactly equal to the free photon propagator,* with coupling constant α_0. Consequently, the absorptive part of the photon proper self-energy vanishes; i.e., we have

$$\langle 0|j_\mu(x)j_\nu(y)|0\rangle=0\,, \tag{62}$$

where j_μ is the electromagnetic current operator. By exploiting positivity of the absorptive part of the full photon propagator, Federbush and Johnson[8,20] have shown that the vanishing of the two-point function in Eq. (62) implies that $j_\mu(x)$ annihilates the vacuum, and hence the general $2n$-point current correlation function vanishes as well,

$$\langle 0|T[j_{\mu_1}(x_1)j_{\mu_2}(x_2)\cdots j_{\mu_{2n}}(x_{2n})]|0\rangle=0,\quad n\geq 2\,. \tag{63}$$

Equation (63) is the essential tool which allows us to simplify the eigenvalue condition. Let us take the difference between the photon self-energy part π_c evaluated at four-momenta q^2 and q'^2. Since the full photon propagator is equal to the free photon propagator in the massless theory, this difference may be calculated from the skeleton diagrams of Fig. 4, and according to Eq. (58) is given by

$$\pi_c(q^2)-\pi_c(q'^2)=f(\alpha_0)\ln(q^2/q'^2)\,. \tag{64}$$

The contributions to Eq. (64) may be divided into two basic types: those containing a single closed fermion loop [Fig. 4(a)] and those containing two or more closed fermion loops [Fig. 4(b)]. The sum of contributions of the second type can be recast as a sum involving current correlation functions which have been linked together by photon lines, and therefore vanishes by Eq. (63). Thus, the vanishing of the logarithmic term in Eq. (64) implies that the sum of contributions of the first type must vanish by itself, which gives the simplified eigenvalue condition

$$F^{[1]}(\alpha_0)=0\,. \tag{65}$$

Clearly, the same argument applied to Eq. (63) shows that *the sum of single closed fermion loop contributions to the general $2n$-point current correlation function ($n\geq 2$) must vanish by itself when the coupling is α_0 and the fermion is massless,* a result which will be of great utility in the next section. We stress in closing that the powerful results which we have just described are consequences of positivity of the spectral function of the photon propagator. In particular, since the single closed fermion loop contributions to π_c do not by themselves have a positive spectral function, the methods which we have used cannot be used to prove the converse result that a zero of $F^{[1]}[y]$ is necessarily a zero of $f(y)$ and $\psi(y)$.[21]

2. Asymptotic Electron Propagator and Finiteness of Z_2 and m_0

To analyze the asymptotic electron propagator, JBW employ the simplified model described above, in which the asymptotically vanishing part h of the photon propagator is neglected. Each internal photon is thus represented by a free propagator, coupling with the asymptotic coupling strength α_0. In this model it is straightforward to determine the asymptotic behavior of the renormalized electron propagator and of the renormalization constants Z_2 and m_0, either by using renormalization group methods[7] or by use of the Callan-Symanzik equation,[18] with the results

$$\tilde{S}_F'(p)^{-1}\underset{p\to\infty}{\sim}F_1(\alpha_1)C_1(\mu^2/m^2,\alpha_1)\left(-\frac{p^2}{m^2}\right)^{\gamma(\alpha_1)/2}\times\left[\not p-mF_2(\alpha_1)C_2(\mu^2/m^2,\alpha_1)\left(-\frac{p^2}{m^2}\right)^{-\delta(\alpha_1)/2}\right], \tag{66}$$

$$Z_2=C_1(\mu^2/m^2,\alpha_1)\left(\frac{\Lambda^2}{m^2}\right)^{\gamma(\alpha_1)/2},$$

$$m_0=C_2(\mu^2/m^2,\alpha_1)m\left(\frac{\Lambda^2}{m^2}\right)^{-\delta(\alpha_1)/2}.$$

In writing Eq. (66) we have used the fact that in the

JBW model the mapping $q(\alpha)$ is effectively the unit mapping $q(\alpha)=\alpha$, and so Eq. (30) tells us that

$$\alpha_0 = q(\alpha_1) = \alpha_1 . \tag{67a}$$

The function $\delta(\alpha_1)$ is defined in Eq. (41), while the definition of $\gamma(\alpha_1)$ is given in Appendix B. The transformation properties of Eq. (17) under changes in the gauge parameter ξ can be explicitly worked out,[18] and for the exponents γ and δ we find (primed quantities refer to gauge parameter ξ', unprimed to gauge parameter ξ)

$$\begin{aligned} \gamma' - \gamma &= \frac{\alpha_1}{2\pi}(\xi' - \xi), \\ \delta' - \delta &= 0 . \end{aligned} \tag{67b}$$

Thus, if we choose ξ' to satisfy

$$(\alpha_1/2\pi)(\xi' - \xi) + \gamma(\alpha_1, \xi) = 0 ,$$

then we have $\gamma' \equiv \gamma(\alpha_1, \xi') = 0$ and the electron wave function renormalization Z_2' remains finite as $\Lambda \to \infty$. Furthermore, if $\delta(\alpha_1) > 0$, the electron bare mass m_0 vanishes in the limit of infinite cutoff, indicating that the physical mass of the electron is entirely electromagnetic in origin. The apparent logarithmic divergence of m_0 in perturbation theory results only when

$$m_0 = C_2(\mu^2/m^2, \alpha_1) m \exp[-\tfrac{1}{2}\delta(\alpha_1)\ln(\Lambda^2/m^2)] \tag{68}$$

is expanded in a power series in $\alpha_1 = \alpha_0$ and illegally truncated at a finite order. Thus, in the model with the photon propagator replaced by its finite asymptotic part, all perturbation theory infinities can be eliminated, provided only that $\delta(\alpha_1) > 0$.

A little caution is required, however, in applying the results of the JBW model to the full theory, where the photon propagator contains the nonasymptotic piece h in addition to the asymptotic part α_0. Because the renormalization counterterms which are subtracted in going from the unrenormalized to the renormalized electron propagator are evaluated at the nonasymptotic four-momentum $\not{p} = m$, it is easy to see that h makes a nonvanishing contribution to the asymptotic renormalized electron propagator. Thus Eq. (66) does not necessarily apply to the full theory. In Appendix B we analyze the effect of h on the asymptotic behavior of $\tilde{S}_F'(p)^{-1}$. Assuming that h vanishes asymptotically as a power of $-q^2/m^2$, we find that the *form* of Eq. (66) and of the exponents γ and δ are unaltered, all of the effects of h being confined to changes in the constants C_1 and C_2.[22] Hence, when h vanishes as a power, the conclusions obtained from the JBW model regarding the finiteness of Z_2 and m_0 apply to the full theory as well.

III. THE ESSENTIAL SINGULARITY AND ITS CONSEQUENCES

We continue in the present section to work with the "vacuum-polarization-insertion-wise" summation scheme described above in Sec. II D 1. We show that the argument leading to the simplified eigenvalue condition of Eq. (65) has the further implication that $F^{[1]}(y)$ vanishes at $y = \alpha_0$ with a zero of infinite order, i.e., an essential singularity. We find that as a result, the nonasymptotic piece h of the photon propagator vanishes asymptotically much more slowly than any power of $-q^2/m^2$, and we discuss consequences of this both for the eigenvalue condition and for the asymptotic behavior of the electron propagator.

A. Existence of an Essential Singularity

Since our argument makes extensive use of the properties of the single-fermion-loop $2n$-point functions, we begin by introducing a compact notation for these. Let us denote the sum of single-fermion-loop contributions to the photon proper self-energy by

$$\pi_c^{[1]}(q^2; m, y) , \tag{69}$$

where we have explicitly indicated the dependence on the fermion mass m and on the coupling constant y. The series of diagrams defining $\pi_c^{[1]}$ has, of course, already been exhibited in Fig. 1. According to the results of Sec. II D, when $-q^2/m^2$ approaches infinity $\pi_c^{[1]}$ has the asymptotic behavior

$$\pi_c^{[1]}(q^2; m, y) = G^{[1]}(y) + F^{[1]}(y)\ln(-q^2/m^2) + \text{vanishing terms} , \tag{70}$$

and the assumption that the Gell-Mann–Low function ψ vanishes at $y = \alpha_0$ implies that the coefficient of the logarithm in Eq. (70) also vanishes for this value of the coupling,

$$F^{[1]}(\alpha_0) = 0 . \tag{71}$$

Let us next denote the sum of single-fermion-loop contributions to the general $2n$-point current correlation function ($n \geq 2$) by

$$\begin{aligned} T_{2n}^{[1]} &= T_{\mu_1 \cdots \mu_{2n}}^{[1]}(q_1, \ldots, q_{2n}; m, y) , \\ q_1 + \cdots + q_{2n} &= 0 ; \end{aligned} \tag{72}$$

the series of diagrams defining $T_{2n}^{[1]}$ is shown in Fig. 5. In each order in the power series expansion in y, all distinct permutations of external and internal photon vertices are included in $T_{2n}^{[1]}$. As a result, $T_{2n}^{[1]}$ is independent of the gauge parameter ξ appearing in the internal photon propagators and satisfies current conservation with respect to the external photon indices,

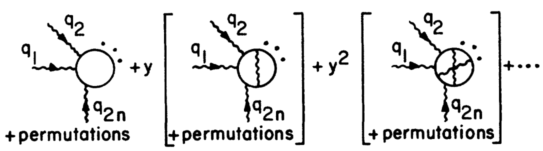

FIG. 5. Sum of single-fermion-loop diagrams which defines the $2n$-point function $T_{2n}^{[1]}$ appearing in Eq. (72), with the dependence on the coupling constant y indicated explicitly.

$$q_1^{\mu_1}T^{[1]}_{\mu_1\cdots\mu_{2n}}=q_2^{\mu_2}T^{[1]}_{\mu_1\cdots\mu_{2n}}=\cdots=q_{2n}^{\mu_{2n}}T^{[1]}_{\mu_1\cdots\mu_{2n}}=0\,. \tag{73}$$

As was shown in Sec. II D, when the fermion mass m is zero and when y is equal to α_0, the general $2n$-point current correlation function vanishes,

$$T^{[1]}_{\mu_1\cdots\mu_{2n}}(q_1,\ldots,q_{2n};0,\alpha_0)=0\,. \tag{74}$$

Finally, let us define a modified two-point function

$$\pi_{2n}^{[1]}(q^2;m,y) \tag{75}$$

by the procedure of linking $2n-2=2(n-1)$ external vertices of the general $2n$-point function with $n-1$ free photon propagators and integrating over the four-momenta carried by these propagators, thus leaving a vacuum-polarization-like tensor of second rank. Because we have enforced current conservation [Eq. (73)] and because there are no photon self-energy insertions, this second-rank tensor has only an over-all logarithmic divergence which can be made finite by a single subtraction. A simple way to perform the subtraction is to use the usual Pauli-Villars procedure of taking the difference of Eq. (75) for two distinct values of the fermion mass, giving the finite expression

$$\begin{aligned}[\pi_{2n}^{[1]}(q^2;m,y)-\pi_{2n}^{[1]}(q^2;m',y)](-q^2g_{\mu\nu}+q_\mu q_\nu)=\int\frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_{n-1}}{(2\pi)^4}\left(-\frac{ig^{\mu_1\mu_2}}{q_1^{\,2}}\right)\cdots\left(-\frac{ig^{\mu_{2n-3}\mu_{2n-2}}}{q_{n-1}^{\,2}}\right)\\ \times[T^{[1]}_{\mu_1\mu_2\cdots\mu_{2n-3}\mu_{2n-2}\mu\nu}(q_1,-q_1,\ldots,q_{n-1},-q_{n-1},q,-q;m,y)\\ -T^{[1]}_{\mu_1\mu_2\cdots\mu_{2n-3}\mu_{2n-2}\mu\nu}(q_1,-q_1,\ldots,q_{n-1},-q_{n-1},q,-q;m',y)]\,.\end{aligned} \tag{76}$$

For the sake of compactness in writing the internal photon propagators, we have restricted ourselves to the Feynman gauge, a convention to which we will adhere henceforth.

Our next step is to establish the following fundamental identity[23] relating the modified two-point function $\pi_{2n}^{[1]}$ to a derivative of the photon proper self-energy part $\pi_c^{[1]}$,

$$2^{n-1}\frac{d^{n-1}}{dy^{n-1}}[\pi_c^{[1]}(q^2;m,y)-\pi_c^{[1]}(q^2;m',y)]=\pi_{2n}^{[1]}(q^2;m,y)-\pi_{2n}^{[1]}(q^2;m',y)\,. \tag{77}$$

To prove Eq. (77), we develop the right-hand side and the bracket on the left-hand side in power series expansions in y,

$$\pi_c^{[1]}(q^2;m,y)-\pi_c^{[1]}(q^2;m',y)=\sum_{j=0}^{\infty}y^j\pi_{c,j}^{[1]}(q^2;m,m')\,, \tag{78}$$

$$\pi_{2n}^{[1]}(q^2;m,y)-\pi_{2n}^{[1]}(q^2;m',y)=\sum_{j=0}^{\infty}y^j\pi_{2n,j}^{[1]}(q^2;m,m')\,,$$

so that Eq. (77) asserts that

$$2^{n-1}\frac{(j+n-1)!}{j!}\pi_{c,j+n-1}^{[1]}(q^2;m,m')=\pi_{2n,j}^{[1]}(q^2;m,m')\,. \tag{79}$$

To verify Eq. (79), we proceed in two steps: First, we show that the functions on the left- and right-hand side are the same, apart from a multiplicative constant, and then we give a simple combinatoric argument to show that this constant is in fact $2^{n-1}(j+n-1)!/j!$.

To prove the first assertion, we refer to Fig. 1 defining $\pi_c^{[1]}$ as a power series in y. We see that $\pi_{c,j+n-1}^{[1]}(q^2;m,m')$ is just the sum of all distinct single fermion loop vacuum polarization contributions containing exactly $j+n-1$ internal virtual photons (with the logarithmic divergence eliminated by taking the difference of expressions with fermion masses m and m'). Next, we refer to Fig. 5 and Eq. (76) which respectively define $T_{2n}^{[1]}$ and $\pi_{2n}^{[1]}$. Since the y dependence of $\pi_{2n}^{[1]}$ comes entirely from $T_{2n}^{[1]}$, we see that $\pi_{2n,j}^{[1]}$ contains j internal virtual photons (the ones which appear in the y^j term of $T_{2n}^{[1]}$) plus the $n-1$ additional virtual photons inserted by the definition of Eq. (76), or a total of $j+n-1$ in all. Thus $\pi_{2n,j}^{[1]}(q^2;m,m')$ is also a sum of (mass differenced) single fermion loop vacuum polarization diagrams containing exactly $j+n-1$ internal virtual photons. Furthermore, it is readily seen that all of the relevant diagrams appear in the sum with equal weight because $T_{2n}^{[1]}$ is completely symmetric in the variables of the $2n$ external photons. Hence $\pi_{2n,j}^{[1]}$ must be a multiple of $\pi_{c,j+n-1}^{[1]}$, the constant of proportionality K reflecting the fact that in obtaining the two-point function by linking $2n-2$ external vertices of the $2n$-point function, there will be multiple counting and each relevant diagram of the two-point function will appear many times.

To complete the derivation of Eq. (79) we must calculate the proportionality constant. This is easily done by noting that

$$K = N\{\pi^{[1]}_{2n,j}\}/N\{\pi^{[1]}_{c,j+n-1}\}\,, \tag{80}$$

the numerator and denominator in Eq. (80) being the *total* number of distinct Feynman graphs appearing in $\pi^{[1]}_{2n,j}$ and in $\pi^{[1]}_{c,j+n-1}$, respectively. Let us define $N_{2n,j}$ to be the total number of distinct Feynman graphs with j internal virtual photons which contribute to the single fermion loop $2n$-point function. Then from the definitions given above we clearly have

$$\begin{aligned} N\{\pi^{[1]}_{2n,j}\} &= N_{2n,j}\,,\\ N\{\pi^{[1]}_{c,j+n-1}\} &= N_{2,j+n-1}\,. \end{aligned} \tag{81}$$

The combinatorics of calculating $N_{2n,j}$ goes as follows. We hold one external vertex fixed on the fermion loop to define a starting point. There are then $(2n+2j-1)!$ diagrams obtained by permuting the remaining $2n-1$ external vertices and the $2j$ vertices which terminate internal photon lines. However, diagrams obtained by permuting any of the j internal photon lines, or interchanging the ends of any of these lines, are identical, and so we must divide by a factor of $2^j j!$ to get the number of distinct Feynman diagrams. Thus we get[24]

$$N_{2n,j} = \frac{(2n+2j-1)!}{2^j j!}\,, \tag{82}$$

and hence

$$\begin{aligned} K &= \frac{(2n+2j-1)!}{2^j j!}\Big/\frac{[2+2(j+n-1)-1]!}{2^{j+n-1}(j+n-1)!}\\ &= 2^{n-1}(j+n-1)!/j!\,, \end{aligned} \tag{83}$$

completing the proof of Eq. (77).

We now have all the apparatus needed to show the existence of an essential singularity. Let us take the limit $m, m' \to 0$ in Eq. (77), with m/m' and q^2 fixed and with $y=\alpha_0$. The left-hand side can be evaluated from the asymptotic expression in Eq. (70), giving

$$2^{n-1}\frac{d^{n-1}}{dy^{n-1}}F^{[1]}(y)\bigg|_{y=\alpha_0}\ln(m'^2/m^2)\,. \tag{84}$$

To evaluate the limit of the right-hand side, we refer to the definition of

$$\pi^{[1]}_{2n}(q^2; m, y) - \pi^{[1]}_{2n}(q^2; m', y)$$

given in Eq. (76). We would like to be able to interchange the subtraction in the square bracket on the right-hand side with the integrations, giving

$$[\pi^{[1]}_{2n}(q^2; m, y) - \pi^{[1]}_{2n}(q^2; m', y)](-q^2 g_{\mu\nu} + q_\mu q_\nu) = I_m - I_{m'}\,, \tag{85}$$

with

$$I_m = \int \frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_{n-1}}{(2\pi)^4}\left(-\frac{ig^{\mu_1\mu_2}}{q_1^{\,2}}\right)\cdots\left(-\frac{ig^{\mu_{2n-3}\mu_{2n-2}}}{q_{n-1}^{\,2}}\right)T^{[1]}_{\mu_1\mu_2\cdots\mu_{2n-3}\mu_{2n-2}\mu\nu}(q_1, -q_1, \ldots, q_{n-1}, -q_{n-1}, q, -q; m, y)\,. \tag{86}$$

For general values of y, this interchange is not allowed, because I_m is a logarithmically divergent integral of the general type

$$\int_0^\infty \frac{d\rho}{\rho+m^2} \tag{87}$$

and hence the right-hand side of Eq. (85) is an ambiguous expression of the form $\infty-\infty$. When $y=\alpha_0$, however, the situation is different, because Eq. (74) tells us that

$$T^{[1]}_{\mu_1\mu_2\cdots\mu_{2n-3}\mu_{2n-2}\mu\nu}(q_1, -q_1, \ldots, q_{n-1}, -q_{n-1}, q, -q; 0, \alpha_0) = 0 \tag{88}$$

and consequently

$$T^{[1]}_{\mu_1\mu_2\cdots\mu_{2n-3}\mu_{2n-2}\mu\nu}(q_1, -q_1, \ldots, q_{n-1}, -q_{n-1}, q, -q; m, \alpha_0) \tag{89}$$

is proportional to m^2.[25] As a result, the convergence of Eq. (86) is improved by two powers of momentum over what it is for general values of y, and hence when $y=\alpha_0$, I_m becomes a convergent integral of the type[26]

$$\int_0^\infty \frac{cm^2 d\rho}{(\rho+m^2)(\rho+cm^2)}\,. \tag{90}$$

The interchange in Eq. (85) is now legal,[27] and taking the limit $m, m' \to 0$ gives

$$\begin{aligned} \lim_{\substack{m,m'\to 0\\ m/m'\ \text{fixed}}} &[\pi^{[1]}_{2n}(q^2; m, y) - \pi^{[1]}_{2n}(q^2; m', y)](-q^2 g_{\mu\nu} + q_\mu q_\nu)\\ &= \lim_{m\to 0} I_m - \lim_{m'\to 0} I_{m'} = 0\,. \end{aligned} \tag{91}$$

Substituting Eqs. (91) and (84) into Eq. (77) we get, finally, the fundamental result

$$\frac{d^{n-1}}{dy^{n-1}}F^{[1]}(y)\bigg|_{y=\alpha_0}=0\,, \quad n\geq 2\,. \tag{92}$$

It is important to note that Eq. (88) does *not* imply the stronger result

$$\lim_{m\to 0} I_m = 0\,, \tag{93}$$

as is readily seen by taking the $m\to 0$ limit of the specific example in Eq. (90),

$$\lim_{m\to 0}\int_0^\infty \frac{cm^2 d\rho}{(\rho+m^2)(\rho+cm^2)}=\frac{c\ln c}{c-1}\,. \tag{94}$$

Thus, our argument gives us no information about $G^{[1]}(y)$, the fermion-mass independent part of the asymptotic expression for $\pi_c^{[1]}(q^2;m,y)$ given in Eq. (70).

To summarize, we have learned that the function $F^{[1]}(y)$ *and all of its derivatives* are zero at the point $y=\alpha_0$, where α_0 is the zero of the Gell-Mann–Low function $\psi(y)$. In other words, $F^{[1]}$ vanishes with an essential singularity at α_0. It is clear that a similar argument can be applied to the general $2n$-point function $T_{2n}^{[1]}$, by using an identity, analogous to Eq. (77), which relates the derivative $d^m T_{2n}^{[1]}/dy^m$ to an integral over the $(2n+2m)$-point function $T_{2n+2m}^{[1]}$. Thus we additionally learn that when the fermion mass m is zero, the single fermion loop $2n$-point function and all of its y derivatives also vanish at α_0. This fact, together with our result for $F^{[1]}$, gives us information about all of the loop diagrams appearing in the modified skeleton expansion for $f(y)$, from which we learn that f also has an infinite order zero at α_0. Finally, referring to Eq. (61), we conclude that the Gell-Mann–Low function $\psi(y)$ vanishes with an essential singularity at $y=\alpha_0$.[28] In Fig. 6 we summarize the complete chain of reasoning which we have used. Clearly, our conclusion shows that the customary assumption, that α_0 is a simple zero and a point of regularity of ψ, is in fact incorrect.

B. Asymptotic Behavior of h

As we have seen in Sec. II B, the customary assumption about the zero of ψ implies that the nonasymptotic piece h of the photon proper self-energy vanishes with power law behavior,

$$h\sim x^{\psi'(\alpha_0)}\,, \tag{95}$$

as $x=-q^2/m^2$ becomes infinite. Now that we know that ψ actually vanishes with an essential singularity, and not with a simple zero, we must reexamine the reasoning leading to Eq. (95). We give first a general, qualitative argument to show how Eq. (95) must be modified. Let us use the Gell-Mann–Low equation in the form of Eqs. (50) and (51),

$$\ln x=\Phi[\alpha d_c^\infty(x,\alpha)]-\Phi[q(\alpha)]\,,$$
$$\Phi[u]=\int_{c'}^{u}\frac{dz}{\psi(z)}\,. \tag{96}$$

If ψ has a zero at $z=\alpha_0$, then $\Phi[u]$ becomes infinite at $u=\alpha_0$, and hence the large-x behavior of αd_c^∞ is governed by the behavior of Φ in the vicinity of α_0. Now if $\psi(z)$ vanishes more rapidly than α_0-z as $z\to\alpha_0$, then $v=\Phi[u]$ will become infinite faster than $\ln(\alpha_0-u)$ as $u\to\alpha_0$. This implies that $\Phi^{-1}[v]-\alpha_0$ is a function which is weaker than an exponential as $v\to\infty$, or equivalently,

$$\alpha d_c^\infty(x,\alpha)-\alpha_0=\Phi^{-1}[\ln x]-\alpha_0 \tag{97}$$

is weaker than a power law as $x\to\infty$. So we obtain the qualitative conclusion that if ψ vanishes more rapidly than with a simple zero as $z\to\alpha_0$, $h(x,\alpha)$ will decrease more slowly than a power law as $x\to\infty$.

To obtain more specifically the connection between the functional form of ψ near α_0 and that of h near $x=\infty$, we resort to the study of exactly integrable examples. As in the discussion of Eqs. (32)–(35), in constructing these examples we can ignore the fact that $\psi(z)$ vanishes at $z=0$, since this region is not relevant to the asymptotic behavior of h. As our first illustration, we consider the case where ψ vanishes with a zero of finite order higher than the first. Substituting

$$\psi(z)=A(\alpha_0-z)^{1+\epsilon} \tag{98}$$

into the Gell-Mann–Low equation [Eq. (26)] and integrating, we get

$$\alpha d_c^\infty-\alpha_0=\frac{q(\alpha)-\alpha_0}{\{1+A\epsilon[\alpha_0-q(\alpha)]^\epsilon\ln x\}^{1/\epsilon}}\underset{x\to\infty}{\sim}(\ln x)^{-1/\epsilon}\,, \tag{99}$$

$$\psi(\alpha_0)=0 \Rightarrow \begin{array}{l}\text{General } 2n\text{-point}\\ \text{function}(n\geq 2)\\ \text{vanishes at } y=\alpha_0,\\ m=0\end{array} \Rightarrow \begin{array}{c}F^{[1]}(\alpha_0)=0\\ T_{2n}^{[1]}\Big|_{\substack{y=\alpha_0\\ m=0}}=0\end{array} \Rightarrow \begin{array}{c}F^{[1]}(\alpha_0)=0^\infty\\ T_{2n}^{[1]}\Big|_{\substack{y=\alpha_0\\ m=0}}=0^\infty\end{array} \Rightarrow f(\alpha_0)=0^\infty \Rightarrow \psi(\alpha_0)=0^\infty$$

FIG. 6. Chain of reasoning which summarizes the discussion of Secs. II D1 and III A. The abbreviation 0^∞ denotes a zero of infinite order (i.e., an essential singularity) in the y variable.

which, as expected, falls off more slowly than any power of x in the limit $x\to\infty$. As our second illustration, we study the case where ψ vanishes with an essential singularity of the form

$$\psi(z)\sim e^{-A/(\alpha_0-z)^p}\ . \tag{100}$$

To get an exactly integrable expression we multiply Eq. (100) by a power of α_0-z, giving

$$\psi(z)=\frac{(\alpha_0-z)^{p+1}}{ABp}\,e^{-A/(\alpha_0-z)^p}\ . \tag{101}$$

Substituting Eq. (101) into Eq. (26) and doing the z integration, we get

$$\alpha d_c^\infty-\alpha_0=\frac{-A^{1/p}}{(\ln[B^{-1}\ln x+\exp\{A/[\alpha_0-q(\alpha)]^p\}])^{1/p}}$$

$$\underset{x\to\infty}{\sim}(\ln\ln x)^{-1/p}\ . \tag{102}$$

We see that when ψ vanishes with an essential singularity at α_0, the asymptotic vanishing of h in the limit of large x is very slow indeed. In Table I we summarize the connection between the type of zero of ψ and the asymptotic behavior of h that we have inferred from our examples.[29]

C. Consequences of the Slow Decrease of h

In both the justification of the JBW form of the eigenvalue condition [Eq. (59) and the discussion preceding it in Sec. II D] and the derivation of the scaling form of the asymptotic electron propagator [Appendix B] we assume that α_0 is a simple zero, and a point of regularity, of the Gell-Mann–Low function ψ, and that h decreases asymptotically with power-law behavior. Now that we have seen that these assumptions are false, we must reexamine our treatmentof the eigenvalue condition and of the asymptotic behavior of the electron propagator, to study the consequences of the essential singularity which we have found in ψ and of the concomitant very slow asymptotic decrease of h. For the sake of definiteness, we will assume behavior as in Eqs. (101) and (102) with $p=1$, so that h decreases asymptotically as

$$h\sim\frac{1}{\ln\ln(-q^2/m^2)}\ . \tag{103}$$

TABLE I. Connection between behavior of $\psi(z)$ near $z=\alpha_0$ and behavior of $h(x,\alpha)$ near $x=\infty$.

Behavior of ψ near α_0	Asymptotic behavior of h
$\psi'(\alpha_0-z)$	$x^{\psi'}$
$(\alpha_0-z)^{1+\epsilon}$	$(\ln x)^{-1/\epsilon}$
$e^{-A/(\alpha_0-z)^p}$	$(\ln\ln x)^{-1/p}$

This restriction, while convenient to make, is not crucial to the discussion which follows.

Let us first reconsider the eigenvalue condition, picking up our discussion of Sec. II D at the point where we established that logarithmic behavior of a graph contributing to π_c can only be associated with the over-all integration involving all lines in the graph. As we noted, each internal photon line in the over-all integration contributes two parts, a part proportional to α_0 and a part proportional to h. Let us separately group together all contributions to π_c involving no factors of h, all contributions involving exactly one factor of h, all those involving exactly two factors of h, etc., as indicated in Fig. 7. The shaded blobs in Fig. 7, to which the insertions of h are attached, are two-point, four-point, six-point, etc. functions calculated with all internal photons described by free propagators coupling with the asymptotic coupling strength α_0. The piece with no factors of h is just the one retained in our earlier discussion, which, as we have seen, makes the contribution

$$g(\alpha_0)+f(\alpha_0)\ln(-q^2/m^2) \tag{104}$$

to the asymptotic behavior of π_c. Heuristically speaking, the logarithm in Eq. (104) can be thought of as arising from the integral

$$\int_{m^2}^{-q^2}\frac{d\rho}{\rho}\,; \tag{105}$$

in this language, the leading asymptotic behavior of the piece of π_c containing n factors of h corresponds to the integral

$$\int_{m^2}^{-q^2}\frac{d\rho}{\rho}\,h(\rho/m^2,\alpha)^n\ . \tag{106}$$

When h vanishes as a power of ρ for large ρ, the integral in Eq. (106) converges at the upper limit as $-q^2/m^2\to\infty$. Asymptotic finiteness of π_c then only requires the vanishing of the coefficient of the integral in Eq. (105), giving the JBW condition $f(\alpha_0)=0$. When h vanishes much more slowly than a power law, as in Eq. (103), the situation is radically changed.[30] The integral in Eq. (106) is now

$$\int_{m^2}^{-q^2}\frac{d\rho}{\rho[\ln\ln(\rho/m^2)]^n}\ , \tag{107}$$

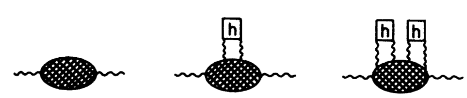

FIG. 7. Grouping of π_c into contributions involving no factor h, exactly one factor h, exactly two factors h, etc. The shaded blobs denote two-point, four-point, six-point, etc. functions calculated with all internal photons described by free propagators coupling with the asymptotic coupling strength α_0.

which for all n is divergent at the upper limit as $-q^2/m^2 \to \infty$. Thus, asymptotic finiteness of π_c requires now that an infinite number of conditions be satisfied: in addition to the coefficient of Eq. (105) vanishing, the coefficient of the contribution represented heuristically by Eq. (107) must vanish for all n. It is remarkable that when α_0 is chosen to be the root of $f(\alpha_0)=0$, this infinity of conditions is in fact satisfied. The reason is the argument based on the Federbush-Johnson theorem given in Eqs. (62)–(63) of Sec. II D 1, which shows that when $f(\alpha_0)=0$ and the fermion mass m is zero, the general $2n$-point current correlation function vanishes for $n \geq 2$. Hence when α_0 satisfies $f(\alpha_0)=0$, each shaded blob in Fig. 7 is proportional to m^2 and therefore contributes a convergence factor m^2/ρ to the integral in Eq. (107). The integral then becomes[25]

$$\int_{m^2}^{-q^2} \frac{d\rho\, m^2}{\rho^2[\ln\ln(\rho/m^2)]^n}, \qquad (108)$$

which is asymptotically finite as $-q^2/m^2 \to \infty$. The asymptotically divergent integral in Eq. (107) of course reappears when α_0 is chosen to have any value other than the root of $f(\alpha_0)=0$. We conclude, then, that Eq. (103) still permits one to deduce the JBW eigenvalue condition $f(\alpha_0)=0$, but only by a more involved mechanism than is required in the case of a power law vanishing of h.

Let us next examine the implications of the essential singularity at α_0 and of Eq. (103) for the argument leading to the scaling form for the asymptotic electron propagator. As we have noted, the approach used to derive the scaling form in Appendix B depends very specifically on the assumptions of regularity of the theory in the vicinity of α_0 and power law vanishing of h. To deal with the situation where α_0 is a point of essential singularity, we give an alternative approach, based on reasoning similar to that which we have just used in our discussion of the eigenvalue condition. Let us consider the *unrenormalized* electron propagator $S_F'(p)^{-1}$ in the limit in which $-p^2$ and the cutoff Λ^2 are both becoming infinite relative to the fermion mass m^2. To study this, we collect together all contributions to the electron proper self-energy involving no factors of h, involving exactly one factor of h, exactly two factors of h, etc., as shown in Fig. 8. As before, the shaded blobs are calculated with all internal photons described by free propagators coupling with the asymptotic coupling strength α_0. The piece with no factors of h is just the unrenormalized electron proper self-energy in the JBW model. A straightforward analysis[31] using the methods of Ref. 17 shows that if this piece alone is retained, the unrenormalized asymptotic electron propagator has the scaling form

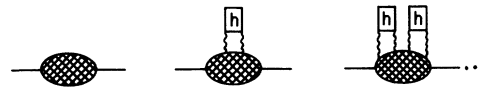

FIG. 8. Grouping of the electron proper self-energy into contributions involving no factors h, exactly one factor h, exactly two factors h, etc. The shaded blobs are calculated with all internal photons described by free propagators coupling with the asymptotic coupling strength α_0.

$$S_F'(p)^{-1} \underset{\substack{-p^2/m^2\to\infty \\ \Lambda^2/m^2\to\infty}}{\sim} F_1(\alpha_1)\left(-\frac{p^2}{\Lambda^2}\right)^{\gamma(\alpha_1)/2} \times\left[\not p - m_0 F_2(\alpha_1)\left(-\frac{p^2}{\Lambda^2}\right)^{-\delta(\alpha_1)/2}\right]. \qquad (109)$$

Together with the fact that S_F' and the scalar vertex Γ_S are multiplicatively renormalizable, Eq. (109) implies[31] the results of Eq. (66) for both the renormalized electron propagator and the renormalization constants Z_2 and m_0, with the modification, already noted in Sec. II D, that the constants C_1 and C_2 in Eq. (66) become dependent on nonasymptotic quantities. We must now examine whether the asymptotic expression of Eq. (109) is modified by the pieces containing one or more factors of h. To this end, it is useful to note that the powers in Eq. (109) arise in perturbation theory from infinite sums of logarithms,

$$\left(-\frac{p^2}{\Lambda^2}\right)^{\gamma(\alpha_1)/2} = \sum_{n=0}^{\infty} \frac{[\frac{1}{2}\gamma(\alpha_1)\ln(-p^2/\Lambda^2)]^n}{n!}, \qquad (110)$$

and heuristically, the logarithms can be thought of as arising from integrals of the form

$$\int_{-p^2}^{\Lambda^2} \frac{d\rho}{\rho}. \qquad (111)$$

In this language, the piece of the electron proper self-energy containing n factors of h will involve integrals of the form

$$\int_{-p^2}^{\Lambda^2} \frac{d\rho}{\rho} h(\rho/m^2, \alpha)^n. \qquad (112)$$

If h vanishes as a power of ρ for large ρ, the integral in Eq. (112) vanishes as $-p^2/m^2$, $\Lambda^2/m^2 \to \infty$, and the scaling form of Eq. (109) is unmodified.[30] On the other hand, if h vanishes as in Eq. (103), then Eq. (112) becomes

$$\int_{-p^2}^{\Lambda^2} \frac{d\rho}{\rho[\ln\ln(\rho/m^2)]^n}, \qquad (113)$$

which does *not* vanish[30] in the limit of asymptotic $-p^2$, Λ^2 and which could therefore give rise to corrections to Eq. (109). We again can salvage the situation if we can use the Federbush-Johnson theorem to argue that the Compton-like shaded blobs in Fig. 8 vanish when $f(\alpha_0)=0$ and the fermion mass m is zero. However, this involves an extension of the Federbush-Johnson theorem outside the charge-zero sector, which is the only place where a satisfactory proof in the case of electrodynamics has been given.[8,32] If such an extension is allowed, we gain a convergence factor m^2/ρ in Eq. (113), giving

$$\int_{-p^2}^{\Lambda^2} \frac{d\rho m^2}{\rho^2[\ln\ln(\rho/m^2)]^n}, \tag{114}$$

which vanishes as $-p^2/m^2$, $\Lambda^2/m^2 \to \infty$.

We conclude, then, that the JBW eigenvalue condition and, possibly, the scaling form for the asymptotic electron propagator remain valid in the presence of the essential singularity, but only by virtue of an additional infinity of conditions being satisfied simultaneously. This, of course, poses troublesome questions of convergence (basically, is $0\times\infty$ effectively 0 in these problems?) which we have not attempted to settle.

IV. LOOPWISE SUMMATION AND AN EIGENVALUE CONDITION FOR α

Up to this point we have consistently employed the "vacuum-polarization-insertion-wise" summation scheme, both in our review of the JBW results in Sec. II D and in our deduction of the presence of an essential singularity in the preceding section. As we have seen, this scheme leads to a one-parameter family of asymptotically finite solutions, in which the asymptotic coupling α_0 is determined to be the zero y_0 of the Gell-Mann–Low function $\psi(y)$ [and simultaneously a zero of the simpler functions $f(y)$ and $F^{[1]}(y)$], while the physical coupling α is a free parameter, restricted only by the condition $\alpha<\alpha_0=y_0$ following from spectral function positivity [see Eq. (129) below.] The usual assumption is that this one-parameter family represents the most general type of asymptotically finite solution which can occur. In the present section, we show that the presence of a simultaneous zero in all of the single fermion-loop diagrams *makes possible one additional asymptotically finite solution,* which has the very appealing feature that the physical coupling α is fixed to be y_0. Our procedure is not strictly deductive, in that we continue to accept the *results* concerning properties of the single fermion-loop diagrams which were found in Sec. IIIA, while dropping the identification of α_0 with y_0 which was made there. We will also introduce a new order of summing the perturbation series, involving "loopwise" rather than "vacuum-polarization-insertion-wise" summation. Specifically, we make the following two assumptions:

(1) The function $F^{[1]}(y)$ defined by Fig. 1 and the $2n$-point current correlation function with zero fermion mass, $T^{[1]}_{\mu_1\cdots\mu_{2n}}(q_1,\ldots,q_{2n};m=0,y)$, vanish simultaneously at $y=y_0$. As we have seen in Sec. III A, the simultaneous vanishing of $F^{[1]}$ and $T^{[1]}_{2n}$ implies that they vanish with a zero of infinite order.

(2) The photon proper self-energy can be correctly obtained by "loopwise" summation. That is, we assume convergence of the sum

$$\pi_c=\sum_{n=1}^{\infty}\pi_c^{[n]}, \tag{115}$$

where $\pi_c^{[n]}$ is the contribution to the photon proper self-energy containing exactly n closed fermion loops. The burden of the present section will be to show that *these two assumptions imply asymptotic finiteness of the photon propagator when the physical fine structure constant is chosen to have the value* $\alpha=y_0$. *Furthermore, we will show that for this particular value of* α *the function* $\beta(\alpha)$ *appearing in the Callan-Symanzik equation vanishes (when summed loopwise) and so the theory has type-1 asymptotic behavior.*

To proceed, we introduce some additional definitions. Let $\beta^{[n]}(\alpha)$ be the contribution to $\beta(\alpha)$ with exactly n closed fermion loops, and let $\pi_c^{[n,r]}$ be the part of $\pi_c^{[n]}$ in which exactly r closed fermion loops remain when all internal photon self-energy parts are shrunk down to points [see Fig. 9.] In terms of these definitions, we can write

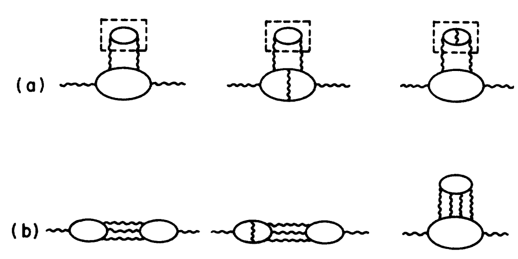

FIG. 9. (a) Typical diagrams contributing to $\pi_c^{[2,1]}$, the part of the two-fermion-loop photon proper self-energy which contains only one fermion loop after the internal photon self-energy part (enclosed by dashed lines) is shrunk down to a point. (b) Typical diagrams contributing to $\pi_c^{[2,2]}$, the part of the two-fermion-loop photon proper self-energy which still contains two fermion loops after shrinking away the internal photon self-energy parts.

$$\beta(\alpha)=\sum_{n=1}^{\infty}\beta^{[n]}(\alpha),$$
$$\pi_c^{[n]}=\sum_{r=1}^{n}\pi_c^{[n,r]}. \tag{116}$$

We now begin our argument by considering the case $n=1$. Because we are dealing with the renormalized theory, the coupling constant which appears is the physical fine structure constant α, and so (using our earlier notation) we must study the asymptotic behavior of $\pi_c^{[1]}(q^2;m,\alpha)$. Referring back to Eq. (70), we see that for asymptotic $-q^2/m^2$ we have

$$\pi_c^{[1]}(q^2;m,\alpha)=G^{[1]}(\alpha)+F^{[1]}(\alpha)\ln(-q^2/m^2) + \text{vanishing terms}; \tag{117}$$

hence choosing $\alpha=y_0$ guarantees the asymptotic finiteness of $\pi_c^{[1]}$. Next we consider the case $n=2$, for which we can write

$$\pi_c^{[2]}=\pi_c^{[2,1]}+\pi_c^{[2,2]}, \tag{118}$$

with the two terms in Eq. (118) corresponding respectively to the diagrams in Figs. 9(a) and 9(b). Because the single fermion-loop vacuum-polarization insertion has already been shown to be asymptotically finite, we can use the argument which was employed above in getting Eq. (58) to show that $\pi_c^{[2,1]}$, as well as $\pi_c^{[2,2]}$, grows asymptotically at worst as a single power of $\ln(-q^2/m^2)$,

$$\pi_c^{[2,1]}(q^2;m,\alpha)=G^{[2,1]}(\alpha)+F^{[2,1]}(\alpha)\ln(-q^2/m^2) + \text{vanishing terms},$$
$$\pi_c^{[2,2]}(q^2;m,\alpha)=G^{[2,2]}(\alpha)+F^{[2,2]}(\alpha)\ln(-q^2/m^2) + \text{vanishing terms}. \tag{119}$$

Furthermore, the same argument tells us that the potential logarithm is associated with the subintegrations involving all lines in $\pi_c^{[2,2]}$, and all lines in $\pi_c^{[2,1]}$ which remain after the internal photon self-energy part has been shrunk down to a point. Clearly, these subintegrations always involve at least one single fermion loop $2j$-point function (with $j\geq 2$) which, we have assumed, vanishes when $\alpha=y_0$ and the fermion mass m is zero. As a result, the potentially dangerous subintegrations are really two powers of momentum more convergent than indicated by naive power counting [cf. Eq. (90)] and hence cannot actually lead to logarithmic asymptotic behavior. So we learn that when $\alpha=y_0$, we have $F^{[2,1]}(\alpha)=F^{[2,2]}(\alpha)=0$, and therefore $\pi_c^{[2]}$ is asymptotically finite. Note that the argument which we have just given does not determine the actual limiting values of $\pi_c^{[1]}$ or $\pi_c^{[2]}$, i.e., we learn nothing about the values of $G^{[1]}(\alpha)$, $G^{[2,1]}(\alpha)$, or $G^{[2,2]}(\alpha)$ at $\alpha=y_0$. This is expected, because the G's depend on the *nonasymptotic* theory (where m cannot be neglected) as a result of the subtraction at $q^2=0$ which renormalizes the photon proper self-energy. Since knowledge of the G's would allow one to calculate α_0 through the formula

$$\sum_{n=1}^{\infty}\sum_{r=1}^{n}G^{[n,r]}(\alpha)=\alpha_0^{-1}-\alpha^{-1}, \tag{120}$$

we see that in our solution with α fixed, α_0 cannot be determined through asymptotic considerations alone.

The next step in the argument is to prove the vanishing of $\beta^{[1]}(\alpha)$ at $\alpha=y_0$. We do this by using the Callan-Symanzik equation in the form given by Eq. (43) which, on substituting Eq. (12) for d_c^{-1}, and dropping the asymptotically vanishing term proportional to $\tilde{\Gamma}_{\gamma\gamma S}$, becomes

$$-\beta(\alpha)+\left[m\frac{\partial}{\partial m}+\beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha}-1\right)\right]\alpha\pi_c\sim 0. \tag{121}$$

The one-fermion-loop part of this equation is

$$-\beta^{[1]}(\alpha)+m\frac{\partial}{\partial m}\alpha\pi_c^{[1]}\sim 0. \tag{122}$$

Substituting Eq. (70) for $\pi_c^{[1]}$ this becomes

$$\beta^{[1]}(\alpha)=-2\alpha F^{[1]}(\alpha), \tag{123}$$

from which we immediately learn that $\beta^{[1]}(\alpha)$ vanishes at $\alpha=y_0$.

We now continue the argument inductively. We assume that when $\alpha=y_0$ the pieces $\pi_c^{[1]},\ldots,\pi_c^{[n]}$ of the photon proper self-energy are asymptotically finite, while the pieces $\beta^{[1]},\ldots,\beta^{[n]}$ of the Callan-Symanzik function β are zero. We wish to extend these assertions to the pieces $\pi_c^{[n+1]}$ and $\beta^{[n+1]}$ which contain one more closed fermion loop. We write

$$\pi_c^{[n+1]}=\sum_{r=1}^{n+1}\pi_c^{[n+1,r]}, \tag{124}$$

where, according to our induction hypothesis and the argument preceding Eq. (58), the piece $\pi_c^{[n+1,r]}$ can grow asymptotically at most as a single power of $\ln(-q^2/m^2)$,

$$\pi_c^{[n+1,r]}(q^2;m,\alpha)=G^{[n+1,r]}(\alpha) + F^{[n+1,r]}(\alpha)\ln(-q^2/m^2) + \text{vanishing terms}. \tag{125}$$

Again, the argument leading to Eq. (125) tells us that the potential logarithm is associated with the subintegration involving all lines in $\pi_c^{[n+1,r]}$ which remain after the internal photon self-energy parts have been shrunk away. This subintegration always involves at least one single fermion-loop $2j$-point function ($j\geq 2$) which, when $\alpha=y_0$, improves the ultraviolet convergence of the subintegration by two powers of momentum and hence prevents a logarithm from actually appearing in Eq. (125). So we conclude that $F^{[n+1,r]}(\alpha)=0$ when $\alpha=y_0$, $r=1,\ldots,n+1$, and hence $\pi_c^{[n+1]}$ is asymptotically finite. To prove the vanishing of $\beta^{[n+1]}(\alpha)$, we consider the part of Eq. (120) involving exactly $n+1$ closed fermion loops,

$$-\beta^{[n+1]}(\alpha)+m\frac{\partial}{\partial m}\alpha\pi_c^{[n+1]} + \sum_{r=1}^{n}\beta^{[r]}(\alpha)\left(\alpha\frac{\partial}{\partial\alpha}-1\right)\alpha\pi_c^{[n+1-r]}\sim 0\,. \tag{126}$$

Using the induction hypothesis on β this equation simplifies, when $\alpha=y_0$, to

$$-\beta^{[n+1]}(\alpha)+m\frac{\partial}{\partial m}\alpha\pi_c^{[n+1]}\sim 0\,. \tag{127a}$$

But asymptotic finiteness of $\pi_c^{[n+1]}$ tells us that

$$m\frac{\partial}{\partial m}\alpha\pi_c^{[n+1]}\sim 0\,, \tag{127b}$$

so we learn that $\beta^{[n+1]}(\alpha)=0$ when $\alpha=y_0$, completing the induction.

To summarize, we have learned, for all n, that $\pi_c^{[n]}$ is asymptotically finite and that $\beta^{[n]}$ vanishes when $\alpha=y_0$. Invoking now our assumption of convergence of the "loopwise" summations in Eq. (115) and Eq. (116), we learn that when $\alpha=y_0$, the total photon proper self-energy π_c is asymptotically finite, and the total Callan-Symanzik function $\beta(\alpha)$ vanishes. The vanishing of the Callan-Symanzik function means that our solution with $\alpha=y_0$ has type-1 asymptotic behavior. According to the discussion of Appendix B, the asymptotic electron propagator must then have the simple scaling form of Eq. (66) (with $\alpha_1=\alpha$), leading, as we have noted, to the possibility of a finite m_0 and Z_2.

In conclusion, we briefly discuss the relation of the asymptotically finite solution which we have just found to the "vacuum-polarization-insertion-wise" summation methods used earlier. As we have seen, in our "loopwise" solution α is determined by the condition $F^{[1]}(\alpha)=0$, with the asymptotic coupling α_0 determined from α by the functions $G^{[n,r]}(\alpha)$ according to Eq. (120). *A priori*, we can say nothing about the value of α_0 except that positivity of the spectral function $w(\rho,\alpha)$ appearing in the Källén-Lehmann representation[33] for the photon propagator,

$$d_c(-q^2/m^2,\alpha)=1+q^2\int_0^\infty w(\rho/m^2,\alpha)\frac{d(\rho/m^2)}{q^2-\rho-i\epsilon}\,, \tag{128}$$

implies the sum rule[34]

$$\alpha_0=\alpha+\int_0^\infty \alpha w(\rho/m^2,\alpha)d(\rho/m^2)\,, \tag{129}$$

and hence $\alpha_0>\alpha$. This inequality raises an apparent paradox when we turn to the "vacuum-polarization-insertion-wise" summation scheme, which if applicable would imply that α_0 obeys the same eigenvalue condition as does α, $F^{[1]}(\alpha_0)=0$. The paradox is resolved, however, when we note that since y_0 is an essential singularity of $F^{[1]}(y)$, the point $\alpha_0>\alpha=y_0$ lies *outside* the radius of convergence of $F^{[1]}(y)$, and so the interchange of limit and sum leading to the eigenvalue condition on α_0 is unjustified. Another way of stating this is obtained by writing down the formal Taylor expansion connecting the eigenvalue conditions for α and α_0,

$$F^{[1]}(\alpha_0)=\sum_{n=0}^{\infty}\frac{(\alpha_0-y_0)^n}{n!}\frac{d^n}{dy^n}F^{[1]}(y)\bigg|_{y=y_0=\alpha}\,. \tag{130}$$

Since $F^{[1]}$ and all its derivatives vanish at y_0, naive application of Eq. (130) tells us that $F^{[1]}(\alpha_0)=0$. This conclusion is of course incorrect, because the Taylor expansion in Eq. (130) attempts the analytic continuation of $F^{[1]}$ outside its region of regularity, and therefore is mathematically meaningless. In other words, because of the essential singularity, we cannot freely rearrange the "loopwise"-summed theory, with $\alpha=y_0$, into a "vacuum-polarization-insertion-wise"-summed theory.

V. DISCUSSION

We have learned that there are two possible ways of having an asymptotically finite electrodynamics. The first is the usual one-parameter family of solutions, in which the asymptotic coupling α_0 is fixed to be y_0 and the physical coupling $\alpha<\alpha_0$ is a free parameter. The second is the unique additional solution found in the preceding section, in which the physical coupling α is fixed to be y_0. *We conjecture that nature in fact chooses this second type of solution, and hence that the fine structure constant may be calculated by determining the location of the infinite order zero y_0 of the function $F^{[1]}(y)$.*[35] [Of course, if the function $F^{[1]}(y)$ does *not* have an infinite-order positive zero, then electrodynamics *cannot* be asymptotically finite.] We can advance two possible reasons why nature may choose the solution which fixes α over the solutions which fix α_0:

(1) The "vacuum-polarization-insertion-wise" summation procedure needed to get the solutions which fix α_0 may be divergent for all nonzero values of α. In other words, electrodynamics may exist only when summed "loopwise," with the spe-

cific choice of physical coupling $\alpha = y_0$.

(2) Both types of solution may be mathematically valid, but there may be stability arguments which tell us that when other interactions (such as weak or gravitational interactions) are switched on, the theory chooses the largest possible value of α, that is $\alpha = y_0$.

We emphasize that we have given no arguments which distinguish which, if either, of these possible reasons is correct.

We conclude the paper by giving a brief, speculative discussion of some further implications of the work of the preceding sections. First, we point out a possible connection of our work with Dyson's[9] old conjecture suggesting singularities in electrodynamics at $\alpha = 0$. Then, we discuss the fact that the conjecture stated at the beginning of this section gives a *species-independent* determination of α, and give an argument based on this which may justify our neglect of strong interaction corrections.

A. Dyson's Conjecture

Dyson has argued that the renormalized perturbation theory of quantum electrodynamics, regarded as a power series in α, cannot have a nonzero radius of convergence. For if it did, the theory would still exist when analytically continued to negative α, which corresponds to a physical situation in which *like* charges, rather than unlike charges, attract. But in the analytically continued theory, the usual vacuum, defined as the state containing no particles, would be unstable. To see this, we note that if we create N electron positron pairs, with N very large, and group the electrons together in one region of space and the positrons together in another separate region, we can create a pathological state in which the negative potential energy of the Coulomb forces exceeds the total rest energy and kinetic energy of the particles. Although this state is separated from the usual vacuum by a high potential barrier (of the order of the rest energy of the $2N$ particles being created), quantum-mechanical tunneling from the vacuum to the pathological state would be allowed, and would lead to an explosive disintegration of the vacuum by spontaneous polarization. This instability means that electrodynamics with negative α cannot be described by well-defined analytic functions; hence the perturbation series of electrodynamics must have zero radius of convergence.

If one assumes, as we do in this paper, that electrodynamics is by itself a complete theory,[36] then physical quantities in electrodynamics are described by well-defined, calculable functions of α when α is positive. According to Dyson's argument however, these functions cannot be continued to negative α, and therefore must have a singularity at $\alpha = 0$. Because the singularity originates in a tunneling phenomenon, and because tunneling amplitudes are typically negative exponentials of a barrier-penetration factor, it is plausible that this singularity should be an essential singularity of the form $e^{-c/\alpha}$.

We can now attempt to make a connection with the results of the preceding two sections. As we recall, we argued there that the single-fermion-loop function $F^{[1]}(\alpha)$ should possess an essential singularity (perhaps of the form $\exp[-c/(y_0 - \alpha)]$, resembling a tunneling amplitude) at the point $\alpha = y_0 > 0$. In establishing a connection with Dyson's work, there appear to be two possibilities. One possibility is that the singularity at y_0 is *not* Dyson's singularity, but that electrodynamics exists for a range of positive α and that $F^{[1]}(\alpha)$ (or perhaps some other function in the theory) has a singularity at $\alpha = 0$ which prevents continuation to negative α. An alternative possibility is that $F^{[1]}(\alpha)$ is regular at $\alpha = 0$, but that the full photon propagator simply does not exist for values of the physical coupling α other than y_0, preventing continuation of the complete theory to negative fine-structure constant. In this case, the singularity of $F^{[1]}$ at y_0 would be a mathematical manifestation of Dyson's argument. In this connection, it is intriguing that the class of single-fermion-loop vacuum-polarization diagrams which we assert to possess an essential singularity are just the simplest diagrams describing the creation of an arbitrarily large number of pairs from the vacuum, and therefore are the simplest diagrams leading to Dyson's pathological state. For, as shown in Fig. 10, the single-fermion-loop diagrams describe the creation of an arbitrary number of pairs from the vacuum, but with only one fermion world line actually present.

B. Species Independence

Up to this point we have assumed the presence of only one species of fermion interacting solely

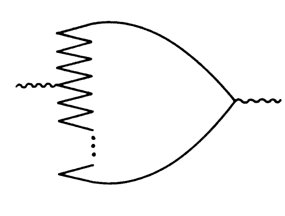

FIG. 10. Ordering in which a single-fermion vacuum-polarization loop diagram describes the creation of an infinite number of pairs from the vacuum. (We have not drawn in any of the internal photons.)

with the electromagnetic field. Let us now consider the more general case in which there are j elementary charged spin-$\frac{1}{2}$ fermion species which, for the moment, we still assume to interact only electromagnetically. Although these fermions may have different masses, the contributions of mass-difference terms to the photon proper self-energy are guaranteed, just by power counting, to be asymptotically finite. Hence to study the effect of having j fermions on the asymptotic behavior of the photon propagator, it suffices to consider the special case in which they all have a common mass m. Then, because each closed fermion loop in the photon proper self-energy appears j times, the piece of π_c containing exactly n closed fermion loops is multiplied by j^n, and so Eq. (115) is modified to read

$$\pi_c = \sum_{n=1}^{\infty} j^n \pi_c^{[n]} . \tag{131}$$

Clearly, because choosing $\alpha = y_0$ makes each of the $\pi_c^{[n]}$ individually asymptotically finite, this choice of coupling makes the total π_c asymptotically finite as well, independent of the species number j. Stated in another way, when j fermion species are present the single fermion loop function determining y_0 is just $jF^{[1]}(y)$, and so the value of y_0 determined is the same as in the one-species case. Thus we reach the important conclusion that *our eigenvalue condition for determining α is independent of the fermion species number*. Whether this species independence is maintained in the presence of elementary charged spin-0 boson fields is not clear. The requirement is obviously that the function $F_B^{[1]}(y)$, defined by summing the single charged boson loop diagrams of Fig. 1 in analogy to our definition of $F^{[1]}(y)$, must vanish with an infinite order zero at the same point y_0 where $F^{[1]}(y)$ vanishes. All that is known about $F^{[1]}(y)$ and $F_B^{[1]}(y)$ at present is the first few terms in their respective power-series expansions,[37]

$$\begin{aligned} -yF^{[1]}(y) &= \frac{2}{3}\left(\frac{y}{2\pi}\right) + \left(\frac{y}{2\pi}\right)^2 - \frac{1}{4}\left(\frac{y}{2\pi}\right)^3 + \cdots , \\ -yF_B^{[1]}(y) &= \frac{1}{6}\left(\frac{y}{2\pi}\right) + \left(\frac{y}{2\pi}\right)^2 + \cdots . \end{aligned} \tag{132}$$

Equation (132) tells us that the functions $F^{[1]}(y)$ and $F_B^{[1]}(y)$ are not identical, but of course says nothing about their behavior when summed to all orders.

Returning, now, to our model with several charged fermion species, let us suppose that some of these fermions have strong interactions mediated by neutral boson exchange (the gluon model).

FIG. 11. Fermion vacuum-polarization loop modified by internal gluon (dashed line) radiative corrections.

Although the bosons do not themselves contribute vacuum-polarization loops, they could modify the fermion vacuum polarization loops when they appear as internal radiative corrections (see Fig. 11.) However, let us now invoke the experimental observation of scaling in deep-inelastic electron scattering, one explanation for which[38] is that the exchanges which mediate the strong interactions are actually much more strongly damped at high four-momentum transfer than is the free boson propagator $(q^2+\mu^2)^{-1}$. If such an explanation proves correct,[39] then vacuum-polarization diagrams with gluon radiative corrections will by themselves be asymptotically finite, and so the presence of strong interactions will not alter our eigenvalue condition for α. Our scheme is clearly incompatible, however, with the presence of fractionally charged fermions such as quarks[40]; all elementary charged fermions must have the same basic electromagnetic coupling $(\pm)\sqrt{\alpha}$.

Note added in proof. R. F. Dashen has pointed out to us that in order y^3 and higher the vacuum polarization structure of charged spin-0 boson electrodynamics will differ from that of the spin-$\frac{1}{2}$ case, as a result of the presence of a boson-boson scattering counterterm in the Lagrangian. Hence the analysis which we have given above for the case of spin-$\frac{1}{2}$ electrodynamics cannot be directly applied to the spin-0 case. The JBW argument for finiteness of the bare mass also fails in spin-0 electrodynamics. [See D. Flamm and P. G. O. Freund, Nuovo Cimento 32, 486 (1964).]

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APPENDIX A: PARTIAL SUMMARY OF NOTATION

JBW	Johnson-Baker-Willey
α	physical coupling (fine-structure constant)
α_w	new coupling constant defined by subtraction at w
α_0	asymptotic coupling constant
α_b	canonical or bare coupling constant, related to α by $\alpha_b = Z_3^{-1}\alpha$
α_1	root of $q(\alpha_1)=\alpha_0$, with $q(y)=yd_c^\infty(1,y)$
Z_3	photon wave-function renormalization constant
m	electron physical mass
$\alpha d_c(-q^2/m^2,\alpha)$	renormalized photon propagator; $d_c(-q^2/m^2,\alpha)=[1+\alpha\pi_c(q^2)]^{-1}$
$h(-q^2/m^2,\alpha)$	difference between αd_c and its asymptotic limit α_0
$\alpha d_c^\infty(-q^2/m^2,\alpha)$	"asymptotic part" of the renormalized photon propagator, obtained by dropping in each order of perturbation theory terms which vanish as $-q^2/m^2\to\infty$, but keeping terms in each order which are constant or increase logarithmically
$\psi(y)$	Gell-Mann–Low function
$F^{[1]}(y)$	coefficient of logarithmically divergent part of the sum of single-fermion-loop vacuum polarization diagrams
y_0	point where $F^{[1]}(y)$ has an infinite-order zero (essential singularity)
Λ	ultraviolet cutoff
μ, μ_0	physical photon mass (infrared cutoff), bare photon mass
$D_F^0(q)_{\mu\nu}$	bare photon propagator
ξ	gauge parameter (coefficient of longitudinal part of photon propagator)
$\pi(q^2)_{\mu\nu}=(-q^2g_{\mu\nu}+q_\mu q_\nu)\pi(q^2)$	photon proper self-energy
$D_F'(q)_{\mu\nu}$	full unrenormalized photon propagator
$\tilde{D}_F'(q)_{\mu\nu}$	full renormalized photon propagator
$\pi_c(q^2)=\lim_{\Lambda\to\infty}[\pi(q^2)-\pi(\mu^2)]$	subtracted photon proper self-energy
x	dimensionless variable $-q^2/m^2$
m_0, Z_2	electron bare mass and wave-function renormalization constant
$\beta(\alpha)$	coefficient of $\partial/\partial\alpha$ in the Callan-Symanzik equation
$f(\alpha_0)$	coefficient of the logarithmically divergent part of the photon proper self-energy in the JBW model
j_μ	electromagnetic current operator
$\tilde{S}_F'(p)$	renormalized electron propagator
$\gamma(\alpha)$, $\delta(\alpha)$	coefficient functions appearing in the Callan-Symanzik equation for the electron propagator
$\pi_c^{[n]}$, $\beta^{[n]}$	parts of π_c, β with exactly n closed fermion loops
$T_{2n}^{[1]}=T_{\mu_1\cdots\mu_{2n}}^{[1]}(q_1,\ldots,q_{2n};m,y)$	single-fermion-loop $2n$-point function ($n\geq 2$) with coupling y
$\pi_{2n}^{[1]}$	modified 2-point function defined as a contraction on $T_{2n}^{[1]}$

$\pi_c^{[n,r]}$	part of $\pi_c^{[n]}$ in which exactly r closed fermion loops remain when all internal photon self-energy parts are shrunk down to points
$w(\rho/m^2, \alpha)$	Källén-Lehmann spectral function for the photon propagator
$F_B^{[1]}(y)$	coefficient of the logarithmically divergent part of the sum of single charged boson loop vacuum polarization diagrams
η	combination $\alpha(\xi-1)$ through which gauge dependence occurs

APPENDIX B: CALLAN-SYMANZIK EQUATIONS AND APPLICATION TO THE ELECTRON PROPAGATOR

In this Appendix we derive the Callan-Symanzik equations for massive photon (i.e., infrared cutoff) spinor electrodynamics in an arbitrary covariant gauge. We are particularly interested in the equations for the electron propagator and the electron-scalar vertex, which can be used to derive the JBW asymptotic form for the electron propagator discussed in Sec. IID2. To begin, we recall that the gauge parameter ξ enters into the theory only via the quantity $\alpha_b D_F^0(q)_{\mu\nu}$, which according to Eqs. (2) and (7b) can be written as

$$\alpha_b D_F^0(q)_{\mu\nu} = \alpha_b\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2-\mu_0^2}\,\frac{-\Lambda^2}{q^2-\Lambda^2} + \alpha(\xi-1)\frac{q_\mu q_\nu}{q^2}\,\frac{1}{q^2-\mu^2}\,\frac{-\Lambda^2}{q^2-\Lambda^2}\,. \tag{B1}$$

In particular, we see that ξ always appears in the combination $\alpha(\xi-1)$, a fact which we exploit by displaying the arguments of the renormalized electron propagator $\tilde{S}_F'^{-1}$ and the electron wave function renormalization Z_2 in the form

$$\begin{aligned} \tilde{S}_F'^{-1} &= \tilde{S}_F'^{-1}[p,\mu,m,\alpha,\eta]\,, \\ Z_2 &= Z_2[\Lambda,\mu,m,\alpha,\eta]\,, \\ \eta &= \alpha(\xi-1)\,. \end{aligned} \tag{B2}$$

To derive the Callan-Symanzik equations for the electron propagator, we start by writing down the equation connecting the renormalized and unrenormalized electron propagators,

$$\begin{aligned} \tilde{S}_F'[p,\mu,m,\alpha,\eta] &= Z_2 S_F'^{-1} \\ &= Z_2[\Lambda,\mu,m,\alpha,\eta](\not p - m_0 - \Sigma)\,, \end{aligned} \tag{B3}$$

with Σ the electron proper self-energy part. We now make independent variations in the physical electron and photon masses m and μ, keeping Λ and α_b fixed and simultaneously making a gauge transformation which keeps $\eta=\alpha(\xi-1)$ fixed. These variations are described by the respective differential operators

$$\begin{aligned} m\frac{d}{dm} &= m\frac{\partial}{\partial m} + \beta_m\alpha\frac{\partial}{\partial\alpha}\,, \\ \mu\frac{d}{d\mu} &= \mu\frac{\partial}{\partial\mu} + \beta_\mu\alpha\frac{\partial}{\partial\alpha}\,, \end{aligned} \tag{B4}$$

with β_m and β_μ defined by

$$\begin{aligned} \alpha\beta_m &= m\frac{d\alpha}{dm} = Z_3^{-1} m\frac{d}{dm}Z_3\,, \\ \alpha\beta_\mu &= \mu\frac{d\alpha}{d\mu} = Z_3^{-1}\mu\frac{d}{d\mu}Z_3\,, \end{aligned} \tag{B5}$$

in analogy with Eq. (42). Applying these differential operators to Eq. (B4), and observing that the unrenormalized propagator $\not p - m_0 - \Sigma$ depends on m and μ implicitly through its dependence on m_0 and μ_0 and explicitly through the factor $1/(q^2-\mu^2)$ in the gauge term, we get the Callan-Symanzik equations for the electron propagator,

$$\begin{aligned} \left(m\frac{\partial}{\partial m}+\alpha\beta_m\frac{\partial}{\partial\alpha}+\gamma_m\right)\tilde{S}_F'^{-1} &= -(1+\delta_m)\tilde{\Gamma}_S + \mu^2\alpha\beta_m\tilde{\Gamma}_{S'}\,, \\ \left(\mu\frac{\partial}{\partial\mu}+\alpha\beta_\mu\frac{\partial}{\partial\alpha}+\gamma_\mu\right)\tilde{S}_F'^{-1} &= -\delta_\mu\tilde{\Gamma}_S + \mu^2(\alpha\beta_\mu-2)\tilde{\Gamma}_{S'} + 2\mu^2(\xi-1)\tilde{\Gamma}_{S''}\,. \end{aligned} \tag{B6}$$

In writing this equation we have introduced the following additional definitions:

$$\begin{aligned} Z_2^{-1} m\frac{d}{dm}Z_2 &= -\gamma_m\,, \\ Z_2^{-1}\mu\frac{d}{d\mu}Z_2 &= -\gamma_\mu\,, \\ m_0^{-1} m\frac{d}{dm}m_0 &= 1+\delta_m\,, \\ m_0^{-1}\mu\frac{d}{d\mu}m_0 &= \delta_\mu\,, \\ \tilde{\Gamma}_S &= m_0 Z_2\left(1+\frac{\partial\Sigma}{\partial m_0}\right), \end{aligned} \tag{B7}$$

$$\tilde{\Gamma}_{S'} = \frac{Z_2}{Z_3}\frac{\partial \Sigma}{\partial \mu_0{}^2},$$

$$\tilde{\Gamma}_{S''} = Z_2 \Gamma_{S''}.$$

The vertex part $\Gamma_{S''}$ is defined as the sum of terms in which each internal photon propagator $\alpha_b D_F^0(q)_{\mu\nu}$ is replaced in succession by

$$\alpha(q_\mu q_\nu/q^2)(q^2-\mu^2)^{-2}[-\Lambda^2/(q^2-\Lambda^2)]. \tag{B8}$$

Note that the derivative $\partial/\partial\alpha$ in Eq. (B6) acts only on the α dependence explicitly displayed in Eq. (B2) and not on the α dependence which is implicitly present as a result of the dependence on η. Let us now simplify Eq. (B6) in two ways. First we pass to the region of asymptotic $-p^2/m^2$, which allows us to drop the terms $\tilde{\Gamma}_{S'}$ and $\tilde{\Gamma}_{S''}$ on the right-hand side, since these vanish asymptotically. Secondly, we observe that we are really only interested in keeping the infrared cutoff μ^2 where it appears in divergent terms proportional to a power of $\ln\mu^2$. We get these divergent terms correctly even if we neglect those terms in Eq. (B6) which vanish as $O(\mu^2(\ln\mu^2)^n)$ as $\mu^2 \to 0$. Making these simplifications and adding the second equation in Eq. (B6) to the first gives the desired form of the Callan-Symanzik equations for the asymptotic electron propagator,

$$\left[m\frac{\partial}{\partial m}+\mu\frac{\partial}{\partial\mu}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}+\gamma(\alpha,\eta)\right]\tilde{S}_F'^{-1}\sim-[1+\delta(\alpha)]\tilde{\Gamma}_S,$$
$$\left[\mu\frac{\partial}{\partial\mu}+\gamma_\mu\right]\tilde{S}_F'^{-1}\sim 0, \tag{B9}$$

where[41]

$$\beta(\alpha)=\beta_m|_{\mu^2=0},$$
$$\delta(\alpha)=\delta_m|_{\mu^2=0}, \tag{B10}$$
$$\gamma(\alpha,\eta)=(\gamma_m+\gamma_\mu)|_{\mu^2=0}.$$

A precisely analogous procedure[18] yields the Callan-Symanzik equations for the asymptotic electron-scalar vertex,

$$\left[m\frac{\partial}{\partial m}+\mu\frac{\partial}{\partial\mu}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}+\gamma(\alpha,\eta)-\delta(\alpha)\right]\tilde{\Gamma}_S\sim 0,$$
$$\left[\mu\frac{\partial}{\partial\mu}+\gamma_\mu\right]\tilde{\Gamma}_S\sim 0. \tag{B11}$$

Finally, in the limit as $\mu^2\to 0$ Eq. (B7) for Z_2 and m_0 can be rewritten in the form

$$\left[m\frac{\partial}{\partial m}+\mu\frac{\partial}{\partial\mu}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}+\gamma(\alpha,\eta)\right]Z_2=0,$$
$$\left[\mu\frac{\partial}{\partial\mu}+\gamma_\mu\right]Z_2=0, \tag{B12}$$
$$\left[m\frac{\partial}{\partial m}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}-\delta(\alpha)\right]\left(\frac{m_0}{m}\right)=0,$$

closely analogous to Eq. (45b) for Z_3 given in the text.

Let us now use Eqs. (B9)–(B12) to study the asymptotic behavior of $\tilde{S}_F'$ and the large $-\Lambda$ behavior of m_0 and Z_2 in the cases of type-1 and type-2 asymptotic behavior (cf. Sec. II B).

Type 1. In this case the physical coupling α is equal to the value α_1 which satisfies $q(\alpha_1)=\alpha_0$, $\beta(\alpha_1)=0$ and, as shown in Sec. II C, the asymptotic renormalized photon propagator αd_c^∞ is exactly equal to α_0. Because $\beta(\alpha)=0$, the $\partial/\partial\alpha$ terms disappear from Eqs. (B9)–(B12), and so these equations become the simplified Callan-Symanzik equations used in the analysis of Ref. 17 (apart from the change that the asymptotic coupling α_0 used in Ref. 17 is replaced now by the physical coupling $\alpha=\alpha_1$). For the asymptotic behavior of $\tilde{S}_F'(p)$ and the large $-\Lambda$ behavior of m_0 and Z_2 we thus get the scaling expressions of Eq. (66). Furthermore, we find the gauge transformation properties derived in Ref. 17 to be in accord with the conclusion which we have reached above, that the gauge parameter ξ appears only in the combination $\eta = \alpha(\xi-1)$.

Type 2. In this case $\alpha\neq\alpha_1$ and so $\beta(\alpha)\neq 0$. We proceed to analyze the asymptotic behavior under the conventional assumption that α_0 is a simple zero, and a point of regularity, of the Gell-Mann-Low function ψ, or equivalently[42] [cf. Eq. (53)] that α_1 is a simple zero and a point of regularity of β. As we have seen in Eqs. (32)–(35), this assumption corresponds to power law vanishing of the nonasymptotic piece h of the renormalized photon propagator. To study Eqs. (B9)–(B11) for $\tilde{S}_F'$ and $\tilde{\Gamma}_S$, we separate out the γ-matrix structure by writing

$$\tilde{S}_F'^{-1}=\not{p}F+mG,$$
$$m\tilde{\Gamma}_S=\not{p}H+mJ, \tag{B13}$$

which gives the equations

$$\left[m\frac{\partial}{\partial m}+\mu\frac{\partial}{\partial\mu}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}+\gamma(\alpha,\eta)\right]F\sim 0,$$
$$\left[m\frac{\partial}{\partial m}+\mu\frac{\partial}{\partial\mu}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}+\gamma(\alpha,\eta)\right]mG\sim-[1+\delta(\alpha)]mJ, \tag{B14}$$
$$\left[m\frac{\partial}{\partial m}+\mu\frac{\partial}{\partial\mu}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}+\gamma(\alpha,\eta)-\delta(\alpha)\right]J\sim 0.$$

The first of these three differential equations has the general integral

$$F=\exp\left[-\int_{L_F}^{\alpha}\frac{dz\,\gamma(z,\eta)}{z\beta(z)}\right]\times\Phi_F\left[\ln\left(\frac{-p^2}{m^2}\right)+\int_{L_F}^{\alpha}\frac{2dz}{z\beta(z)}\,,\,\mu^2/m^2,\eta\right], \quad (B15)$$

with $\Phi_F[u, \mu^2/m^2, \eta]$ an arbitrary function of its arguments. Let us now consider the behavior of Eq. (B15) as $\alpha \to \alpha_1$. Since β has a zero at $z=\alpha_1$, the argument of the exponential prefactor and the argument u of the function Φ_F both become infinite. The only way for the function F to remain regular at $\alpha=\alpha_1$ is for the singularities of the exponential and of Φ_F at $\alpha=\alpha_1$ to precisely cancel. This can happen only if Φ_F has the following asymptotic behavior as u becomes infinite,

$$\Phi_F[u,\mu^2/m^2,\eta]\underset{u\to\infty}{\sim}C_F(\mu^2/m^2,\alpha_1,\eta)\times\exp[\tfrac{1}{2}\gamma(\alpha_1,\eta)u]\,. \quad (B16)$$

If we assume Eq. (B16), then when α is near α_1 we get

$$F\approx\exp\left[\int_{L_F}^{\alpha}dz\frac{\gamma(\alpha_1,\eta)-\gamma(z,\eta)}{z\beta(z)}\right]\times\text{finite terms}, \quad (B17)$$

which is regular because β vanishes with only a simple zero at $z=\alpha_1$. Let us now consider what happens as $-p^2/m^2$ becomes infinite, with α fixed at its physical value, different from α_1. Again u becomes infinite, this time because of the term $\ln(-p^2/m^2)$ in Eq. (B15), and so invoking Eq. (B16) gives us

$$F\sim C_F(\mu^2/m^2,\alpha_1,\eta)\exp\left[\int_{L_F}^{\alpha}\frac{\gamma(\alpha_1,\eta)-\gamma(z,\eta)}{z\beta(z)}\right]\times\left(\frac{-p^2}{m^2}\right)^{\gamma(\alpha_1,\eta)/2}. \quad (B18)$$

Thus, we see that even when $\alpha\neq\alpha_1$, *in the asymptotic limit F exhibits scaling behavior with a scaling exponent γ characteristic of the value α_1 at which β vanishes*.[43] An identical argument can be used to integrate the equations for G and J in Eq. (B14) and those for Z_2 and m_0/m in Eq. (B12), and finally the equation

$$\mu\frac{\partial}{\partial\mu}(\tilde{S}_F'^{-1}/Z_2)=\frac{1}{Z_2^{\,2}}\left(Z_2\mu\frac{\partial}{\partial\mu}\tilde{S}_F'^{-1}-\tilde{S}_F'^{-1}\mu\frac{\partial}{\partial\mu}Z_2\right)\sim 0 \quad (B19)$$

can be used to relate the μ dependence of the resulting constants of integration. The procedure unfolds in complete analogy[44] with the treatment of the JBW model given in Ref. 17, and the results obtained are of the same form as in Eq. (66), apart from the more complex structure of the integration constants seen in Eq. (B18).

To conclude, we reemphasize that in order to derive Eq. (B18), we need the twin assumptions of a simple zero in β and of regularity of the theory around α_1. If β vanishes more rapidly than with a simple zero at α_1, the exponential factor in Eq. (B17) is still not regular at α_1, and so the argument for requiring Φ_F to have the particular asymptotic form given in Eq. (B16) is no longer compelling. For an alternative derivation of scaling behavior of the asymptotic electron propagator, which may be valid even when ψ (or equivalently, β) has a higher-order zero, see Sec. III C of the text.

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[1] Particle Data Group, Rev. Mod. Phys. 43, S1 (1971).

[2] There are already stringent limits on the possible variation of α on a cosmological time scale. See P. J. Peebles and R. H. Dicke, Phys. Rev. 128, 2006 (1962); F. J. Dyson, Phys. Rev. Letters 19, 1291 (1967); A. Peres, *ibid.* 19, 1293 (1967); J. N. Bahcall and M. Schmidt, *ibid.* 19, 1294 (1967).

[3] See, for example, S. Weinberg, Phys. Rev. Letters 19, 1264 (1967); 27, 1688 (1971); L. D. Landau, in *Niels Bohr and the Development of Physics* (McGraw-Hill, New York, 1955), p. 60; A. Salam, paper presented at the Fifteenth International Conference on High-Energy Physics, Kiev, U.S.S.R., 1970 (unpublished).

[4] See the historical footnote on p. 607 of S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, Evanston, Ill., 1961).

[5] R. Jost and J. M. Luttinger, Helv. Phys. Acta 23, 201 (1950).

[6] M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

[7] K. Johnson, M. Baker, and R. Willey, Phys. Rev. 136, B111 (1964); K. Johnson, R. Willey, and M. Baker, *ibid.* 163, 1699 (1967); M. Baker and K. Johnson, *ibid.* 183, 1292 (1969); M. Baker and K. Johnson, Phys. Rev. D 3, 2516 (1971); 3, 2541 (1971).

[8] P. G. Federbush and K. Johnson, Phys. Rev. 120, 1296 (1960). See also R. Jost, in *Lectures on Field Theory and the Many-Body Problem,* edited by E. R. Caianiello (Academic, New York, 1961); K. Pohlmeyer, Commun. Math. Phys. 12, 204 (1969). The usual derivation of the Federbush-Johnson theorem depends heavily on assumptions of positivity and locality. As a result, the derivation cannot be directly applied to quantum electrodynamics, where, if one quantizes in the Lorentz gauge to guarantee locality, one must use the Gupta-Bleuler negative metric, while if one quantizes in the radiation gauge to

guarantee positivity, one loses local commutativity. It appears that this problem can be circumvented, and that a satisfactory proof for the case of electrodynamics can be given [F. Strocchi (unpublished)], at least in the charge-zero sector.

[9]B. Simon, in Proceedings of the 1972 Coral Gables Conference (unpublished), conjectures that field theories, as a function of coupling constant, will prove to be Borel summable. This would make it possible to determine them uniquely from a knowledge of their formal power series expansions. A cautionary remark is necessary here: As discussed by Simon, Borel summability of a function such as $F^{[1]}(y)$ cannot be deduced from the formal power-series coefficients alone – additional information about $F^{[1]}(y)$ is needed. In the same article, Simon also discusses a possible connection of Borel summability with the conjecture of F. J. Dyson, Phys. Rev. 85, 631 (1952), concerning singularities in field theory at zero coupling constant. However, one should note that in the specific examples of nonanalytic behavior given by Simon, the nonanalyticity arises from mass renormalization, not from the fermion vacuum polarization effects discussed both by Dyson and by us in the present paper.

[10]C. G. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970); and in *Springer Tracts in Modern Physics,* edited by G. Höhler (Springer, Berlin, 1971), Vol. 57, p. 222.

[11]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965). Throughout this paper, we follow wherever possible the Bjorken-Drell notation and metric conventions.

[12]Note that this is *not* just a "leading logarithm" approximation: *All* terms in each order of perturbation theory which do not vanish asymptotically are retained.

[13]A discussion of effects of dropping this assumption has been given by M. Astaud and B. Jouvet, Nuovo Cimento 63A, 5 (1969), and M. Astaud, *ibid.* 66A, 111 (1970).

[14]However, when $\alpha_w d_c[x,w,\alpha_w]$ is reexpanded as a power series in α, there will be an m dependence, introduced by the appearance of the nonasymptotic point zero in Eq. (18).

[15]T. Kinoshita, J. Math. Phys. 3, 650 (1962); T. D. Lee and M. Nauenberg, Phys. Rev. 133, B1549 (1964). See also A. Sirlin, Phys. Rev. D 5, 436 (1972), who discusses the connection between the infrared cancellation theorems and the Callan-Symanzik equations.

[16]M. Baker and K. Johnson, Phys. Rev. 183, 1292 (1969).

[17]We follow the approach of S. Coleman (unpublished) and A. Sirlin, Ref. 15.

[18]S. L. Adler and W. A. Bardeen, Phys. Rev. D 4, 3045 (1971). The quantity called $\delta(\alpha)$ in Eq. (41) is denoted by α in the work of Adler and Bardeen.

[19]S. Weinberg, Phys. Rev. 118, 838 (1960).

[20]See also K. Pohlmeyer, Commun. Math. Phys. 12, 204 (1969). Note that although the $2n$-point current correlation functions vanish for $n \geq 2$, the Federbush-Johnson theorem does *not* require the vanishing of the dispersive part of the photon proper self-energy (the case $n=1$), which would imply that $\alpha_0 = \alpha$. A heuristic way of understanding this is to note the annihilation of the vacuum by the electromagnetic current implies the vanishing of the absorptive part of the general $2n$-point current correlation function ($n \geq 1$). This implies that in momentum space the correlation function must be a polynomial function of its four-momentum arguments. Gauge invariance tells us that this polynomial must contain as factors the four-momenta $k_1 \cdots k_{2n}$ of the $2n$ photons, and hence contains only terms of degree $2n$ or higher. On the other hand, Weinberg's theorem tells us that the amplitude behaves at worst as $(\text{momentum})^{4-2n} \times$ logarithms as all four-momenta are scaled to infinity. Thus the minimum degree $2n$ of the polynomial must satisfy the inequality $2n \leq 4-2n$, which is compatible only with $n=1$. Hence the $2n$-point current correlation function with $n \geq 2$ must vanish, while the two-point function is a polynomial of the form $(-q^2 g_{\mu\nu} + q_\mu q_\nu) \times$ constant, consistent with our initial assumption that the photon propagator αd_c is equal to the asymptotic value $\alpha_0 \neq \alpha$.

[21]It is still possible, of course, that the converse result is true, and might be provable if the analyticity structure of the single-fermion-loop diagram as a function of the external photon four-momenta is taken into account.

[22]See also S. L. Adler and W. A. Bardeen, erratum to Ref. 18 (to be published).

[23]Equation (77) is the generalization of an identity, due to M. L. Goldberger, which formally relates an integral over the virtual Compton amplitude to the coupling constant derivative of the electron self-energy part in quantum electrodynamics. I am grateful to R. F. Dashen for reminding me of that identity and for suggesting its applicability to vacuum polarization loops.

[24]This formula also gives the correct fractional weights for vacuum diagrams (the case $n=0$, $j \geq 1$).

[25]Even though Eq. (88) is true only by virtue of taking a nonperturbative sum of diagrams to infinite order, we are *assuming,* with no attempt at justification, that the difference between Eq. (89) and Eq. (88) vanishes for small m as it would in perturbation theory. The proportionality to m^2, rather than just to m, follows from the fact that because of charge conjugation invariance $T^{[1]}$ is an even function of m.

[26]In writing Eq. (90) we make no commitment as to the size of the mass cm^2 which effectively cuts off the integral – it could in principle be exceedingly large, since it arises from a cancellation of asymptotically dominant terms which involves all orders of perturbation theory. Experimental tests of electrodynamics which are sensitive to vacuum polarization effects could be used to set a lower limit on the possible value of cm^2.

[27]One might ask why, even for $y \neq \alpha_0$, one cannot simply add and subtract

$$T^{[1]}_{\mu_1\mu_2\cdots\mu_{2n-3}\mu_{2n-2}\mu\nu}(q_1,-q_1,\ldots,q_{n-1},-q_{n-1},q,-q;0,y)$$

in the integrand of Eq. (76), thus leading to the following modified version of Eq. (85),

$$[\pi^{(1)}_{2n}(q^2;m,y) - \pi^{(1)}_{2n}(q^2;m',y)](-q^2 g_{\mu\nu} + q_\mu q_\nu) = \tilde{I}_m - \tilde{I}_{m'},$$

with

$$\tilde{I}_m = \int \frac{d^4q_1}{(2\pi)^4} \cdots \frac{d^4q_{n-1}}{(2\pi)^4} \left(-\frac{ig^{\mu_1\mu_2}}{q_1^2}\right) \cdots \left(-\frac{ig^{\mu_{2n-3}\mu_{2n-2}}}{q_{n-1}^2}\right)$$
$$\times [T^{[1]}_{\mu_1\mu_2\cdots\mu_{2n-3}\mu_{2n-2}\mu\nu}(q_1,-q_1,\ldots,q_{n-1},-q_{n-1},q,-q;m,y)$$
$$- T^{[1]}_{\mu_1\mu_2\cdots\mu_{2n-3}\mu_{2n-2}\mu\nu}(q_1,-q_1,\ldots,q_{n-1},-q_{n-1},q,-q;0,y)].$$

The answer is that although $\tilde{I}_m$ is now ultraviolet convergent, the subtraction term makes $\tilde{I}_m$ a logarithmically divergent integral in the infrared of the general type

$$\int_0^\infty d\rho\left(\frac{1}{\rho+m^2}-\frac{1}{\rho}\right)=\int_0^\infty\frac{-m^2}{\rho(\rho+m^2)}\,.$$

Thus, the modified version of Eq. (85) is still an ambiguous expression of the form $\infty-\infty$. The significance of the special condition, Eq. (74), which holds when $y=\alpha_0$ is that it improves the ultraviolet behavior of I_m without simultaneously making the infrared behavior worse. This feature has been incorporated in the illustrative example given in Eq. (90).

[28]After this work was completed, we learned that K. Johnson (unpublished) knew a related argument suggesting a zero of infinite order in $\psi(y)$, obtained by working directly with the modified skeleton expansion described in Sec. II D1.

[29]Equations (99) and (102) clearly illustrate the distinction between type-1 and type-2 asymptotic behavior. When $\alpha_0\neq q(\alpha)$, the asymptotic behavior is type 2, and Eqs. (99) and (102) can both be developed as power series in $\ln x$. When $\alpha_0=q(\alpha)$, Eqs. (99) and (102) both degenerate to $\alpha d_c^\infty=\alpha_0$, as expected for type-1 asymptotic behavior.

[30]The discussion which follows depends only on the fact that the Gell-Mann–Low function vanishes with a zero of infinite order, and does not hinge crucially on the choice of Eq. (103) for h. To see this, we note from Table I that if the Gell-Mann–Low function vanishes with a zero of *finite* order N, the function $h(x,\alpha)$ vanishes asymptotically as $(\ln x)^{-1/(N-1)}$. Consequently, h^n vanishes as $(\ln x)^{-n/(N-1)}$ and the integral in Eq. (106) diverges for all $n\leq N-1$. Letting $N\to\infty$, we learn that in the case of an infinite-order zero of the Gell-Mann–Low function, the integral in Eq. (106) diverges for all n. Note that in making the distinction between the case where h vanishes as a power of x and the case where h vanishes more slowly, it is important to adhere to our convention of "vacuum-polarization-insertion-wise" summation, which requires us to sum the logarithmic series defining d_c^∞ *before* passing to the asymptotic limit. This is particularly important in the case of Eq. (102), where the logarithmic series has only a finite radius of convergence and so cannot be used to describe the asymptotic region.

[31]W. A. Bardeen (unpublished).

[32]The question of whether the Federbush-Johnson theorem can be extended outside the charge-zero sector in electrodynamics is an important one and deserves further study. If it can be extended sufficiently to imply the vanishing of the electron-photon vertex part, then using the Ward identity to relate the vertex part to the asymptotic electron propagator in Eq. (66) implies that

$$\lim_{m\to 0}F_1C_1m^{-\gamma}=0$$

when the eigenvalue condition is satisfied. If $F_1\neq 0$, this equation then tells us that Z_2 must vanish.

[33]G. Källén, Helv. Phys. Acta 25, 417 (1952); H. Lehmann, Nuovo Cimento 11, 342 (1954).

[34]Note that there is no contradiction between the fact that $\alpha_0>\alpha$ and the assertion, essential to the argument of Sec. II D1, that the spectral function vanishes as $m^2\to 0$. For illustrative purposes let us follow the example of Eq. (90) and take

$$w(x,\alpha)=c/[(1+x)(1+cx)]$$

with $c>0$. Then Eq. (129) becomes

$$\alpha_0=\alpha+\int_0^\infty\frac{c\alpha m^2d\rho}{(\rho+m^2)(\rho+cm^2)}=\alpha+c\alpha\frac{\ln c}{c-1}>\alpha\,,$$

but *for fixed* ρ the spectral function (i.e., the integrand) vanishes as $m^2\to 0$.

[35]The fact that y_0 appears as an *infinite-order* zero of $F^{[1]}$ means that it may be possible to determine y_0 from the limiting behavior of the nth term in the perturbation expansion for $F^{[1]}$ as $n\to\infty$. For example, suppose that $F^{[1]}$ actually has a convergent power series expansion around $y=0$ with radius of convergence y_0,

$$F^{[1]}(y)=\sum_{n=0}^{\infty}c_ny^n\,.$$

Then y_0 is given by the limit formula

$$y_0=\lim_{n\to\infty}|c_n|^{1/n}\,.$$

Since c_n describes the fermion loop with n internal virtual photons, it is thus conceivable that y_0 can be computed in a semiclassical (large-photon-number) calculation.

[36]Dyson (Ref. 9) also considers the alternative possibility, that electrodynamics by itself is not a complete theory, and becomes consistent only when other interactions are taken into account.

[37]The sixth-order result for $F^{[1]}$ is due to J. L. Rosner, Phys. Rev. Letters 17, 1190 (1966), and Ann. Phys. (N.Y.) 44, 11 (1967). The fourth-order expansion for $F_B^{[1]}(y)$ is due to Z. Białynicka-Birula, Bull. Acad. Polon. Sci. 13, 369 (1965). Conflicting results in the fourth-order boson calculation have been claimed by I.-J. Kim and C. R. Hagen, Phys. Rev. D 2, 1511 (1970). However, D. Sinclair (unpublished) has located an error in the work of Kim and Hagen which, when corrected, gives Białynicka-Birula's result. Sinclair has also rechecked this result independently by Rosner's method of calculation.

[38]For a review of this point of view, see D. J. Gross and S. B. Treiman, Phys. Rev. D 4, 1059 (1971).

[39]For an alternative explanation, which regards scaling as an intermediate energy manifestation of compositeness of the nucleon, see S. D. Drell and T. D. Lee, Phys. Rev. D 5, 1738 (1972).

[40]Since $F_B^{[1]}(y)$ is different from $F^{[1]}(y)$ our scheme could, in principle, accommodate a fractionally charged elementary boson.

[41]The renormalization constants m_0 and Z_3 are gauge-invariant and are also infrared finite as $\mu^2\to 0$. These properties account, respectively, for the facts that $\delta(\alpha)$ and $\beta(\alpha)$ are independent of η and that δ_μ and β_μ vanish as $\mu^2\to 0$. In Ref. 18, it is shown that the gauge dependence of $\gamma(\alpha,\eta)$ is strictly additive, i.e., $\gamma(\alpha,\eta)-\gamma(\alpha,\eta')=(\eta-\eta')/(2\pi)$.

[42]We assume that the mapping $q(\alpha)$ is well behaved near α_1, in particular, that $q'(\alpha_1)\neq 0$.

[43]This was first pointed out by K. G. Wilson, Phys. Rev. D 3, 1818 (1971). Our treatment is suggested by the procedure of C. G. Callan, Ref. 10.

[44]In particular, comparison of the second and third equations in Eq. (B14) shows that $G=-J$ is a particular integral of the differential equation for G, and a simple application of Weinberg's theorem shows that one cannot add a solution of the homogeneous equation

$$\left[m\frac{\partial}{\partial m}+\mu\frac{\partial}{\partial\mu}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}+\gamma(\alpha,\eta)\right]mG=0$$

to the particular solution.