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Anomalous Magnetic Moment of the Charged Intermediate Vector Boson*

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Assuming that the intermediate vector boson for weak interactions has no strong interactions, we reinvestigate the Corben-Schwinger problem of the scattering of a charged vector meson with anomalous magnetic moment by an external static Yukawa potential, or in particular a static Coulomb field. Meixner's corner condition of finite integrated energy in every bounded region is applied in the neighborhood of the singularity of the Yukawa potential, and it is found that the Corben-Schwinger problem, although nonrenormalizable, possesses a finite, unique solution if and only if the anomalous magnetic moment is larger than unity. It is therefore a characteristic of nonrenormalizable theories that there may be lower bounds to coupling constants. Experimental determination of the anomalous magnetic moment of the intermediate vector boson is thus of great importance. Furthermore, because certain differential equations happen to be solvable in terms of Bessel functions, an explicit perturbation expansion for weak coupling can be given for this scattering process.

I. INTRODUCTION

In the near future before the publication of this paper, the 500-BeV accelerator at the National Accelerator Laboratory is expected to operate, and an experiment to search for the intermediate vector boson[1] for weak interactions has already been approved. Let us assume that the intermediate vector boson is found, then a theory is needed for such a particle. As suggested by a number of authors, the intermediate vector boson may have strong interactions.[2] However, since the mass of the muon remains a total mystery, we believe that there is no compelling reason why the intermediate vector meson should have strong interactions. Let us, therefore, assume that the intermediate vector boson has no strong interactions.

Since the weak currents carry a charge, the intermediate vector boson must have electromagnetic interactions. The theory of a charged vector particle has been known for a long time.[3] By simple power counting, such a theory is *nonrenormalizable*. This is the reason for Lee and Yang[4] to propose, nearly a decade ago, their ξ-limiting formalism. This formalism was used by Lee,[5] and Bernstein and Lee,[6] to study various properties of the intermediate vector boson and the neutrino, and also muon decay. As particularly emphasized by Lee,[5] the existence of the limit $\xi \to 0$ is a pure assumption.

On the other hand, we have been pursuing a program of learning about the high-energy scattering of hadrons from quantum field theory.[7] In addition to the predictions of increasing total cross sections[8] and distribution of pionization product,[9] *close similarity between the field-theoretic results and potential scattering,* suitably interpreted, *is found.*[10] In particular, we have recently studied the scattering of a charged vector particle by a static external potential.[11] The purpose of this study is in no way related to the possible existence of the intermediate vector boson, and is in fact to gain a better understanding of exponentiation[10] in high-energy scattering. In the course of this work, however, it is realized that our understanding of the charged vector particle is remarkably incomplete: In addition to our ignorance about the nonrenormalizable nature of the quantum theory, we do not even know much about the classical field of a charged vector particle.

It is the purpose of this paper to study the classical problem of the scattering of a charged vector particle by a Yukawa potential. In the absence of anomalous magnetic moment, the divergence in this case is the same as that encountered in the quantum field theory of a charged vector particle. There are several motivations for studying this classical problem. First, a better understanding of the classical field is perhaps useful for quantization, since historically a thorough knowledge of the classical electromagnetic field precedes by many years the quantum theory of the photon. Secondly, a solution of this classical problem is in any case needed because of the similarity, at high energies, between field theory and potential scattering.[10] Thirdly, from a more general point of view, this classical problem gives a natural example of a nonrenormalizable theory. If we insist on using the Born approximation, worse and worse divergences are found in higher and higher orders. However, since we are dealing with a set of partial differential equations, we are in no way restricted to the Born approximation.

This scattering problem was first treated, over thirty years ago, by Corben and Schwinger[12] in connection with the problem of the "mesotron." (Our generalization of the Coulomb field to the Yukawa potential is of no importance.) This remarkable work was carried out long before the understanding of renormalization, but they were unfortunately concerned mainly with two cases where the anomalous magnetic moment κ is either zero or unity. Here we shall study directly the differential equation for scattering, and *no divergence is found* provided that κ satisfies the relation

$$\kappa > 1 , \qquad (1.1)$$

which excludes both cases of Corben and Schwinger. Thus the nonrenormalizability must be blamed on the inadequacy of the Born approximation. In this way, we hope to have a beginning in understanding the intermediate vector meson.

2. FIELD EQUATIONS

Let ϕ_μ denote the field of a charged vector meson. Since we are interested in the classical theory, ϕ_μ is a set of four c-number functions of space and time. Let A_μ denote a known external electromagnetic field, then the field equations for ϕ_μ are[4]

$$\left(\frac{\partial}{\partial x_\mu} - ieA_\mu\right) G_{\mu\nu} - \phi_\nu + ie\kappa\phi_\mu F_{\mu\nu} = 0 , \qquad (2.1)$$

where

$$G_{\mu\nu} = \left(\frac{\partial}{\partial x_\mu} - ieA_\mu\right)\phi_\nu - \left(\frac{\partial}{\partial x_\nu} - ieA_\nu\right)\phi_\mu \qquad (2.2)$$

and

$$F_{\mu\nu} = (\partial/\partial x_\mu)A_\nu - (\partial/\partial x_\nu)A_\mu . \qquad (2.3)$$

In (2.1), κ is a constant, and the mass of the charged vector meson has been taken to be unity.

For the case of a static external field, $\vec{A}=0$, and the time component of A_μ, denoted as V, is independent of time. Thus there is invariance under time translation, and the time dependence of ϕ_μ can be taken as e^{-iEt}. Thus, from here on, we consider ϕ_μ to be functions of the space coordinates and E. Taking separately the time and space components of (2.1), we get

$$(\nabla^2 - 1)\phi_0 - i\nabla\cdot(E - eV)\vec{\phi} + ie\kappa(\nabla V)\cdot\vec{\phi} = 0 \qquad (2.4)$$

and

$$[(E-eV)^2 + \nabla^2 - 1]\vec{\phi} - \nabla(\nabla\cdot\vec{\phi}) + i(E-eV)\nabla\phi_0 + ie\kappa\phi_0\nabla V = 0 . \qquad (2.5)$$

These are the partial differential equations we shall deal with in this paper, where

$$eV = ge^{-\mu r}/r \qquad (2.6)$$

is the Yukawa potential.

Equations (2.4) and (2.5) are four coupled second-order partial differential equations. From these four equations, we can get one first-order partial differential equation by the standard procedure of obtaining the Lorentz condition for time-independent problems. Take the divergence of (2.5) and use (2.4) to eliminate the $\nabla^2\phi_0$ term:

$$\nabla\cdot\vec{\phi} - i(E-eV)\phi_0 + e(1-\kappa)(\nabla V)\cdot[(E-eV)\vec{\phi} + i\nabla\phi_0] - ie\kappa\phi_0\nabla^2 V = 0 . \qquad (2.7)$$

This equation (2.7) turns out to be extremely useful.

The incident field ϕ_μ^{inc} satisfies (2.4) and (2.5) with $eV=0$. If the z axis is chosen to be the direction of propagation of the incident plane wave, then

$$\phi_0^{\text{inc}} = u_0 e^{ikz}$$

and $\qquad (2.8)$

$$\vec{\phi}^{\text{inc}} = \vec{u}\, e^{ikz} ,$$

where $k=(E^2-1)^{1/2}$ and

$$u_0 E - u_3 k = 0 . \qquad (2.9)$$

Since (2.4) and (2.5) are linear, it is convenient to separate the present scattering problem into two parts: one corresponding to longitudinal polarization of the incident wave and the other corresponding to transverse polarization. The incident fields are then, respectively, the following.

Longitudinal polarization:

$$\phi_0^{\text{inc}} = ke^{ikz} , \qquad (2.10a)$$

$$\phi_1^{\text{inc}} = \phi_2^{\text{inc}} = 0 , \qquad (2.10b)$$

and

$$\phi_3^{\rm inc} = E e^{ikz} . \tag{2.10c}$$

Transverse polarization:

$$\phi_1^{\rm inc} = e^{ikz} , \tag{2.11a}$$

$$\phi_2^{\rm inc} = ie^{ikz} , \tag{2.11b}$$

and

$$\phi_0^{\rm inc} = \phi_3^{\rm inc} = 0 . \tag{2.11c}$$

Note that in writing down (2.11) we have chosen a circularly polarized incident wave.

3. PARTIAL-WAVE EXPANSION FOR LONGITUDINAL POLARIZATION

We first treat the case of (2.10), where the incident plane wave is longitudinally polarized. In this case, there is no preferred direction in the transverse xy plane. Let (r, θ, ϕ) be the spherical coordinates, then the field quantities are all independent of ϕ. Moreover, $\vec{\phi}$ has no ϕ component.

In spherical coordinates, (2.4), (2.5), and (2.7) are explicitly

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \phi_0}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\phi_0}{\partial\theta} - \phi_0 - \frac{i}{r^2}\frac{\partial}{\partial r} r^2(E-eV)\phi_r - \frac{i}{r\sin\theta}(E-eV)\frac{\partial}{\partial\theta}\sin\theta\,\phi_\theta + ie\kappa\frac{dV}{dr}\phi_r = 0 , \tag{3.1}$$

$$[(E-eV)^2-1]\phi_r + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\phi_r}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\phi_\theta - \frac{\partial}{\partial r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\phi_\theta + i(E-eV)\frac{\partial\phi_0}{\partial r} + ie\kappa\frac{dV}{dr}\phi_0 = 0 , \tag{3.2}$$

$$[(E-eV)^2-1]\phi_\theta + \frac{1}{r^2}\frac{\partial}{\partial r} r^2\frac{\partial\phi_\theta}{\partial r} - \frac{1}{r}\frac{\partial^2\phi_r}{\partial r\partial\theta} + \frac{i}{r}(E-eV)\frac{\partial\phi_0}{\partial\theta} = 0 , \tag{3.3}$$

and

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2\phi_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\phi_\theta - i(E-eV)\phi_0 + e(1-\kappa)\frac{dV}{dr}\left((E-eV)\phi_r - i\frac{\partial\phi_0}{\partial r}\right) - ie\kappa\phi_0\frac{1}{r^2}\frac{d}{dr}r^2\frac{dV}{dr} = 0 . \tag{3.4}$$

Remembering that the spherical Bessel functions are defined by

$$j_n(z) = (\tfrac{1}{2}\pi/z)^{1/2} J_{n+1/2}(z) , \tag{3.5}$$

we take the partial-wave expansion of the field components to be

$$\phi_0 = \sum_{n=0}^{\infty}(2n+1)i^n\Phi_{0n}(r)P_n(\cos\theta) ,$$

$$\phi_r = \sum_{n=0}^{\infty}(2n+1)i^{n-1}\Phi_{rn}(r)P_n(\cos\theta) , \tag{3.6}$$

and

$$\phi_\theta = \sum_{n=1}^{\infty}(2n+1)i^{n-1}\Phi_n(r)P_n^1(\cos\theta) ,$$

where the associated Legendre polynomial $P_n^1(\cos\theta)$ is simply $(d/d\theta)P_n(\cos\theta)$. A comparison of (3.6) with (2.10) shows that

$$\Phi_{0n}^{\rm inc}(r) = kj_n(kr)$$

for $n \geq 0$,

$$\Phi_{rn}^{\rm inc}(r) = Ej_n'(kr) \tag{3.7}$$

for $n \geq 0$, and

$$\Phi_n^{\rm inc}(r) = E(kr)^{-1}j_n(kr)$$

for $n > 0$.

It is slightly more convenient to use

$$\eta = \kappa - 1 \tag{3.8}$$

instead of κ. When (3.6) is substituted into (3.1)–(3.4), the ordinary differential equations for the partial waves are found to be

$$\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}-\frac{n(n+1)}{r^2}-1\right)\Phi_{0n}-\left(\frac{1}{r^2}\frac{d}{dr}r^2(E-eV)-e(1+\eta)\frac{dV}{dr}\right)\Phi_{rn}+n(n+1)r^{-1}(E-eV)\Phi_n=0 \tag{3.9}$$

for $n\geq 0$,

$$-\left((E-eV)\frac{d}{dr}+e(1+\eta)\frac{dV}{dr}\right)\Phi_{0n}+\left((E-eV)^2-1-\frac{n(n+1)}{r^2}\right)\Phi_{rn}+n(n+1)\frac{1}{r^2}\frac{d}{dr}r\Phi_n=0 \tag{3.10}$$

for $n\geq 0$,

$$-\frac{1}{r}(E-eV)\Phi_{0n}-\frac{1}{r}\frac{d}{dr}\Phi_{rn}+\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+(E-eV)^2-1\right)\Phi_n=0 \tag{3.11}$$

for $n\geq 1$, and

$$\left[E-eV+e\eta\frac{dV}{dr}\frac{d}{dr}+e(1+\eta)\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{dV}{dr}\right)\right]\Phi_{0n}+\left(\frac{1}{r^2}\frac{d}{dr}r^2-e\eta\frac{dV}{dr}(E-eV)\right)\Phi_{rn}-\frac{n(n+1)}{r}\Phi_n=0 \tag{3.12}$$

for $n\geq 0$. These four equations (3.9)–(3.12) are of course not independent. More precisely, (3.12) can be obtained from

$$-(E-eV)\times(3.9)-r^{-2}(d/dr)r^2\times(3.10)+n(n+1)r^{-1}\times(3.11).$$

We shall study in some detail this set of coupled ordinary differential equations in this paper.

4. PARTIAL-WAVE EXPANSION FOR TRANSVERSE POLARIZATION

The case of transverse polarization is only slightly more complicated. By (2.11), the spherical components of the incident plane wave are

$$\begin{aligned}\phi_0^{\text{inc}}&=0,\\ \phi_r^{\text{inc}}&=\sin\theta\, e^{ikz}e^{i\phi},\\ \phi_\theta^{\text{inc}}&=\cos\theta\, e^{ikz}e^{i\phi},\end{aligned} \tag{4.1}$$

and

$$\phi_\phi^{\text{inc}}=ie^{ikz}e^{i\phi}.$$

Therefore $\partial/\partial\phi=i$, and (2.4), (2.5), and (2.7) are

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\phi_0}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\phi_0}{\partial\theta}-\frac{1}{r^2\sin^2\theta}\phi_0-\phi_0-\frac{i}{r^2}\frac{\partial}{\partial r}r^2(E-eV)\phi_r-\frac{i}{r\sin\theta}(E-eV)\left(\frac{\partial}{\partial\theta}\sin\theta\,\phi_\theta+i\phi_\phi\right)+ie\kappa\frac{dV}{dr}\phi_r=0, \tag{4.2}$$

$$[(E-eV)^2-1]\phi_r+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\phi_r}{\partial\theta}-\frac{1}{r^2\sin^2\theta}\phi_r-\frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\,\phi_\theta+i\phi_\phi\right)-\frac{\partial}{\partial r}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\,\phi_\theta+i\phi_\phi\right)+i(E-eV)\frac{\partial\phi_0}{\partial r}+ie\kappa\frac{dV}{dr}\phi_0=0, \tag{4.3}$$

$$[(E-eV)^2-1]\phi_\theta+\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\phi_\theta}{\partial r}-\frac{1}{r^2\sin^2\theta}\phi_\theta-\frac{1}{r}\frac{\partial^2\phi_r}{\partial r\partial\theta}-\frac{i}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}+\cot\theta\right)\phi_\phi+\frac{i}{r}(E-eV)\frac{\partial\phi_0}{\partial\theta}=0, \tag{4.4}$$

$$[(E-eV)^2-1]\phi_\phi+\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\phi_\phi}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\phi_\phi}{\partial\theta}-\frac{1}{r^2\sin^2\theta}\phi_\phi-\frac{i}{r\sin\theta}\frac{\partial\phi_r}{\partial r}-\frac{i}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}-\cot\theta\right)\phi_\theta-\frac{1}{r\sin\theta}(E-eV)\phi_0=0, \tag{4.5}$$

and

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\phi_r+\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\,\phi_\theta+i\phi_\phi\right)-i(E-eV)\phi_0+e(1-\kappa)\frac{dV}{dr}\left((E-eV)\phi_r+i\frac{\partial\phi_0}{\partial r}\right)-ie\kappa\phi_0\frac{1}{r^2}\frac{d}{dr}r^2\frac{dV}{dr}=0. \tag{4.6}$$

It is important to note that the only combination of ϕ_θ and ϕ_ϕ that appears in (4.2), (4.3), and (4.6) is

$$(r\sin\theta)^{-1}[(\partial/\partial\theta)\sin\theta\,\phi_\theta+i\phi_\phi]. \tag{4.7}$$

This expression (4.7) is the transverse divergence (i.e., the divergence in the θ and ϕ directions) of $\vec{\phi}$. Since ϕ_0, ϕ_r, and (4.7) all transform as scalars under rotation, we have the partial-wave expansions

$$\phi_0=\sum_{n=1}^{\infty}(2n+1)i^n\,\Phi_{0nt}(r)P_n^1(\cos\theta)e^{i\phi},$$

$$\phi_r=\sum_{n=1}^{\infty}(2n+1)i^{n-1}\Phi_{rnt}(r)P_n^1(\cos\theta)e^{i\phi}, \tag{4.8}$$

and

$$\csc\theta\left(\frac{\partial}{\partial\theta}\sin\theta\,\phi_\theta+i\phi_\phi\right)=-\sum_{n=1}^{\infty}n(n+1)(2n+1)i^{n-1}\Phi_{nt}(r)P_n^1(\cos\theta)e^{i\phi}.$$

A comparison with (4.1) shows that

$$\Phi_{0nt}^{\rm inc}(r)=0,$$

$$\Phi_{0nt}^{\rm inc}(r)=-(kr)^{-1}j_n(kr), \tag{4.9}$$

and

$$\Phi_{nt}^{\rm inc}(r)=-[n(n+1)]^{-1}[j_n'(kr)+(kr)^{-1}j_n(kr)].$$

The substitution of (4.8) into (4.2), (4.3), and (4.6) shows that $\Phi_{0nt}(r)$, $\Phi_{rnt}(r)$, and $\Phi_{nt}(r)$ satisfy exactly the same ordinary differential equations as $\Phi_{0n}(r)$, $\Phi_{rn}(r)$, and $\Phi_n(r)$, namely, (3.9)–(3.12) for $n\geq 1$. This result is of course to be expected as a consequence of rotational invariance. The boundary conditions (4.9) for the transverse case are quite different from (3.7) of the longitudinal case.

The present case of transverse polarization is, however, more complicated in that we still have to determine ϕ_0 and ϕ_ϕ themselves. By (4.8), it is sufficient to find ϕ_θ. For this purpose, we write (4.4) in the form

$$[(E-eV)^2-1]\phi_\theta+\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\phi_\theta}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\phi_\theta}{\partial\theta}+\frac{2}{r^2}\cot\theta\frac{\partial\phi_\theta}{\partial\theta}-\frac{2}{r^2}\phi_\theta$$
$$=\frac{1}{r}\frac{\partial^2\phi_r}{\partial r\partial\theta}-\frac{i}{r}(E-eV)\frac{\partial\phi_0}{\partial\theta}+\frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}+\cot\theta\right)\left(\frac{\partial}{\partial\theta}\sin\theta\,\phi_\theta+i\phi_\phi\right). \tag{4.10}$$

When (4.8) is substituted into (4.10), we get

$$\text{Right-hand side of (4.10)}=\sum_{n=1}^{\infty}(2n+1)i^{n-1}\left[r^{-1}\frac{\partial}{\partial r}\Phi_{rnt}(r)+r^{-1}(E-eV)\Phi_{0nt}(r)-n(n+1)r^{-2}\Phi_{nt}(r)\right]e^{i\phi}\frac{d}{d\theta}P_n^1(\cos\theta)$$
$$-2r^{-2}\sum_{n=1}^{\infty}n(n+1)(2n+1)i^{n-1}\Phi_{nt}(r)e^{i\phi}\cot\theta\,P_n^1(\cos\theta)$$
$$=-\sum_{n=1}^{\infty}i^n\Big\{(n-1)^2\left[r^{-1}\frac{\partial}{\partial r}\Phi_{r(n-1)t}(r)+r^{-1}(E-eV)\Phi_{0(n-1)t}(r)-n(n+1)r^{-2}\Phi_{(n-1)t}(r)\right]$$
$$+(n+2)^2\left[r^{-1}\frac{\partial}{\partial r}\Phi_{r(n+1)t}(r)+r^{-1}(E-eV)\Phi_{0(n+1)t}(r)-n(n+1)r^{-2}\Phi_{(n+1)t}(r)\right]\Big\}$$
$$\times\csc\theta\,P_n^1(\cos\theta)e^{i\phi}. \tag{4.11}$$

In writing down (4.11), we have used the identities

$$(2n+1)\cot\theta\, P_n^1(\cos\theta)=n\csc\theta\, P_{n+1}^1(\cos\theta)+(n+1)\csc\theta\, P_{n-1}^1(\cos\theta)$$

and (4.12)

$$(2n+1)\frac{d}{d\theta}P_n^1(\cos\theta)=n^2\csc\theta\, P_{n+1}^1(\cos\theta)-(n+1)^2\csc\theta\, P_{n-1}^1(\cos\theta)\,.$$

In view of (4.11), the partial-wave expansion of ϕ_θ must be taken to be

$$\phi_\theta=\sum_{n=1}^{\infty} i^n\Phi_{\theta n}(r)\csc\theta\, P_n^1(\cos\theta)e^{i\phi}\,. \tag{4.13}$$

On the one hand, a comparison of (4.13) with (4.1) shows that

$$\Phi_{\theta n}^{\rm inc}(r)=(2n+1)\frac{d}{d(kr)}(kr)^{-1}j_n(kr)\,; \tag{4.14}$$

on the other hand, the substitution of (4.13) into (4.10) yields that

$$\text{Left-hand side of (4.10)}=\sum_{n=1}^{\infty} i^n\csc\theta\, P_n^1(\cos\theta)e^{i\phi}\left[r^{-2}\frac{d}{dr}r^2\frac{d}{dr}+(E-eV)^2-1-n(n+1)r^{-2}\right]\Phi_{\theta n}(r)\,, \tag{4.15}$$

because of the identity

$$\left(\csc\theta\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+2\cot\theta\frac{d}{d\theta}\right)\csc\theta\, P_n^1(\cos\theta)=-(n-1)(n+2)\csc\theta\, P_n^1(\cos\theta)\,. \tag{4.16}$$

Therefore, once $\Phi_{0nt}(r)$, $\Phi_{rnt}(r)$, and $\Phi_{nt}(r)$ are known for $n\geq 1$ by solving (3.9)–(3.12) together with the boundary conditions (4.9), $\Phi_{\theta n}(r)$ can be found from the following second-order ordinary differential equation with the boundary condition (4.14):

$$\left[r^{-2}\frac{d}{dr}r^2\frac{d}{dr}+(E-eV)^2-1-n(n+1)r^{-2}\right]\Phi_{\theta n}(r)$$
$$=-(n-1)^2\left[r^{-1}\frac{d}{dr}\Phi_{r(n-1)t}(r)+r^{-1}(E-eV)\Phi_{0(n-1)t}(r)-n(n+1)r^{-2}\Phi_{(n-1)t}(r)\right]$$
$$-(n+2)^2\left[r^{-1}\frac{d}{dr}\Phi_{r(n+1)t}(r)+r^{-1}(E-eV)\Phi_{0(n+1)t}(r)-n(n+1)r^{-2}\Phi_{(n+1)t}(r)\right]$$

for $n\geq 1$. (4.17)

Finally, we list more explicitly the various boundary conditions as $r\to\infty$:

$\Phi_{0nt}(r)=$ outgoing waves ,

$\Phi_{rnt}(r)=$ outgoing waves ,

$\Phi_{nt}(r)=-[n(n+1)]^{-1}j_n'(kr)+$ outgoing waves , (4.18)

and

$\Phi_{\theta n}(r)=$ outgoing waves .

5. CASE OF *s* WAVE

We first consider the case of s-wave scattering, i.e., the case $n=0$ with reference to (3.9)–(3.12). This case is different from all other cases $n\geq 1$. More precisely, we are basically dealing with a differential equation of the second order for $n=0$, but of the fourth order for $n\geq 1$.

Let us set $n=0$ in (3.9), (3.10), and (3.12):

$$(r^{-2}Dr^2D-1)\Phi_{00}-[r^{-2}Dr^2(E-eV)-e(1+\eta)(dV/dr)]\Phi_{r0}=0\,, \tag{5.1}$$

$$-[(E-eV)D+e(1+\eta)(dV/dr)]\Phi_{00}+[(E-eV)^2-1]\Phi_{r0}=0\,, \tag{5.2}$$

and

$$[E-eV+e\eta(dV/dr)D+e(1+\eta)(r^{-2}Dr^2DV)]\Phi_{00}+[r^{-2}Dr^2-e\eta(dV/dr)(E-eV)]\Phi_{r0}=0\,, \tag{5.3}$$

where $D=d/dr$. If (5.2) is solved for Φ_{r0} and the result substituted into (5.1), we get a second-order differential equation for Φ_{00}:

$$\{r^{-2}Dr^2D+2e(dV/dr)[(E-eV)^2-1]^{-1}[(E-eV)D+e(1+\eta)(dV/dr)] + e(1+\eta)(E-eV)(r^{-2}Dr^2DV)-e^2\eta(1+\eta)(dV/dr)^2+(E-eV)^2-1\}\Phi_{00}=0\,. \quad (5.4)$$

When V is given by (2.6), (5.4) is interesting in several ways.

(A) The factor $r^{-2}Dr^2Dr$ contains a term proportional to $\delta(\vec{r})$. This δ function is absent in (5.4) if and only if $\eta=-1$.

(B) In case $\eta=0$, the presence of this δ function implies that

$$\Phi_{00}=O(r) \quad (5.5)$$

as $r\to 0$.

(C) Roughly speaking, (5.4) describes the scattering from a singular potential[13] of the form r^{-4}. This singular potential is repulsive when

$$\eta(1+\eta)>0\,, \quad (5.6)$$

i.e., either $\eta>0$ or $\eta<-1$. This is the origin of the condition on the anomalous magnetic moment.

It is thus clear that we must study (5.1)–(5.3) in the region of small r. By (2.6), if $r\gg g$, then eV is small. We therefore concentrate on the case

$$r=O(g)\,. \quad (5.7)$$

Let us define, for $g>0$,

$$R=r/g \quad (5.8)$$

and keep only the leading order in r or equivalently in g. Thus

$$eV\sim R^{-1}-\mu g\,. \quad (5.9)$$

Since the energy E appears only in the combination $E-eV$, define

$$E'=E+\mu g \quad (5.10)$$

so that

$$E-eV=E'-R^{-1}\,. \quad (5.11)$$

An inspection of (5.1), for example, shows that in this region under consideration Φ_{00} is smaller than Φ_{r0} by a factor of g. If we define

$$\Phi_{10}=\Phi_{00}/g\,, \quad (5.12)$$

then the leading terms of (5.1)–(5.3) are, respectively,

$$R^{-2}(d/dR)R^2(d/dR)\Phi_{10}-[R^{-2}(d/dR)R^2(E'-R^{-1})+(1+\eta)R^{-2}]\Phi_{r0}=0\,, \quad (5.13)$$

$$-[(E'-R^{-1})d/dR-(1+\eta)R^{-2}]\Phi_{10}+[(E'-R^{-1})^2-1]\Phi_{r0}=0\,, \quad (5.14)$$

and

$$[-\eta R^{-2}d/dR-4\pi(1+\eta)\delta(\vec{R})]\Phi_{10}+[R^{-2}(d/dR)R^2+\eta R^{-2}(E'-R^{-1})]\Phi_{r0}=0\,. \quad (5.15)$$

Although these equations (5.13)–(5.15) look complicated, they are in fact extremely simple, as we shall show now. If the δ function in (5.15) is neglected [a justified step if $\Phi_0(0)=0$], then

$$d\Phi_{10}/dR=[\eta^{-1}(d/dR)R^2+E'-R^{-1}]\,\Phi_{r0}\,. \quad (5.16)$$

If (5.16) is substituted into (5.13), we get

$$(d/dR)R^2(d/dR)R^2\Phi_{r0}-\eta(1+\eta)\Phi_{r0}=0\,. \quad (5.17)$$

It is remarkable that E' does not appear in (5.17). As we shall see in Sec. 7, this miracle occurs for all n.

The explicit solution of (5.13), (5.14), and (5.17) depends on the sign of $\eta(1+\eta)$. We list the various cases:

If $\eta=0$, then

$$\Phi_{r0}\sim C_{01}R^{-2} \tag{5.18}$$

and

$$\Phi_{10}\sim -[(E'^2-1)C_{01}+E'C_{02}]+C_{02}R^{-1}+C_{01}R^{-2}\,. \tag{5.19}$$

If $\eta=-1$, then

$$\Phi_{r0}\sim C_{03}(E'-R^{-1})R^{-2} \tag{5.20}$$

and

$$\Phi_{10}\sim C_{04}-C_{03}R^{-1}[(E'^2-1)-E'R^{-1}+\tfrac{1}{3}R^{-2}]\,. \tag{5.21}$$

If $\eta(1+\eta)>0$, then

$$\Phi_{r0}\sim C_{05}R^{-2}\exp\{-[\eta(1+\eta)]^{1/2}R^{-1}\}+C_{06}R^{-2}\exp\{[\eta(1+\eta)]^{1/2}R^{-1}\} \tag{5.22}$$

and

$$\Phi_{10}\sim C_{05}\{[\eta(1+\eta)]^{-1/2}(E'-R^{-1})+(1+\eta)^{-1}\}\exp\{-[\eta(1+\eta)]^{1/2}R^{-1}\} + C_{06}\{-[\eta(1+\eta)]^{-1/2}(E'-R^{-1})+(1+\eta)^{-1}\}\exp\{[\eta(1+\eta)]^{1/2}R^{-1}\}\,. \tag{5.23}$$

If $\eta(1+\eta)<0$, then

$$\Phi_{r0}\sim C_{07}R^{-2}\exp\{i[-\eta(1+\eta)]^{1/2}R^{-1}\}+C_{08}R^{-2}\exp\{-i[-\eta(1+\eta)]^{1/2}R^{-1}\} \tag{5.24}$$

and

$$\Phi_{10}\sim C_{07}\{i[-\eta(1+\eta)]^{-1/2}(E'-R^{-1})+(1+\eta)^{-1}\}\exp\{i[-\eta(1+\eta)]^{1/2}R^{-1}\} + C_{08}\{-i[-\eta(1+\eta)]^{-1/2}(E'-R^{-1})+(1+\eta)^{-1}\}\exp\{-i[-\eta(1+\eta)]^{1/2}R^{-1}\}\,. \tag{5.25}$$

In (5.18)–(5.25) C_{01}, C_{02}, ..., C_{08} are constants to be determined from the boundary conditions. Half of these constants are given by the normalization of the incident wave, while the rest are fixed by the boundary condition at $r=0$. In Sec. 6 we study this condition at the origin.

6. ENERGY DENSITY

In order to determine the constants in (5.18)–(5.25), a precise formulation of the boundary condition at the origin is needed. For this purpose, let us recall the analogous but more familiar situation in classical electromagnetic theory. In particular, consider the scattering of a classical electromagnetic wave by a perfectly conducting half-plane of zero thickness. If no condition is imposed on the behavior of the scattered electromagnetic field near the edge of the half-plane, Maxwell's equations can be satisfied with an infinite number of undetermined constants in the solution. In order to get a unique solution, it is simplest to use the Meixner condition[14] that the energy density is integrable over any bounded region.

This condition that the energy density is integrable needs some clarification. It is well known that the Hamiltonian density, commonly designated as H or T_{00}, is not unique. In particular, it can be changed by the addition of a divergence in the three space variables. On the other hand, the condition of being integrable can be changed by the addition of such a divergence. For example, consider the two expressions

$$\nabla f^*\cdot\nabla f \tag{6.1}$$

and

$$\nabla f^*\cdot\nabla f-2\nabla\cdot(f\,\nabla f)\,. \tag{6.2}$$

When $f=r^{-2/3}$, (6.2) is integrable over a neighborhood of the origin while (6.1) is not. Therefore, with reference to the condition that the energy density is integrable, the term "energy density" must refer to a specific form of H or T_{00}.

In classical electromagnetic theory, "energy density" refers unambiguously to

$$\mathcal{E}_{\rm em}=\tfrac{1}{2}(\vec{E}^2+\vec{B}^2)\,, \tag{6.3}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic field vectors. In particular, this form (6.3) is manifestly gauge-invariant and positive definite. The boundary condition for the above scattering problem by a perfectly conducting half-plane is that

$$\int \mathcal{E}_{\rm em}\, d\vec{r} < \infty \tag{6.4}$$

over any bounded region.

In this section we find an expression for the energy density of a charged vector-meson field similar to (6.3). Although the procedure is well known, we present it in some detail here because the result is of crucial importance. We begin with the Lagrangian density for the interacting charged vector-meson field and electromagnetic field[4]:

$$\mathfrak{L} = -\tfrac{1}{2}(\partial A_\mu/\partial x_\nu)(\partial A_\mu/\partial x_\nu) - \tfrac{1}{2} G^\star_{\mu\nu} G_{\mu\nu} - \phi^\star_\mu \phi_\mu - ie\kappa F_{\mu\nu}\phi^\star_\mu \phi_\nu\,, \tag{6.5}$$

where $G_{\mu\nu}$ and $F_{\mu\nu}$ are given by (2.2) and (2.3), respectively, and the superscript $\star$ denotes Hermitian conjugate times $(-1)^n$ with n = number of "4" subscripts.[4] The field equations are (2.1) and

$$\Box A_\nu = ie\kappa(\partial/\partial x_\mu)(\phi^\star_\nu \phi_\mu - \phi^\star_\mu \phi_\nu) + ie(\phi^\star_\mu G_{\mu\nu} + G^\star_{\mu\nu}\phi_\mu)\,, \tag{6.6}$$

where $\Box = \partial^2/\partial x_\mu \partial x_\mu$. In particular, from (6.6),

$$\Box A_0 = ie\kappa \nabla\cdot(\phi_0^*\vec{\phi} - \vec{\phi}^*\phi_0) - ie\{\vec{\phi}^*\cdot[(\nabla - ie\vec{A})\phi_0 + (\partial/\partial t + ieA_0)\vec{\phi}] - [(\nabla + ie\vec{A})\phi_0 + (\partial/\partial t - ieA_0)\vec{\phi}^*]\cdot\vec{\phi}\}\,. \tag{6.7}$$

The Lagrangian density (6.5) is more explicitly

$$\mathfrak{L} = \mathfrak{L}_{\rm em} + \mathfrak{L}_1\,, \tag{6.8}$$

where

$$\mathfrak{L}_{\rm em} = -\tfrac{1}{2}(\partial A_0/\partial t)^2 + \tfrac{1}{2}(\partial\cdot\vec{A}/\partial t)^2 + \tfrac{1}{2}(\nabla A_0)^2 - \tfrac{1}{2}(\nabla\vec{A})^2 \tag{6.9}$$

is the electromagnetic part of the Lagrangian density, while

$$\begin{aligned}\mathfrak{L}_1 = &-\tfrac{1}{2}[(\partial/\partial x_i + ieA_i)\phi_j^* - (\partial/\partial x_j + ieA_j)\phi_i^*][(\partial/\partial x_i - ieA_i)\phi_j - (\partial/\partial x_j - ieA_j)\phi_i]\\ &+[(\nabla + ie\vec{A})\phi_0^* + (\partial/\partial t - ieA_0)\vec{\phi}^*]\cdot[(\nabla - ie\vec{A})\phi_0 + (\partial/\partial t + ieA_0)\vec{\phi}]\\ &+\phi_0^*\phi_0 - \vec{\phi}^*\cdot\vec{\phi} + ie\kappa(\nabla A_0 + \partial\vec{A}/\partial t)\cdot(\vec{\phi}^*\phi_0 - \phi_0^*\vec{\phi}) - ie\kappa(\nabla\times\vec{A})\cdot(\vec{\phi}^*\times\vec{\phi})\,.\end{aligned} \tag{6.10}$$

Therefore, the usual Hamiltonian density is[4]

$$\begin{aligned}\mathcal{H} = &-(\partial A_0/\partial t)^2 + [\partial\vec{A}/\partial t + ie\kappa(\vec{\phi}^*\phi_0 - \phi_0^*\vec{\phi})]\cdot\partial\vec{A}/\partial t + [(\nabla + ie\vec{A})\phi_0^* + (\partial/\partial t - ieA_0)\vec{\phi}^*]\cdot\partial\vec{\phi}/\partial t\\ &+(\partial\vec{\phi}^*/\partial t)\cdot[(\nabla - ie\vec{A})\phi_0 + (\partial/\partial t + ieA_0)\vec{\phi}] - \mathfrak{L}\\ = &\,\mathcal{H}_{\rm em} + \mathcal{H}_1\,,\end{aligned} \tag{6.11}$$

where

$$\mathcal{H}_{\rm em} = -\tfrac{1}{2}(\partial A_0/\partial t)^2 + \tfrac{1}{2}(\partial\cdot\vec{A}/\partial t)^2 - \tfrac{1}{2}(\nabla A_0)^2 + \tfrac{1}{2}(\nabla\vec{A})^2 \tag{6.12}$$

is the electromagnetic part of the Hamiltonian density, while

$$\begin{aligned}\mathcal{H}_1 = &[(\nabla + ie\vec{A})\phi_0^* + (\partial/\partial t - ieA_0)\vec{\phi}^*]\cdot\partial\vec{\phi}/\partial t + (\partial\vec{\phi}^*/\partial t)\cdot[(\nabla - ie\vec{A})\phi_0 + (\partial/\partial t + ieA_0)\vec{\phi}]\\ &+ ie\kappa(\partial\vec{A}/\partial t)\cdot(\vec{\phi}^*\phi_0 - \phi_0^*\vec{\phi}) - \mathfrak{L}_1\,.\end{aligned} \tag{6.13}$$

The important point is the following. If A_μ is considered to be a fixed external field, then for the Hamiltonian density we get just $\mathcal{H}_1$ of (6.13) but not the $\mathcal{H}_{\rm em}$ part. But, since $\partial/\partial t$ appears by itself, $\mathcal{H}_1$ is clearly not gauge-invariant. In fact, the nonsensical condition that $\mathcal{H}_1$ is integrable leads to most peculiar results.

In order to solve this problem of the lack of gauge invariance, we observe that the $\mathcal{H}_{\rm em}$ of (6.12) is not the same as the $\mathcal{E}_{\rm em}$ of (6.3). Moreover, the difference is not a three-divergence:

$$\mathcal{H}_{\rm em} - \mathcal{E}_{\rm em} = A_0\Box A_0 - \nabla\cdot\{A_0(\nabla A_0 + \vec{A}) + [(\nabla\cdot\vec{A}) - (\vec{A}\cdot\nabla)]\vec{A}\}\,. \tag{6.14}$$

In the presence of the charged vector-meson field, $\Box A_0$ is not zero but given by (6.7). Therefore, by (6.11) and (6.13),

$$\mathcal{H}=\mathcal{E}_{\rm em}+\mathcal{H}_2-\nabla\cdot\{A_0(\nabla A_0+\vec{A})+[(\nabla\cdot\vec{A})-(\vec{A}\cdot\nabla)]\vec{A}+ie\kappa A_0(\vec{\phi}^*\phi_0-\phi_0^*\vec{\phi})\}, \tag{6.15}$$

where[12]

$$\mathcal{H}_2=\tfrac{1}{2}[(\partial/\partial x_i+ieA_i)\phi_j^*-(\partial/\partial x_j+ieA_j)\phi_i^*][(\partial/\partial x_i-ieA_i)\phi_j-(\partial/\partial x_j-ieA_j)\phi_i]$$
$$+[(\partial/\partial t-ieA_0)\vec{\phi}^*]\cdot[(\partial/\partial t+ieA_0)\vec{\phi}]-[(\nabla+ie\vec{A})\phi_0^*]\cdot[(\nabla-ie\vec{A})\phi_0]-\phi_0^*\phi_0+\vec{\phi}^*\cdot\vec{\phi}+ie\kappa(\nabla\times\vec{A})\cdot(\vec{\phi}^*\times\vec{\phi}) \tag{6.16}$$

is gauge-invariant.

This density $\mathcal{H}_2$ is still not what we need because of its lack of similarity to the $\mathcal{E}_{\rm em}$ of (6.3) even when $e=0$. It is necessary to add another three-divergence because, by (2.1),

$$[(\partial/\partial t-ieA_0)\vec{\phi}^*]\cdot[(\partial/\partial t+ieA_0)\vec{\phi}]-[(\nabla+ie\vec{A})\phi_0^*]\cdot[(\nabla-ie\vec{A})\phi_0]$$
$$-[(\nabla+ie\vec{A})\phi_0^*+(\partial/\partial t-ieA_0)\vec{\phi}^*]\cdot[(\nabla-ie\vec{A})\phi_0+(\partial/\partial t+ieA_0)\vec{\phi}]$$
$$=-2\,\mathrm{Re}\nabla\cdot\{\phi_0^*[(\nabla-ie\vec{A})\phi_0+(\partial/\partial t+ieA_0)\vec{\phi}]\}+2\,\mathrm{Re}\phi_0^*[\phi_0-ie\kappa\vec{\phi}\cdot(\nabla A_0+\partial\vec{A}/\partial t)], \tag{6.17}$$

and hence

$$\mathcal{H}_2=\mathcal{E}_V-2\nabla\cdot\mathrm{Re}\{\phi_0^*[(\nabla-ie\vec{A})\phi_0+(\partial/\partial t+ieA_0)\vec{\phi}]\}, \tag{6.18}$$

where

$$\mathcal{E}_V=\tfrac{1}{2}[(\partial/\partial x_i+ieA_i)\phi_j^*-(\partial/\partial x_j+ieA_j)\phi_i^*][(\partial/\partial x_i-ieA_i)\phi_j-(\partial/\partial x_j-ieA_j)\phi_i]$$
$$+[(\nabla+ie\vec{A})\phi_0^*+(\partial/\partial t-ieA_0)\vec{\phi}^*]\cdot[(\nabla-ie\vec{A})\phi_0+(\partial/\partial t+ieA_0)\vec{\phi}]$$
$$+\phi_0^*\phi_0+\vec{\phi}^*\cdot\vec{\phi}+ie\kappa(\nabla A_0+\partial\vec{A}/\partial t)\cdot(\vec{\phi}^*\phi_0-\phi_0^*\vec{\phi})+ie\kappa(\nabla\times\vec{A})\cdot(\vec{\phi}^*\times\vec{\phi})\,. \tag{6.19}$$

From (6.15) and (6.18), the Hamiltonian density $\mathcal{H}$ is equal to $\mathcal{E}_{\rm em}+\mathcal{E}_V$ plus divergence terms. Field equations have been used in reaching this conclusion.

We shall use the $\mathcal{E}_V$ of (6.19) as the energy density for the charged vector-meson field, and hence the desired boundary condition is that

$$\int \mathcal{E}_V\,d\vec{r}<\infty \tag{6.20}$$

over any bounded region, analogous to (6.4) for classical electromagnetic fields. It is interesting to note the close relation between (6.10) and (6.19).

7. RESULT FROM s-WAVE SCATTERING

Some simple results can be obtained readily from (5.18)–(5.25). First, (5.19) is clearly inconsistent with (5.5). Therefore, we get[12]

$$\eta\neq 0\,. \tag{7.1}$$

Although this is a very weak result, it shows that we can indeed get nontrivial statements about the magnetic moment.

Secondly, even without knowing the precise nature of the boundary condition at the origin, we must have

$$C_{06}=0 \tag{7.2}$$

to avoid exponentially increasing field near $r=0$. Thus, for $\eta(1+\eta)>0$,

$$\Phi_{r0}\sim C_{05}R^{-2}\exp\{-[\eta(1+\eta)]^{1/2}R^{-1}\} \tag{7.3}$$

and

$$\Phi_{00}\sim gC_{05}\{[\eta(1+\eta)]^{-1/2}(E'-R^{-1})+(1+\eta)^{-1}\}\exp\{-[\eta(1+\eta)]^{1/2}R^{-1}\}. \tag{7.4}$$

In order to study the cases $-1\leq\eta<0$, the more precise formulation of the boundary condition at the origin, as given by (6.19) and (6.20), is needed. For the present case, $\vec{A}=0$ and $A_0=V$ which is a function of r only; therefore, (6.19) can be slightly simplified to

$$\mathcal{E}_V=(\nabla\times\vec{\phi}^*)\cdot(\nabla\times\vec{\phi})+[\nabla\phi_0^*+i(E-eV)\vec{\phi}^*]\cdot[\nabla\phi_0-i(E-eV)\vec{\phi}]+\phi_0^*\phi_0+\vec{\phi}^*\cdot\vec{\phi}+ie\kappa(dV/dr)(\phi_r^*\phi_0-\phi_0^*\phi_r)\,. \quad (7.5)$$

Since this energy density is quadratic in the ϕ's, the various partial waves contribute essentially independently. By (3.6), the part due to the s wave is

$$\mathcal{E}_0=[d\Phi_{00}^*/dr-(E-eV)\Phi_{r0}^*][d\Phi_{00}/dr-(E-eV)\Phi_{r0}]+\Phi_{00}^*\Phi_{00}+\Phi_{r0}^*\Phi_{r0}-e\kappa(dV/dr)(\Phi_{r0}^*\Phi_{00}+\Phi_{00}^*\Phi_{r0})\,. \quad (7.6)$$

For the region of small r analyzed in Sec. 5, the term $\Phi_{00}^*\Phi_{00}$ is negligible, and hence by (5.8) and (5.12)

$$\mathcal{E}_0\sim[d\Phi_{10}^*/dR-(E'-R^{-1})\Phi_{r0}^*][d\Phi_{10}^*/dR-(E'-R^{-1})\Phi_{r0}]+\Phi_{r0}^*\Phi_{r0}+(1+\eta)R^{-2}(\Phi_{10}^*\Phi_{r0}+\Phi_{r0}^*\Phi_{10})\,. \quad (7.7)$$

Consider first the case $\eta=-1$. The substitution of (5.20) and (5.21) into (7.7) gives

$$\mathcal{E}_0\sim|C_{03}|^2R^{-4}[(E'-R^{-1})^2+1]\,. \quad (7.8)$$

In order to satisfy the condition that

$$\int_0^R R^2dR\,\mathcal{E}_0<\infty\,, \quad (7.9)$$

we must have

$$C_{03}=0\,. \quad (7.10)$$

Thus, by (5.20) and (5.21), the solution for $\eta=-1$ is

$$\Phi_{r0}\sim 0 \quad (7.11)$$

and

$$\Phi_{00}\sim gC_{04}\,. \quad (7.12)$$

We next consider the case $\eta(1+\eta)<0$. The substitution of (5.24) and (5.25) into (7.7) gives

$$\mathcal{E}_0\sim(|C_{07}|^2+|C_{08}|^2)R^{-4}[(E'-R^{-1})^2-(1-2\eta)/\eta]+\text{oscillatory terms}\,. \quad (7.13)$$

Consequently, (7.9) cannot be satisfied unless $C_{07}=C_{08}=0$. In other words, no solution is possible for the s wave if $\eta(1+\eta)<0$.

The conclusion is therefore reached that, in order that a solution exists for the s wave, either

$$\eta>0 \quad\text{or}\quad \eta\leqslant-1\,. \quad (7.14)$$

The approximate solutions for small r are given by (7.3) and (7.4), or (7.11) and (7.12). This is the desired result from the problem of s-wave scattering.

8. HIGHER PARTIAL WAVES

In Secs. 5 and 7, we have analyzed in detail the case $n=0$, and found that the scattering problem has a solution only if the magnetic moment of the vector meson satisfies (7.14). In this section, we generalize the above considerations for $n=0$ to the cases $n\geqslant 1$. This generalization is by no means trivial.

We begin with (3.9)–(3.12). With (5.8) and modified (5.12) in the form

$$\Phi_{1n}=\Phi_{0n}/g\,, \quad (8.1)$$

the leading terms [in the same sense as (5.13)–(5.15)] of (3.9)–(3.12) are, respectively,

$$[R^{-2}(d/dR)R^2(d/dR)-n(n+1)R^{-2}]\Phi_{1n}-[R^{-2}(d/dR)R^2(E'-R^{-1})+(1+\eta)R^{-2}]\Phi_{rn}+n(n+1)R^{-1}(E'-R^{-1})\Phi_n=0\,, \quad (8.2)$$

$$-n(n+1)R^{-2}\Phi_{rn}+n(n+1)R^{-2}(d/dR)R\Phi_n=0\,, \quad (8.3)$$

$$-R^{-1}(d/dR)\Phi_{rn}+R^{-2}(d/dR)R^2(d/dR)\Phi_n=0\,, \quad (8.4)$$

and

$$[-\eta R^{-2}(d/dR)-4\pi(1+\eta)\delta(\vec{R})]\Phi_{1n}+[R^{-2}(d/dR)R^2+\eta R^{-2}(E'-R^{-1})]\Phi_{rn}-n(n+1)R^{-1}\Phi_n=0\,. \quad (8.5)$$

Note that (8.4) is a trivial consequence of (8.3), which is simply

$$\Phi_{rn} = (d/dR)R\Phi_n . \tag{8.6}$$

Because of this redundancy, (3.12) and hence (8.5) are essential.

Let

$$f = R\Phi_n . \tag{8.7}$$

We want to get a fourth-order differential equation for f by eliminating Φ_{1n}. For this purpose, we first substitute (8.5), with the δ function omitted, into (8.2) to get

$$\Phi_{1n} = \{[n(n+1)\eta]^{-1}(d/dR)R^2(d/dR)R^2(d/dR) - (1+\eta)[n(n+1)]^{-1}(d/dR) - \eta^{-1}(d/dR)R^2 + (E' - R^{-1})\}f . \tag{8.8}$$

When (8.8) is substituted back into (8.5), we get the desired equation

$$\{(d^2/dR^2)R^2(d/dR)R^2(d/dR) - n(n+1)[(d^2/dR^2)R^2 + (d/dR)R^2(d/dR) - \eta R^{-2}] - \eta(1+\eta)d^2/dR^2 + n^2(n+1)^2\}f = 0 . \tag{8.9}$$

Note again here the miracle that E' does not appear in (8.9).

It is a further miracle that this fourth-order differential equation (8.9) can be solved exactly in terms of Bessel functions. However, before dealing with the exact solution, we study first some of the simple properties of (8.9). This differential equation (8.9) has an irregular singular point at $R=0$. In the vicinity of this irregular singular point, the behaviors of the four solutions, when (7.14) is satisfied, are listed as follows:

$$\exp\{[\eta(1+\eta)]^{1/2}/R\}, \quad \exp\{-[\eta(1+\eta)]^{1/2}/R\}, \quad R^{(1+2\nu)/2}, \text{ and } R^{(1-2\nu)/2} \tag{8.10}$$

for $\eta > 0$ or $\eta < -1 - [4n(n+1)]^{-1}$, where

$$\nu = \tfrac{1}{2}[1 + 4n(n+1)(1+\eta)^{-1}]^{1/2} ; \tag{8.11}$$

$$\exp\{[\eta(1+\eta)]^{1/2}/R\}, \quad \exp\{-[\eta(1+\eta)]^{1/2}/R\}, \quad R^{(1+2i\nu')/2}, \quad \text{and} \quad R^{(1-2i\nu')/2} \tag{8.12}$$

for $-1 - [4n(n+1)]^{-1} < \eta < -1$, where

$$\nu' = \tfrac{1}{2}[-1 - 4n(n+1)(1+\eta)^{-1}]^{1/2} ; \tag{8.13}$$

$$\exp\{[\eta(1+\eta)]^{1/2}/R\}, \quad \exp\{-[\eta(1+\eta)]^{1/2}/R\}, \quad R^{1/2}, \quad \text{and} \quad R^{3/2} \tag{8.14}$$

for $\eta = -1 + [4n(n+1)]^{-1}$; and, as given by Tamm,[15]

$$\begin{aligned} &R^{-1/4}\exp\{2[n(n+1)]^{1/4}R^{-1/2}\}, \quad R^{-1/4}\exp\{-2[n(n+1)]^{1/4}R^{-1/2}\} , \\ &R^{-1/4}\exp\{2i[n(n+1)]^{1/4}R^{-1/2}\}, \quad \text{and} \quad R^{-1/4}\exp\{-2i[n(n+1)]^{1/4}R^{-1/2}\} \end{aligned} \tag{8.15}$$

for $\eta = -1$.

Once again we face the problem of applying the boundary condition at the origin. We consider first the case $\eta = -1$ and follow the procedure of Sec. 7. Since

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 2\delta_{mn}/(2n+1)$$

and (8.16)

$$\int_{-1}^{1} P_m^1(x)P_n^1(x)dx = 2n(n+1)\delta_{mn}/(2n+1) ,$$

we get from (3.6) and (7.5) that

$$\begin{aligned} \mathcal{E}_n = (2n+1)\{&r^{-2}n(n+1)[d(r\Phi_n^*)/dr - \Phi_{rn}^*][d(r\Phi_n)/dr - \Phi_{rn}] + [d\Phi_{0n}^*/dr - (E - eV)\Phi_{rn}^*][d\Phi_{0n}/dr - (E - eV)\Phi_{rn}] \\ &+ n(n+1)[r^{-1}\Phi_{0n}^* - (E - eV)\Phi_n^*][r^{-1}\Phi_{0n} - (E - eV)\Phi_n] + \Phi_{0n}^*\Phi_{0n} + \Phi_{rn}^*\Phi_{rn} + n(n+1)\Phi_n^*\Phi_n \\ &- e\kappa(dV/dr)(\Phi_{0n}^*\Phi_{rn} + \Phi_{rn}^*\Phi_{0n})\} . \end{aligned} \tag{8.17}$$

Note that (7.6) is a special case of (8.17). We want to keep only terms of order g^0 in (8.17); thus for example the term

$$\Phi_{0n}^*\Phi_{0n} = g^2\Phi_{1n}^*\Phi_{1n}$$

can be neglected. By (3.10), we get

$$(d/dr)(r\Phi_n) - \Phi_{rn} = r^2[n(n+1)]^{-1}\{(E-eV)d\Phi_{0n}/dr - [(E-eV)^2-1]\Phi_{rn}\}$$
$$\sim g^2R^2[n(n+1)]^{-1}\{(E-eV)d\Phi_{1n}/dR - [(E-eV)^2-1]\Phi_{rn}\} \tag{8.18}$$

and hence the term

$$r^{-2}n(n+1)[(d/dr)(r\Phi_n^*) - \Phi_{rn}^*][(d/dr)(r\Phi_n) - \Phi_{rn}]$$

is also negligible. Therefore

$$\mathcal{E}_n \sim (2n+1)\{[d\Phi_{1n}^*/dR - (E'-R^{-1})\Phi_{rn}^*][d\Phi_{1n}/dR - (E'-R^{-1})\Phi_{rn}]$$
$$+n(n+1)[R^{-1}\Phi_{1n}^* - (E'-R^{-1})\Phi_n^*][R^{-1}\Phi_{1n} - (E'-R^{-1})\Phi_n]$$
$$+\Phi_{rn}^*\Phi_{rn} + n(n+1)\Phi_n^*\Phi_n + (1+\eta)R^{-2}(\Phi_{1n}^*\Phi_{rn} + \Phi_{rn}^*\Phi_{1n})\}\ . \tag{8.19}$$

When $\eta=-1$, the last term of (8.19) vanishes, and $\mathcal{E}_n$ is explicitly a sum of non-negative terms. Suppose now f is

$$R^{-1/4}\exp\{\pm 2i[n(n+1)]^{1/4}R^{-1/2}\}\ . \tag{8.20}$$

Then by (8.8)

$$R^{-1}\Phi_{1n} - (E'-R^{-1})\Phi_n \sim -[n(n+1)]^{-1}R^3d^3f/dR^3 = O(R^{-7/4}) \tag{8.21}$$

which is not square integrable in the neighborhood of $R=0$. Therefore, of the four forms of (8.15), only the second one is admissible. Since two additional boundary conditions, specifying the incident waves, need to be satisfied for $r\to\infty$, we can get no consistent solution for this case of $\eta=-1$.[15] In other words, from $n\geq 1$ we get

$$\eta \neq -1 \quad \text{or} \quad \kappa\neq 0\ . \tag{8.22}$$

Attention is next directed to (8.10). If Φ_{1n} is not highly oscillatory, then by (8.5) it must be finite as $R\to 0$. A comparison of (8.8) and (8.10) thus shows that the third form is admissible only if

$$\nu \geq \tfrac{1}{2}\ . \tag{8.23}$$

Consequently, by (8.11), the boundary conditions can be satisfied only if either

$$\eta>0 \quad \text{or} \quad -1\leq\eta\leq -1-[4n(n+1)]^{-1}\ . \tag{8.24}$$

Since (8.24) must hold for all n, it is equivalent to

$$\eta>0 \quad \text{or} \quad \eta=-1\ . \tag{8.25}$$

We combine (8.22) and (8.25) to get our final conclusion,

$$\eta>0\ , \tag{8.26}$$

which is just (1.1) in terms of κ.

In Appendixes A–F, we solve (8.9) and study the solutions.

9. BOUNDARY CONDITIONS FOR TRANSVERSE POLARIZATION

The above result (8.26) has been obtained by considering only the case where the incident plane wave is longitudinally polarized. In this section we show that the more complicated case of transverse polarization gives no further restriction.

When (8.26) is satisfied, the second and third forms of (8.10) are admissible. Let us consider first the third form. In this case, we get by (8.7), (8.6), and (8.8), respectively, that, as $R\to 0$,

$$\Phi_n = O(R^{\nu-1/2})\ , \tag{9.1}$$

$$\Phi_{rn} = O(R^{\nu-1/2})\ , \tag{9.2}$$

and

$$\Phi_{1n} = O(R^{\nu-1/2})\ . \tag{9.3}$$

In Sec. 4, we have found that, for $n\geq 1$, $\Phi_{0nt}(r)$, $\Phi_{rnt}(r)$, and $\Phi_{nt}(r)$ for the case of transverse polarization satisfy exactly the same ordinary differential equations as $\Phi_{0n}(r)$, $\Phi_{rn}(r)$, and $\Phi_n(r)$ for longitudinal polarization. Therefore, by (5.8), (8.1), and (9.1)–(9.3) we get

$$\Phi_{nt}(r) = O(r^{\nu-1/2}), \tag{9.4}$$

$$\Phi_{rnt}(r) = O(r^{\nu-1/2}), \tag{9.5}$$

and

$$\Phi_{0nt}(r) = O(r^{\nu-1/2}) \tag{9.6}$$

as $r \to 0$. Therefore we get from (4.8), (9.5), and (9.6) that, as $r \to 0$,

$$\phi_0 = O(r^{\nu_0-1/2})$$

and (9.7)

$$\phi_r = O(r^{\nu_0-1/2}),$$

where

$$\nu_0 = \min\nu = \frac{1}{2}\left(1 + \frac{8}{\kappa}\right)^{1/2}. \tag{9.8}$$

In particular, (9.7) implies that the last term

$$ie\kappa(dV/dr)(\phi_r^*\phi_0 - \phi_0^*\phi_r)$$

in the energy density $\mathcal{E}_V$ of (7.5) is of the order of $r^{2\nu_0-3}$ and hence integrable. The remaining four terms are each non-negative and hence must be separately integrable.

Attention is now turned toward the differential equation (4.17) for $\Phi_{\theta n}(r)$. This differential equation has no analog in the simpler case of longitudinal polarization. For small r, (4.17) is approximately

$$\begin{aligned}[R^{-2}(d/dR)R^2(d/dR) - n(n+1)R^{-2}]\Phi_{\theta n} \\ = -(n-1)^2[R^{-1}(d/dR)\Phi_{r(n-1)t} - n(n+1)R^{-2}\Phi_{(n-1)t}] \\ - (n+2)^2[R^{-1}(d/dR)\Phi_{r(n+1)t} - n(n+1)R^{-2}\Phi_{(n+1)t}].\end{aligned} \tag{9.9}$$

By (9.4) and (9.5), a particular solution of (9.9) is of the order of $r^{\nu-1/2}$, while the two solutions of the corresponding homogeneous equation are of the orders of r^n and r^{-n-1} (modified by e^2). Of these two, the one with the behavior of r^{-n-1} is not acceptable because, for example, the second term on the right-hand side of (7.5) is then nonintegrable. Thus the solution of (9.8) has the property

$$\Phi_{\theta n}(r) = O(r^{\nu-1/2}) \tag{9.10}$$

as $r \to 0$ with one constant to be determined by the condition of outgoing waves as $r \to \infty$.

The conclusion is therefore reached from (9.5), (9.10), and (4.8) that

$$\vec{\phi} = O(r^{\nu_0-1/2}) \tag{9.11}$$

as $r \to 0$. Together with (9.7), (9.11) shows that the energy density $\mathcal{E}_V$ of (7.5) is indeed integrable in this case. Entirely similar considerations show that the same conclusion holds for the third form of (8.10). Thus (8.26) is sufficient for both the longitudinal polarization and the transverse polarization.

10. DISCUSSIONS

In this paper we have studied in detail the problem of the scattering of a charged vector particle *without electric dipole moment* by an external static Yukawa potential, or in particular a static Coulomb field. This problem is a natural example of a nonrenormalizable theory, and furthermore, for small momentum transfers, is closely related to the field-theoretic problem of scattering, through electromagnetic interactions, of a charged vector particle by a charged particle of spin 0 or $\frac{1}{2}$. Our main result from studying this problem of potential scattering is that the anomalous magnetic moment κ, instead of being an arbitrary coupling constant, must be larger than unity. We emphasize that *parity is conserved* in this problem of potential scattering.

In this way we have shown that, for nonrenormalizable theories, there are in general restrictions on the values of coupling constants. This result $\kappa > 1$ arises in a way which is by no means straightforward. In Secs. 5 and 7 where we deal with s-wave scattering, we find first that κ cannot be in the range from zero to unity. It is clear that this range is excluded to avoid singular attractive potentials that lead to infinite energy content in every neighborhood of the origin. At this stage the particular case $\kappa = 0$ remains admissible. It is the investigation of higher partial waves in Sec. 8 that the case $\kappa \leq 0$ is also excluded. The reason for this exclusion is much less clear and a detailed study of the behavior of the solution near the origin is necessary.

The most striking feature of our result is that κ cannot be too small (and in fact has to be larger than unity). In other words, *that the vector meson has one interaction* (the electric charge) *implies that it must have a sizable second interaction* (the anomalous magnetic moment). There is thus a similarity in theme to our previous argument, that on the basis of the ξ-limiting formalism of Lee and Yang,[4] the charged intermediate boson has an electric dipole moment and hence violates parity.[16] However, unlike that earlier work,[16] the present result on κ is rigorous within the context of potential scattering, no assumption being made about the existence of any limit such as $\xi \to 0$.

In the renormalizable case of $\pi\pi$ scattering, Martin and collaborators have obtained bounds on various physical quantities.[17] However, the problems and the results are fundamentally different in that case of $\pi\pi$ scattering and in the present case. First, while the $\pi\pi$ case is intrinsically nonlinear due to the role played by unitarity, the present vector-meson case is linear. The more important difference is the nature of the resulting restrictions: The $\pi\pi$ scattering amplitude cannot be too large but can be arbitrarily small; but, for given $e \neq 0$, the coupling constant $e\kappa$ cannot be too small. This may be a fundamental difference between renormalizable and nonrenormalizable theories.

Consider the behavior of the total field near the origin for $\kappa > 1$. It follows from (3.6), (7.3), (7.4), and (9.1)–(9.3) that (9.7) and (9.11) hold for both longitudinal and transverse polarizations. These should be compared with the incident plane waves (2.10) and (2.11), and we find in particular

$$|\vec{\phi}| \ll |\vec{\phi}^{\,\mathrm{inc}}| \; . \tag{10.1}$$

In other words, near the singularity of the external Yukawa potential, the fields are quenched by a factor

$$r^{\nu_0 - 1/2} \, , \tag{10.2}$$

where ν_0 is given by (9.8). For $\kappa \to 1$, this factor (10.2) approaches r. Such a factor is intimately related to the fact that the present scattering problem is entirely finite. In this connection, it will be extremely interesting to study the Lee model[18] with an additional nonrenormalizable interaction between the N and θ particles.

We conclude with two technical remarks. First, the coupling constant g is really

$$g = \pm e^2 \, . \tag{10.3}$$

We have considered in detail only the case of the plus sign. However, the fundamental system of equations (3.9)–(3.12) is invariant under

$$V \to -V \, , \tag{10.4}$$

$$E \to -E \, , \quad \text{and} \quad \Phi_{0n} \to -\Phi_{0n} \, . \tag{10.5}$$

Since the sign of E does not play any important role, there is no new feature for the case of the minus sign in (10.3). Thus, when $\kappa > 1$, there is no divergence for *both* signs of the external field.

Secondly, for the purpose of getting the result $\kappa > 1$, there is no need to introduce the scale g and hence the variable R through (5.8). Instead, it is sufficient to study directly the behavior of the partial waves near the origin $r = 0$. However, the fact that (8.9) can be solved in terms of Bessel functions as shown in Appendix A guarantees that it is possible to develop a systematic perturbation theory. In a separate paper, we shall use the solutions $\bar{f}_1$ and $\bar{f}_2$ of (A31) and (A26), respectively, to study the scattering when e is small and $\kappa > 1$.

11. SPECULATIONS

We *assume* that our results are valid not only for the limiting situation neglecting recoil effects and radiative corrections but also for both the field-theoretic case and the real world involving the scattering of charged intermediate bosons for transmitting weak interaction. We speculate briefly about its possible implications.

First, the intermediate boson, being charged, must have a sizeable anomalous magnetic moment κ. The presence of such a κ changes the production cross section by both neutrinos[19] and photons,[20] and it is in particular possible to measure once the particle is found experimentally. On the basis of entirely different arguments involving the electromagnetic form factor of the neutrino, Bernstein and Lee[6] have reached the conclusion that $\kappa = 0$ by assuming that the sum of a certain series has a finite limiting value which is different from zero. Since the present result contradicts this answer of Bernstein and Lee, it is extremely exciting to have an experimental measurement of this value κ.

We speculate further on the more theoretical side. As given by (10.2), the field at small distances may be quenched due to the presence of the nonrenormalizable interaction. Even though this quenching is only a power of r, it is independent of e and may be sufficient to remove many, if not all, of the ultraviolet divergences. Therefore, *with suitable restrictions on the coupling constants*, a good nonrenormalizable theory can be "better" than a renormalizable theory, and may even be finite.

For the present case of the potential scattering of a charged vector meson, the restriction, written in the form $e\kappa > e$, expresses a linear inequality between two coupling constants. In more complicated and realistic cases, there is presumably no reason why the relation should be linear. Let us consider one of many possibilities involving two coupling constants g_1 and g_2 as following: By studying one process, we get a restriction $g_1 \leq g_2$; by studying another process, we get a nonlinear relation $g_2 \leq f(g_1)$. Suppose that the result of plotting this function is of the form shown in Fig. 1, then we reach the conclusion that either $g_1 = g_2 = 0$ or (g_1, g_2) must be in the cross-hatched area. In other words, either it is a free field or the coupling constants are not small. In particular, if it is not a free field, neither g_1 nor g_2 can be zero in this case. It is most intriguing to think about the questions: Why is there such a bewildering num-

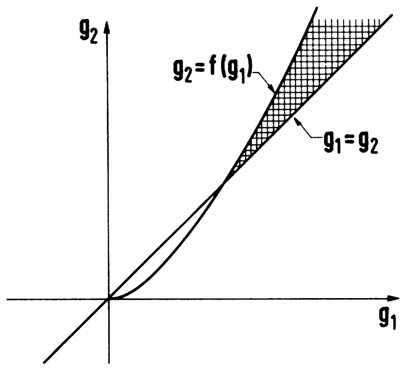

FIG. 1. A possible situation with two coupling constants.

ber of different couplings in nature? Why is strong interaction strong?

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APPENDIX A

In this appendix we solve (8.9) exactly. Let

$$f = R^{1/2}\tilde{f}, \tag{A1}$$

then

$$R^{-1/2}(d^2/dR^2)R^2(d/dR)R^2(d/dR)f = T_0R^{3/2}(d/dR)R^2(d/dR)R^{1/2}\tilde{f}$$
$$= \tfrac{1}{2}(T_0R^4T_0 + T_0R^2T_0R^2 - 2T_0R^2)\tilde{f}\,, \tag{A2}$$

where

$$T_0 = R^{-1/2}(d^2/dR^2)R^{1/2}$$
$$= d^2/dR^2 + R^{-1}d/dR - \tfrac{1}{4}R^{-2}\,. \tag{A3}$$

The fourth-order ordinary differential equation (8.9) is therefore

$$[\tfrac{1}{2}T_0R^4T_0 + \tfrac{1}{2}T_0R^2T_0R^2 - T_0R^2 - \tfrac{1}{2}n(n+1)(3T_0R^2 + R^2T_0 - 2 - 2\eta R^{-2}) - \eta(1+\eta)T_0 + n^2(n+1)^2]\tilde{f} = 0\ . \tag{A4}$$

It is therefore clear that we ought to define the differential operator

$$T_1 = T_0 - n(n+1)(\eta+1)^{-1}R^{-2}$$
$$= d^2/dR^2 + R^{-1}d/dR - \nu^2R^{-2}, \tag{A5}$$

where (8.11) has been used. In terms of T_1, (A4) reduces to

$$[\tfrac{1}{2}T_1R^4T_1 + \tfrac{1}{2}T_1R^2T_1R^2 - T_1R^2 - \tfrac{1}{2}n(n+1)\eta(1+\eta)^{-1}(3T_1R^2 + R^2T_1 - 2) - \eta(1+\eta)T_1 + n^2(n+1)^2\eta^2(1+\eta)^{-2}]\tilde{f} = 0. \tag{A6}$$

Since the differential operator in (A6) is a polynomial in R^2 and T_1, it is natural to apply the Fourier-Bessel transformation

$$\tilde{F}(\zeta) = \int_0^\infty R\,dR\,\tilde{f}(R)J_\nu(\zeta R) \tag{A7}$$

and the inversion formula

$$\tilde{f}(R) = \int_0^\infty \zeta\,d\zeta\,\tilde{F}(\zeta)J_\nu(\zeta R). \tag{A8}$$

We have made the assumption that the Fourier-Bessel transform of $\tilde{f}(R)$ exists, but this temporary assumption will later be removed by a standard procedure in treating differential equations. If we use $\Longleftrightarrow$ to denote a pair of Fourier-Bessel transformations, then

$$\tilde{f} \Longleftrightarrow \tilde{F} \tag{A9}$$

and furthermore by the Bessel differential equation

$$T_1\tilde{f} \Leftrightarrow -\zeta^2\tilde{F} \tag{A10}$$

and

$$R^2\tilde{F} \Leftrightarrow -T\tilde{F}\,, \tag{A11}$$

where T is the same as T_1 except for a change of variable from R to ζ

$$\begin{aligned} T &= \zeta^{-1/2}(d^2/d\zeta^2)\zeta^{1/2} - n(n+1)(1+\eta)^{-1}\zeta^{-2} \\ &= d^2/d\zeta^2 + \zeta^{-1}d/d\zeta - \nu^2\zeta^{-2}. \end{aligned} \tag{A12}$$

By (A10) and (A11), the equation for $\tilde{F}$ is

$$[\tfrac{1}{2}\zeta^2 T^2\zeta^2 + \tfrac{1}{2}\zeta^2 T\zeta^2 T - \zeta^2 T - \tfrac{1}{2}n(n+1)\eta(1+\eta)^{-1}(3\zeta^2 T + T\zeta^2 - 2) + \eta(1+\eta)\zeta^2 + n^2(n+1)^2\eta^2(1+\eta)^{-2}]\,\tilde{F} = 0. \tag{A13}$$

When (A12) is substituted into (A13), we get

$$\begin{aligned} \{\tfrac{1}{2}\zeta^{3/2}(d^4/d\zeta^4)\zeta^{5/2} + \tfrac{1}{2}\zeta^{3/2}(d^2/d\zeta^2)\zeta^2(d^2/d\zeta^2)\zeta^{1/2} - \zeta^{3/2}(d^2/d\zeta^2)\zeta^{1/2} \\ - \tfrac{1}{2}n(n+1)[3\zeta^{3/2}(d^2/d\zeta^2)\zeta^{1/2} + \zeta^{-1/2}(d^2/d\zeta^2)\zeta^{5/2} - 2] + \eta(1+\eta)\zeta^2 + n^2(n+1)^2\}\tilde{F} = 0. \end{aligned} \tag{A14}$$

In (A14), η appears only in the next-to-last term, and all the dependence on η can be removed by changing the scale of ζ.

Let

$$\tilde{F} = \zeta^{-1}\tilde{F}_1 \tag{A15}$$

and

$$\zeta = \tfrac{1}{4}[\eta(1+\eta)]^{-1/2}\zeta_1^{\,2}\,, \tag{A16}$$

then a straightforward but somewhat tedious calculation from (A14) gives that

$$\{[d^2/d\zeta_1^{\,2} + \zeta_1^{-1}d/d\zeta_1 - (2n+1)^2\zeta_1^{-2}]^2 + 1\}\,\tilde{F}_1 = 0. \tag{A17}$$

Therefore two possible solutions are given by

$$\tilde{F}_1 = J_{2n+1}(\zeta_1 e^{\pm i\pi/4}) \tag{A18}$$

or

$$\tilde{F} = \zeta^{-1}J_{2n+1}(2\eta^{1/4}(1+\eta)^{1/4}\zeta^{1/2}e^{\pm i\pi/4}). \tag{A19}$$

The substitution into (A8) then gives

$$\tilde{f}(R) = \int_0^\infty d\zeta J_\nu(\zeta R)\,J_{2n+1}(2\eta^{1/4}(1+\eta)^{1/4}\zeta^{1/2}e^{\pm i\pi/4}). \tag{A20}$$

The right-hand side of (A20) exists in the sense of Abel summability.

Let

$$\zeta R = \mp i\tfrac{1}{2}\xi^2\,, \tag{A21}$$

then the right-hand side of (A20) can be rewritten in the form

$$\tilde{f}(R) = \mp i e^{\mp i\nu\pi/2}R^{-1}\int_0^{\infty e^{\pm i\pi/4}} \xi d\xi\, I_\nu(\tfrac{1}{2}\xi^2)\,J_{2n+1}(x\xi), \tag{A22}$$

where

$$x = 2^{1/2}[\eta(1+\eta)]^{1/4}R^{-1/2}\,. \tag{A23}$$

In particular, one possible linear combination from (A22) is

$$R^{-1}\int_C \xi d\xi\, J_{2n+1}(x\xi)\,I_\nu(\tfrac{1}{2}\xi^2), \tag{A24}$$

where C is the contour shown in Fig. 2.

We are now ready to remove the temporary assumption that the Fourier-Bessel transform of $\tilde{f}(R)$ exists. The expression (A24) has the property that it is zero if the I_ν is replaced by K_ν. With reference to (A17), the same differential

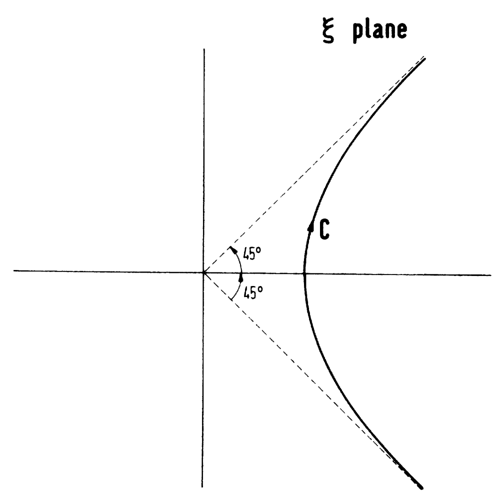

FIG. 2. The contour C of integration.

equation is still satisfied if the I_{2n+1} is replaced by Y_{2n+1}, I_{2n+1}, or K_{2n+1}. Therefore, the most general solution of (8.9) is given by

$$f(R)=R^{-1/2}\bar{f}(R), \tag{A25}$$

where $\bar{f}(R)$ is a linear combination of the following four functions:

$$\bar{f}_2=-\tfrac{1}{2}\pi i\int_C \xi d\xi K_{2n+1}(x\xi)I_\nu(\tfrac{1}{2}\xi^2), \tag{A26}$$

$$\bar{f}_3=-\tfrac{1}{2}\pi i\int_C \xi d\xi Y_{2n+1}(x\xi)I_\nu(\tfrac{1}{2}\xi^2), \tag{A27}$$

$$\bar{f}_4=-\tfrac{1}{2}\pi i\sec(\nu\pi)\int_C \xi d\xi J_{2n+1}(x\xi)I_\nu(\tfrac{1}{2}\xi^2), \tag{A28}$$

and

$$\bar{f}_5=-\tfrac{1}{2}\pi i\sec(\nu\pi)\int_C \xi d\xi I_{2n+1}(x\xi)I_\nu(\tfrac{1}{2}\xi^2). \tag{A29}$$

For $\eta>0$, these four expressions are all real, and are linearly independent.

With reference to (8.10), the admissible solutions are those $f(R)$ that behave like a linear combination of

$$\exp(-\tfrac{1}{2}x^2) \text{ and } x^{-1-2\nu} \tag{A30}$$

as $x\to\infty$. Of the above four $\bar{f}$'s only $\bar{f}_2$ satisfies (A30), but $\bar{f}_3$, $\bar{f}_4$, and $\bar{f}_5$ by themselves do not. There is, however, a linear combination of $\bar{f}_3$, $\bar{f}_4$, and $\bar{f}_5$ that does satisfy (A30), namely,

$$\bar{f}_1=\tfrac{1}{2}\pi\lim_{\epsilon\to 0+}\int_\epsilon^{\infty e^{i\pi/4}}\xi d\xi H^{(1)}_{2n+1}(x\xi)I_\nu(\tfrac{1}{2}\xi^2)+\int_\epsilon^{\infty e^{-i\pi/4}}\xi d\xi H^{(2)}_{2n+1}(x\xi)I_\nu(\tfrac{1}{2}\xi^2), \tag{A31}$$

where the contours of integration are taken to be in the right half-plane. For $\eta>0$, the desired $f(R)$ is given by (A25), where $\bar{f}(R)$ is a linear combination of $\bar{f}_1$ and $\bar{f}_2$.

Needless to say, the differential equation (8.9) is originally solved by a totally different method. The present procedure, devised after knowing the answer, is much more elegant. In Appendixes B–F we give some simple properties of the solution.

When $\kappa=0$ or 1, the solutions can be written directly in terms of Bessel functions and modified Struve's functions[21] (needed only for $\kappa=1$) without further integration. We shall not study further these particularly simple cases of Corben and Schwinger[12] because (1.1) is not satisfied.

APPENDIX B

We discuss here briefly the reason for introducing the factors $\sec(\nu\pi)$ in (A28) and (A29). Let $F_0(x\xi)$ be any function analytic in the right half-plane and bounded at the origin, then

$$\begin{aligned}\int_C \xi d\xi F_0(x\xi)I_\nu(\tfrac{1}{2}\xi^2)&=\left(\int_{\infty e^{-i\pi/4}}^0+\int_0^{\infty e^{i\pi/4}}\right)\xi d\xi F_0(x\xi)I_\nu(\tfrac{1}{2}\xi^2)\\&=\tfrac{1}{2}e^{-i\nu\pi/2}\int_{\infty e^{-i\pi/4}}^0\xi d\xi F_0(x\xi)[H_\nu^{(1)}(\tfrac{1}{2}\xi^2e^{i\pi/2})+H_\nu^{(2)}(\tfrac{1}{2}\xi^2e^{i\pi/2})]\\&\quad+\tfrac{1}{2}e^{i\nu\pi/2}\int_0^{\infty e^{i\pi/4}}\xi d\xi F_0(x\xi)[H_\nu^{(1)}(\tfrac{1}{2}\xi^2e^{-i\pi/2})+H_\nu^{(2)}(\tfrac{1}{2}\xi^2e^{-i\pi/2})]\\&=\pi^{-1}i\left[\cos(\nu\pi)\int_0^\infty\xi d\xi F_0(x\xi)K_\nu(\tfrac{1}{2}\xi^2)+\int_0^\infty\xi d\xi F_0(-ix\xi)K_\nu(\tfrac{1}{2}\xi^2)+\int_0^\infty\xi d\xi F_0(ix\xi)K_\nu(\tfrac{1}{2}\xi^2)\right].\end{aligned} \tag{B1}$$

For (A28) and (A29), let F_0 be, respectively, J_{2n+1} and I_{2n+1}. For these two special cases, the last two terms on the right-hand side of (B1) cancel each other. Therefore an over-all factor of $\cos(\nu\pi)$ appears. We introduce the factor $\sec(\nu\pi)$ specifically to cancel this $\cos(\nu\pi)$ so that $\bar{f}_4$ and $\bar{f}_5$ are not identically zero when 2ν is an odd integer. Otherwise we fail to get four linearly independent solutions in this special case.

By (B1) we have the following alternative expressions for $\bar{f}_4$ and $\bar{f}_5$:

$$\bar{f}_4=\tfrac{1}{2}\int_0^\infty\xi d\xi J_{2n+1}(x\xi)K_\nu(\tfrac{1}{2}\xi^2) \tag{B2}$$

and

$$\bar{f}_5=\tfrac{1}{2}\int_0^\infty\xi d\xi I_{2n+1}(x\xi)K_\nu(\tfrac{1}{2}\xi^2). \tag{B3}$$

APPENDIX C

In this appendix we discuss the relation between $\bar{f}_1$, $\bar{f}_3$, $\bar{f}_4$, and $\bar{f}_5$ as given by (A31), (A27), (A28), and (A29), respectively. First we note that for $\bar{f}_4$ and $\bar{f}_5$ the contour C of integration can be deformed to the two 45° lines, namely, from $\infty e^{-i\pi/4}$ to 0 and then to $\infty e^{i\pi/4}$. From (A29), let

$$\xi = \pm i\xi' . \tag{C1}$$

Then

$$\bar{f}_5 = -\tfrac{1}{2}\pi i \sec(\nu\pi)\left(\int_0^{\infty e^{i\pi/4}} - \int_0^{\infty e^{-i\pi/4}}\right)\xi d\xi\, I_{2n+1}(x\xi) I_\nu(\tfrac{1}{2}\xi^2)$$

$$= -\tfrac{1}{2}\pi i \sec(\nu\pi) i(-1)^n\left(-e^{i\pi\nu}\int_0^{\infty e^{-i\pi/4}} \xi' d\xi' J_{2n+1}(x\xi') I_\nu(\tfrac{1}{2}\xi'^2) - e^{-i\pi\nu}\int_0^{\infty e^{i\pi/4}} \xi' d\xi' J_{2n+1}(x\xi') I_\nu(\tfrac{1}{2}\xi'^2)\right). \tag{C2}$$

This expression (C2) can be combined with the $\bar{f}_4$ of (A28) to give

$$\sin(\nu\pi)\bar{f}_4 + (-1)^n \bar{f}_5 = -\tfrac{1}{2}\pi\left(\int_0^{\infty e^{i\pi/4}} + \int_0^{\infty e^{-i\pi/4}}\right)\xi d\xi\, J_{2n+1}(x\xi) I_\nu(\tfrac{1}{2}\xi^2), \tag{C3}$$

because $-i\cos(\nu\pi) = -ie^{i\pi\nu} + \sin(\nu\pi) = -ie^{-i\pi\nu} - \sin(\nu\pi)$. Since by (A31)

$$\bar{f}_1 = \tfrac{1}{2}\pi\left(\int_0^{\infty e^{i\pi/4}} + \int_0^{\infty e^{-i\pi/4}}\right)\xi d\xi\, J_{2n+1}(x\xi) I_\nu(\tfrac{1}{2}\xi^2) + \tfrac{1}{2}\pi i \int_C \xi d\xi\, Y_{2n+1}(x\xi) I_\nu(\tfrac{1}{2}\xi^2), \tag{C4}$$

we get immediately from (A27), (C3), and (C4) that

$$\bar{f}_1 = -\bar{f}_3 - \sin(\nu\pi)\bar{f}_4 - (-1)^n \bar{f}_5 . \tag{C5}$$

This is the desired relation.

APPENDIX D

We next discuss the behavior of $\bar{f}_2$, $\bar{f}_3$, $\bar{f}_4$, and $\bar{f}_5$ for small x, or large R. In this case, we can simply expand the Bessel functions of order $2n+1$. We therefore need to compute the following integral:

$$C(\tau) = \int_C \xi d\xi (\tfrac{1}{2}\xi)^\tau I_\nu(\tfrac{1}{2}\xi^2) . \tag{D1}$$

Assume temporarily that

$$0 > \tau > -2 - 2\nu . \tag{D2}$$

Then[22]

$$C(\tau) = \left(\int_{\infty e^{-i\pi/4}}^{0} + \int_0^{\infty e^{i\pi/4}}\right)\xi d\xi (\tfrac{1}{2}\xi)^\tau I_\nu(\tfrac{1}{2}\xi^2)$$

$$= \int_{-i\infty}^{i\infty} d\xi (\tfrac{1}{2}\xi)^{\tau/2} I_\nu(\xi)$$

$$= 2i\cos\tfrac{1}{2}\pi(\tfrac{1}{2}\tau + \nu)\int_0^\infty d\xi(\tfrac{1}{2}\xi)^{\tau/2} J_\nu(\xi)$$

$$= 2\pi i[\Gamma(\tfrac{1}{2} - \tfrac{1}{2}\nu - \tfrac{1}{4}\tau)\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu - \tfrac{1}{4}\tau)]^{-1} . \tag{D3}$$

By analytic contimation, (D3) also holds when (D2) is not satisfied. Therefore, by (A28) and (A29),

$$\bar{f}_4 = \pi^2 \sec(\nu\pi)\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+2n+1}}{m!\,\Gamma(m+2n+2)\Gamma(\tfrac{1}{4} - \tfrac{1}{2}m - \tfrac{1}{2}n - \tfrac{1}{2}\nu)\Gamma(\tfrac{1}{4} - \tfrac{1}{2}m - \tfrac{1}{2}n + \tfrac{1}{2}\nu)}$$

$$= \sum_{m=0}^{\infty}(-1)^m x^{2m+2n+1}\,\Gamma(\tfrac{3}{4} + \tfrac{1}{2}m + \tfrac{1}{2}n + \tfrac{1}{2}\nu)\Gamma(\tfrac{3}{4} + \tfrac{1}{2}m + \tfrac{1}{2}n - \tfrac{1}{2}\nu)/[m!\,\Gamma(m+2n+2)], \tag{D4}$$

and similarly

$$\bar{f}_5=\sum_{m=0}^{\infty} x^{2m+2n+1}\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}m+\tfrac{1}{2}n+\tfrac{1}{2}\nu)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}m+\tfrac{1}{2}n-\tfrac{1}{2}\nu)/[m!\,\Gamma(m+2n+2)]\,. \tag{D5}$$

For $\eta>0$, by (8.11) ν is always less than $n+\frac{1}{2}$. Therefore $\bar{f}_4$ and $\bar{f}_5$ are always linearly independent.

Similar formulas can be written down for $\bar{f}_2$ and $\bar{f}_3$; $\ln x$ together with derivative of Γ functions appear. The two leading terms are

$$\bar{f}_2\sim\tfrac{1}{2}\pi^2\left(\frac{(2n)!\,x^{-2n-1}}{\Gamma(\frac{3}{4}+\frac{1}{2}n-\frac{1}{2}\nu)\Gamma(\frac{3}{4}+\frac{1}{2}n+\frac{1}{2}\nu)}-\frac{(2n-1)!\,x^{-2n+1}}{\Gamma(\frac{1}{4}+\frac{1}{2}n-\frac{1}{2}\nu)\Gamma(\frac{1}{4}+\frac{1}{2}n+\frac{1}{2}\nu)}\right) \tag{D6}$$

and

$$\bar{f}_3\sim-\pi\left(\frac{(2n)!\,x^{-2n-1}}{\Gamma(\frac{3}{4}+\frac{1}{2}n-\frac{1}{2}\nu)\Gamma(\frac{3}{4}+\frac{1}{2}n+\frac{1}{2}\nu)}+\frac{(2n-1)!\,x^{-2n+1}}{\Gamma(\frac{1}{4}+\frac{1}{2}n-\frac{1}{2}\nu)\Gamma(\frac{1}{4}+\frac{1}{2}n+\frac{1}{2}\nu)}\right)\,. \tag{D7}$$

The expansions (D4)–(D7) also show that $\bar{f}_2$, $\bar{f}_3$, $\bar{f}_4$, and $\bar{f}_5$ are always linearly independent.

APPENDIX E

In this appendix, we give the asymptotic behaviors of $\bar{f}_1,\ldots,\bar{f}_5$ for large x or small R.

We begin with the $\bar{f}_1$ of (A31). When x is large, the important contribution to the integral comes from small ξ, and thus we can use

$$I_\nu(\tfrac{1}{2}\xi^2)\sim(\tfrac{1}{2}\xi)^{2\nu}/\Gamma(\nu+1)\,. \tag{E1}$$

Consequently the contours of integration can be shifted to the real axis[22]

$$\begin{aligned}\bar{f}_1&\sim\pi\int_0^\infty \xi d\xi J_{2n+1}(x\xi)(\tfrac{1}{2}\xi)^{2\nu}/\Gamma(\nu+1)\\&=2\pi[\Gamma(\nu+1)]^{-1}[\Gamma(n+\nu+\tfrac{3}{2})/\Gamma(n-\nu+\tfrac{1}{2})]x^{-2-2\nu}\,.\end{aligned} \tag{E2}$$

In particular this verifies that (A30) is satisfied by $\bar{f}_1$.

Consider next the $\bar{f}_2$ of (A26). On the positive real axis, this integrand has a minimum near $\xi=x$ for large x. Thus the asymptotic formulas for K_{2n+1} and I_ν may be used to yield

$$\begin{aligned}\bar{f}_2&\sim-\tfrac{1}{2}\pi i\int_C(2x\xi)^{-1/2}d\xi\,\exp(-x\xi+\tfrac{1}{2}\xi^2)\\&\sim\tfrac{1}{2}\pi^{3/2}x^{-1}\exp(-\tfrac{1}{2}x^2)\,.\end{aligned} \tag{E3}$$

Again (A30) is satisfied.

The asymptotic behavior of $\bar{f}_4$ is easily obtained from (B2) in the same way as $\bar{f}_1$:

$$\begin{aligned}\bar{f}_4&\sim\tfrac{1}{4}\Gamma(\nu)\int_0^\infty \xi d\xi J_{2n+1}(x\xi)(\tfrac{1}{2}\xi)^{-2\nu}\\&=\tfrac{1}{2}\Gamma(\nu)[\Gamma(n-\nu+\tfrac{3}{2})/\Gamma(n+\nu+\tfrac{1}{2})]x^{-2+2\nu}\,.\end{aligned} \tag{E4}$$

And the asymptotic behavior of $\bar{f}_5$ is given by (B3) as

$$\begin{aligned}\bar{f}_5&\sim\tfrac{1}{2}\int_0^\infty d\xi(2x\xi)^{-1/2}\exp(x\xi-\tfrac{1}{2}\xi^2)\\&\sim\tfrac{1}{2}\pi^{1/2}x^{-1}\exp(\tfrac{1}{2}x^2)\,.\end{aligned} \tag{E5}$$

Finally, by (C5), (E2), (E4), and (E5) the asymptotic behavior of $\bar{f}_3$ is

$$\bar{f}_3\sim(-1)^{n+1}\tfrac{1}{2}\pi^{1/2}x^{-1}\exp(\tfrac{1}{2}x^2)\,. \tag{E6}$$

APPENDIX F

In the formulas (A31) and (A26)–(A29) for $\bar{f}_1,\ldots,\bar{f}_5$ each integrand is a product of two Bessel functions. We can trade two Bessel functions for one confluent hypergeometric function. A number of the results in the preceding appendixes are first obtained through this representation by an integral of a confluent hypergeometric function, which we discuss very briefly here.

For definiteness, we treat $\bar{f}_4$ through the expression (B2). Since[23]

$$K_\nu(\tfrac{1}{2}\xi^2)=\int_0^\infty dt\cosh(\nu t)\exp(-\tfrac{1}{2}\xi^2\cosh t)\,, \qquad \text{(F1)}$$

we get[24]

$$\begin{aligned}\bar{f}_4&=\tfrac{1}{2}\int_0^\infty dt\cosh(\nu t)\int_0^\infty \xi d\xi J_{2n+1}(x\xi)\exp(-\tfrac{1}{2}\xi^2\cosh t)\\ &=\Gamma(n+\tfrac{3}{2})[(2n+1)!\,]^{-1}x^{2n+1}\int_0^\infty dt\cosh(\nu t)(2\cosh t)^{-n-3/2}\,\Phi(n+\tfrac{1}{2},2n+2;\tfrac{1}{2}x^2\operatorname{sech}t)\exp(-\tfrac{1}{2}x^2\operatorname{sech}t)\,, \qquad \text{(F2)}\end{aligned}$$

where Φ is the confluent hypergeometric function. One of many advantages of (F2) is that the argument of Φ is finite from 0 to $\frac{1}{2}x^2$.

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[22]See Eq. (31) on p. 52 of Ref. 21.

[23]See Eq. (21) on p. 82 of Ref. 21.

[24]See Eq. (22) on p. 50 of Ref. 21.