Pulses of Gravitational Radiation of a Particle Falling Radially into a Schwarzschild Black Hole*

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Using the Regge-Wheeler-Zerilli formalism of fully relativistic linear perturbations in the Schwarzschild metric, we analyze the radiation of a particle of mass m falling into a Schwarzschild black hole of mass $M \gg m$. The detailed shape of the energy pulse and of the tide-producing components of the Riemann tensor at large distances from the source are given, as well as the angular distribution of the radiation. Finally, analysis of the energy going down the hole indicates the existence of a divergence; implications of this divergence as a testing ground of the approximation used are examined.

In a recent series of investigations Zerilli,[1] Davis and Ruffini,[2] and Davis, Ruffini, Press, and Price[3] have analyzed the problem of a particle falling radially into a Schwarzschild black hole. In this paper this process is analyzed further. We are concerned with the features of the burst of the components of the Riemann tensor significant in the use of a detector and of the angular distribution of gravitational radiation. General and apparently contradictory considerations on the structure of a burst of gravitational radiation in black-hole physics were presented by Gibbons and Hawking[4] and by Press.[5] Some of the major features predicted in these two treatments are found indeed to be present in the detailed analysis of the physical example under consideration. An analysis for the radiation going into the hole is presented and its implications are examined.

We can expand[6] the perturbations $h_{\mu\nu} = g_{\mu\nu} - (g_{\mu\nu})_{\rm Schw}$ of a Schwarzschild background in spherical harmonics of multipole orders l and m. In our case (a particle falling radially in along the z axis starting at rest from infinity) the "magnetic" and the $m \neq 0$ "electric" terms identically vanish (see Zerilli[1]). We have in this case for large values of the radial distance, in Zerilli's radiation gauge,

$$h_{\mu\nu}(t, r, \theta, \phi) \sim p_{\mu\nu} \sum_l R_l(r, t)\left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta}\right) Y_{l0}(\theta, \phi)/2r \qquad (\mu, \nu = 0, 3). \quad (1)$$

Here $p_{\mu\nu}$ is the polarization tensor with the only nonvanishing components $p_{22} = r^2$ and $p_{33} = -r^2 \sin^2\theta$. Thus the outgoing radiation will be totally polarized with the principal axes in the θ and ϕ directions. The function $R_l(r, t)$ satisfies the Zerilli equation which in Fourier-transformed form gives

$$\frac{d^2 R_l(r, \omega)}{dr^{*2}} + [\omega^2 - V_l(r)]R_l(r, \omega) = S_l(r, \omega). \quad (2)$$

Here $r^* = r + 2M \ln(r/2M - 1)$, $V_l(r)$ is the effective curvature potential, and $S_l(r, \omega)$ is the Fourier-transformed electric source term generated by the incoming particle expressed in tensor harmonics.[2,3] Equation (2) has been numerically integrated with the asymptotic boundary conditions

$$R_l(r, \omega) = \begin{cases} A_l^{\rm out}(\omega)e^{i\omega r^*} & \text{as } r^* \to +\infty \\ A_l^{\rm in}(\omega)e^{-i\omega r^*} & \text{as } r^* \to -\infty, \end{cases} \quad (3)$$

where $A_l^{\rm out}$ is given in the Green's function technique by

$$A_l^{\rm out}(\omega) \propto \int_{-\infty}^{\infty} u_l(r^*, \omega) S_l(r^{*\prime}, \omega)\, dr^{*\prime}, \quad (4a)$$

$$A_l^{\rm in}(\omega) \propto \int_{-\infty}^{\infty} v_l(r^{*\prime}, \omega) S_l(r^{*\prime}, \omega)\, dr^{*\prime}. \quad (4b)$$

Here u_l (v_l) is the solution of the homogeneous equation obtained from (2) specifying a purely ingoing wave at $r^* = -\infty$ (outgoing at $r^* = +\infty$). By a further Fourier transformation we obtain the asymptotic expression

$$R_l^{\rm out}(r^*, t) = \int_{-\infty}^{\infty} A_l^{\rm out}(\omega) e^{i\omega(r^* - t)}\, d\omega. \quad (5)$$

The explicit results of this integration are given in Fig. 1(b) as a function of the retarded time $t - r^*$ for $l = 2$.

The asymptotic expression of the tide-producing components of the Riemann tensor, which is what is measured by gravitational-wave detectors,[7] is easily obtained in the radiation region from

$$R_{\alpha\beta\gamma\delta}=\tfrac{1}{2}(h_{\alpha\delta,\,\beta\gamma}+h_{\beta\gamma,\,\alpha\delta}-h_{\alpha\gamma,\,\beta\delta}-h_{\beta\delta,\,\alpha\gamma}), \tag{6}$$

where the comma (,) means ordinary derivative. If we assume the z axis is pointed along the line of propagation of the wave and the principal axes of polarization [see Eq. (1)] are pointed in the x and y directions, the only nonzero Newtonian tide-producing components of the Riemann tensor are, in our problem, $R^y{}_{0y0}=-R^x{}_{0x0}$, with

$$R^y{}_{0y0}(r^*,t)=\sum_l \ddot{R}_l(r^*,t)W_l(\theta,\phi)/2r\,. \tag{7}$$

Here the dot indicates normal derivative with respect to time. The components of the Riemann tensor for selected l, without their angular dependence factor

$$W_l(\theta)=\left(\frac{\partial^2}{\partial\theta^2}-\cot\theta\,\frac{\partial}{\partial\theta}\right)Y_{l0}(\theta), \tag{8}$$

and the factors $1/(8\pi)^{1/2}r$ are plotted in Fig. 1(c). Finally, we have also computed the outgoing energy flux from the stress-energy pseudotensor which becomes in the asymptotic region

$$t_{01}\sim\frac{1}{16\pi}\sum_{ll'}\dot{R}_l\dot{R}_{l'}W_l(\theta,\phi)W_{l'}(\theta,\phi)/4r^2\,. \tag{9}$$

In Fig. 1(d) we give the outgoing energy flux integrated over all directions for selected values of l. The interference between terms of different l is zero due to the orthogonality of the functions $W_l(\theta,\phi)$. As a check on our entire treatment we have verified that the total energy $\int_{-\infty}^{+\infty}(dE/dt)\,dt$ for every l agrees with the value $\int_{-\infty}^{\infty}(dE/d\omega)\,d\omega$ as given in Ref. 3 within 1%. From (7) we can compute the total flux per steradian; this quantity is plotted in Fig. 2. For pure quadrupole radiation the angular pattern of the radiation has a $\sin^4\theta$ dependence (z axis $\rightarrow\theta=0$). Inclusion of higher multi-

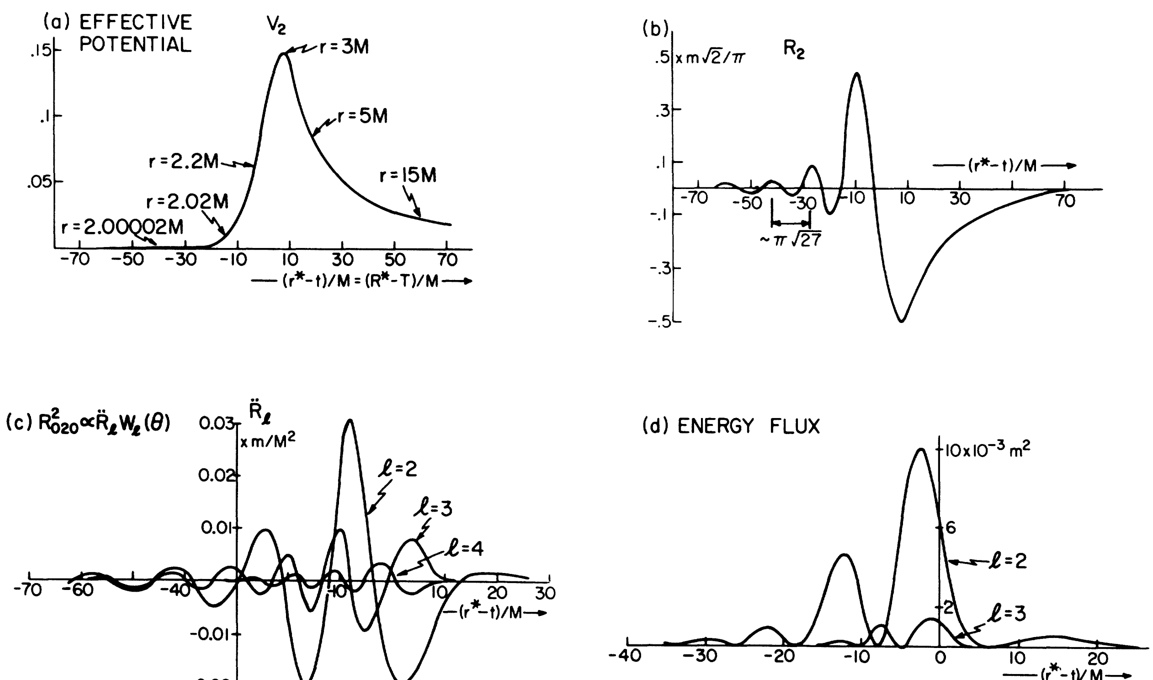

FIG. 1. Asymptotic behavior of the outgoing burst of gravitational radiation compared with the effective potential, as a function of the retarded time $(t-r^*)/M$. (a) Effective potential for $l=2$ in units of M^2 as a function of the retarded time $(t-r^*)/M=(T-R^*)/M$. For selected points the value of the Schwarzschild coordinate r is also given. (b) Radial dependence of the outgoing field $R_l(r,t)$ as a function of the retarded time for $l=2$. (c) $\ddot{R}_l(r^*,t)$ factors of the Riemann tensor components (see text) given as a function of the retarded time for $l=2,3,4$. (d) Energy flux integrated over angles for $l=2,3$; the contributions of higher l are negligible.

poles introduces interference terms which tip the peak of the pattern forward by $7\frac{1}{2}°$. Figure 2 shows that there is no beaming of the radiation.

From the comparison of the different diagrams in Fig. 1 we can distinguish and characterize three different regions in the total energy flux:

(i) $5 \lesssim (r^* - t)/M \lesssim 30$, a precursor,

(ii) $-10 \lesssim (r^* - t)/M \lesssim 5$, a sharp burst,

(iii) $(r^* - t)/M \lesssim -10$, a ringing tail.

The precursor corresponds to the first part of the pulse as produced in the Ruffini-Wheeler approximation.[8] The sharp burst has a width $\sim 10M$ in agreement with the predictions on qualitative ground by Gibbons and Hawking[4] referring to any emission process taking place during the formation of (or capture by) a collapsed object. However, the present results do not support their suggestion that the "number of zeros" of the Riemann tensor could discriminate between sources of different origin since the ringing tail produces many zero crossings of the Riemann tensor. Finally, the oscillating tail has characteristic frequencies $\omega \sim l/\sqrt{27}$ which correspond to the ringing modes of the black hole found by Press.[5] It is interesting, however, that these ringing modes are energetically significant in this physical example only for low values of l, as is clear from Fig. 1(d).

A deeper insight in the three regions (i), (ii), and (iii) can be gained by the study of the effective potential plotted in Fig. 1 as a function of the retarded time $(t - r^*)/M$ referred to the observer at large distances. This is equal to $(T - R^*)/M$ as "seen" by the ingoing particle as the outgoing wave sweeps past it. R^* and T are the particle's position and Schwarzschild coordinate time computed from the geodesic trajectory (with $T = -\infty$ at $R^* = +\infty$, and $T = +\infty$ at $R^* = -\infty$). The implication is that for radiation "directly" emitted outward from the particle and not reflected, one can specify the radial position of the particle when it supposedly "emits" this radiation. Notice, for example, that the peak of the radiation flux occurs near retarded time $(r^* - t)/M = -2$, when the particle itself is at $2.3M$ (very near the horizon indeed), and just inside the peak of the curvature potential, which peaks at $r = 3M$.

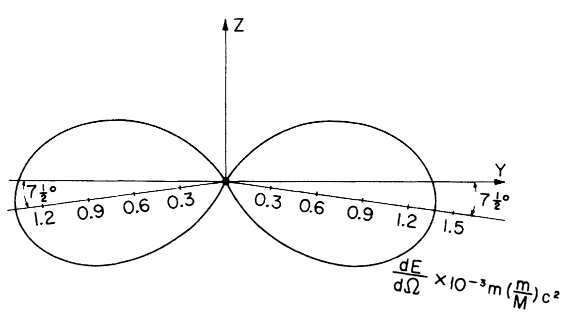

FIG. 2. Angular pattern of the radiation. The particle is supposed to fall down in the z direction which corresponds to $\theta = 0$.

We see that starting from large values of $(r^* - t)/M$ the field $R_l(r, t)$ builds up slowly and thus the energy emission (proportional to $\dot{R}_l^2$) and the Riemann tensor (proportional to $\ddot{R}_l$) gives rise to the very small "precursor" as the particle approaches the effective potential barrier. Concerning the emission of the main burst we have noticed in the evaluation of the integral (4a) for $A_l^{\rm out}(\omega)$ starting from $r = 2M$ that the main contribution came from the interval $2.1 \lesssim r/M \lesssim 10$. The contributions beyond this point were oscillating, and very slowly damped, since the source decreases only as $r^{-1/2}$ for asymptotic distances. This averaging out of the contributions for large values of r is expected from the linearized theory of gravitation. Note that the ringing comes out after the particle is already inside the barrier. Here we cannot be seeing direct radiation from the particle because the driving source S_l is exponentially decaying, and the contributions to $A_l^{\rm out}(\omega)$ for $2 \lesssim r/M \lesssim 2.1$ are very small. The wave emitted in this region for a given mode has a characteristic frequency as expected from the frequency spectrum calculated previously.[3] These facts suggest that part of the energy produced in the strong-burst region (ii) was stored in the "resonant cavity" of the geometry and then slowly released in the ringing modes.

We can now briefly summarize the main results of the analysis of the radiation going into the black hole. We have proceeded as follows: (1) Evaluate the amplitude A of the ingoing wave $R_l(r^*, \omega)$ for $r^* \to -\infty$ (purely ingoing waves), (2) solve for the scattering problem of Eq. (2) without source, imposing a purely ingoing wave of amplitude A at $r^* = -\infty$. As a consequence at $r^* = +\infty$ we have an ingoing wave with amplitude B and an outgoing wave with amplitude C. The energy flux going into the black hole is evaluated at $r^* = +\infty$, subtracting from the energy going in (proportional to $|B|^2$) the energy coming out (proportional to $|C|^2$). From the structure of the homogeneous Eq. (2) we also have $|A|^2 = |B|^2 - |C|^2$; therefore we can evaluate the energy going into the black hole simply by the same expression as used to calculate outgoing energy flux[1]:

$$\frac{dE}{d\omega} = \frac{1}{32\pi}\sum_l l(l-1)(l+1)(l+2)\omega^2 |A_l^{\rm in}|^2 . \qquad (10)$$

The results of this analysis are given in Fig. 3. The spectral distribution for every multipole is

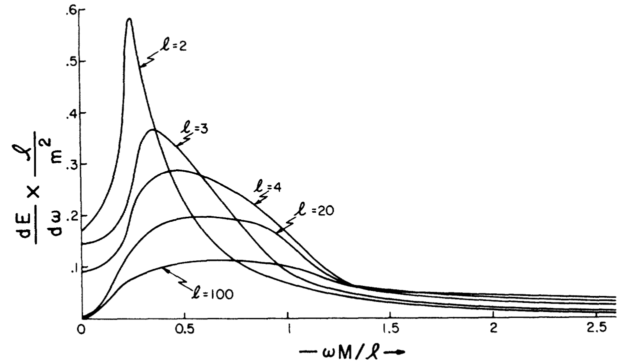

FIG. 3. Energy spectrum of the radiation going into the black hole for selected values of l. The total energy radiated per multipole is roughly constant and $\sim 0.25m^2/M$ for all considered l.

well behaved, and the total energy per multipole is roughly constant $\sim 0.25m^2/M$ (at least up to $l = 100$). This is in contrast to the case of outgoing radiation, where the higher multipoles were exponentially damped.[3] It is plausible to assume that this behavior is indeed valid even for larger l. At first it would seem that the energy summed over all l diverges. This divergence results from the fact that we are treating the incoming object as a point particle. Under this assumption the Regge-Wheeler condition that the perturbation introduced by the particle be small by comparison with the background metric is no longer fulfilled. However, this circumstance is automatically eliminated as soon as a minimum size is assumed for the particle. We take as minimum size $2m$. From the preceding results we have seen that most of the radiation is emitted in the region near the horizon. This and the assumption of the minimum size $2m$ for the particle implies a cutoff in l given by $l_{max} \simeq \frac{1}{2}\pi M/m$. The total energy going in is thus of the order

$$\frac{m^2}{4M}\frac{\pi M}{2m} \sim \frac{\pi m}{8}.$$

It is remarkable that the ingoing radiation does not depend, as does the outgoing radiation, on the ratio m/M, and that the amount of this radiation is a sizable fraction to the total rest mass of the ingoing particle, for an incident particle of minimum size.

Study of the numerical integration of $A_l^{in}(\omega)$ shows that for calculations of ingoing radiation, one can neglect contributions beyond $r = 5M$. Most of the ingoing radiation is generated inside the barrier, where $\omega^2 - V_l(r) < 0$, typically in the interval $2.01 \lesssim r/M \lesssim 3.5$. This large inward burst of energy is therefore generated in a finite Schwarzschild coordinate time interval; it occurs for $-20 < (r^* - t)/M < 15$ and vanishes as $(r^* - t)/M \to -\infty$. It occurs early enough that its reaction on the geodesic path of the incoming particle could affect the nature of the outgoing radiation, because the integral for $A_l^{out}(\omega)$ has significant contributions beginning as $r \gtrsim 2.1M$. Further analysis may give a deeper understanding of this process, and details on exactly how the reaction of ingoing radiation affects the outgoing burst.

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