expression in braces in $\mathfrak{L}$ can be written as $\frac{1}{2}H^{\mu\nu}\times F_{\mu\nu}$, and by (2.7) this is equal to $\frac{1}{2}F^{cb'ad'}F_{ab'cd'}$. In empty space (i.e., when all Φ's and Λ are assumed to be zero) this last expression can be seen, by Eq. (2.9), to be equal to $\frac{1}{2}F^{ab'cd'}F_{ab'cd'}$, or equal to $\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, thus giving the expression[13]

$$-\tfrac{1}{2}(-g)^{1/2}\,\mathrm{Tr}\{F^{\mu\nu}(-\tfrac{1}{2}F_{\mu\nu}+\partial_\nu B_\mu-\partial_\mu B_\nu+[B_\mu, B_\nu])\}+\text{c.c.}$$

for the Lagrangian density (3.2) in free space, which is the Lagrangian density used by Carmeli.[3]

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[9]The covariant derivative $\nabla_\mu \xi_A$ of the spinor ξ_A is $\nabla_\mu\xi_A=\partial_\mu\xi_A-\xi_B\Gamma^B{}_{A\mu}$, where $\Gamma^B{}_{A\mu}$ is the spinor affine connection. The corresponding quantity $\overline{\Gamma}^{B'}{}_{A'\mu}$ deals with the spinor $\xi_{A'}$. Throughout this paper there will be no need to know the explicit form of any affinities.

[10]Some authors denote these vectors as follows: $\sigma^\mu{}_{00'}=l^\mu$, $\sigma^\mu{}_{01'}=m^\mu$, $\sigma^\mu{}_{10'}=\overline{m}^\mu$, and $\sigma^\mu{}_{11'}=n^\mu$.

[11]It will be noted that in the Yang-Mills theory it is the spin affinities which are considered as potentials whereas here the vectors B_μ are defined by Eq. (2.1). Obviously spin affinities are not space-time vectors in the Riemannian space whereas the B's are.

[12]In Ref. 7 the four operators $\partial_{ab'}$ are denoted as follows: $\partial_{00'}=D$, $\partial_{01'}=\delta$, $\partial_{10'}=\overline{\delta}$, and $\partial_{11'}=\Delta$.

[13]The similarity of this expression, which can be written as a second-order Lagrangian density of the form $-\frac{1}{4}(-g)^{1/2}\,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$ +c.c., to that given by Eq. (1.5) of Kibble is obvious. The difference between them is due only to the group structure, which is SL(2,C) in the present case and is the Poincaré group in Kibble's case. This fact explains why we here need to add the complex conjugate term in order to make the Lagrangian density real.

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Models of Static, Cylindrically Symmetric Solutions of the Einstein-Maxwell Field

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Models of static, cylindrically symmetric solutions of the combined Einstein-Maxwell field equations are given. These models consist of extended distributions of matter with surface electric currents and magnetic fields outside the matter. The electric currents serve as sources of the magnetic fields; the distribution of matter as well as the magnetic fields serve as sources of the gravitational field. The magnetic lines of force may be parallel to the axis or circular and centered on the axis. The matter distribution is cylindrically symmetric and may be contained within a central cylinder or a tube centered about the axis. All ordinary physical and geometric requirements are satisfied by the models.

I. INTRODUCTION

The static, cylindrically symmetric source-free solutions of the combined Einstein-Maxwell gravitational and electromagnetic fields are fairly well understood.[1,2] In most cases these solutions are singular along the axis and not singular anywhere else. The singularity along the axis is interpreted as the source of the fields. To avoid singularities, it is necessary to introduce a distribution of the matter region over a finite portion of space. Such is the purpose of this paper.

Four models are discussed (Fig. 1). The first two consist of a cylinder of matter centered along the axis with different external magnetic fields. The other two consist of tubes of matter with different interior and exterior magnetic fields. Section II gives a quick review of the general solutions of the static, cylindrically symmetric solutions of the Einstein-Maxwell field equations. Sections III, IV, V, and VI describe the four models, respectively. All four models satisfy physically reasonable requirements on the strength of the mass and pressure densities. The metric tensor and its first derivative are everywhere continuous.

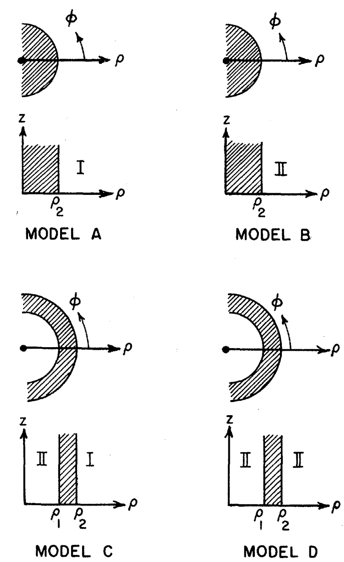

FIG. 1. The four model configurations studied. Side and top views are given. The shading indicates the distribution of matter. ρ is the distance from the axis of symmetry, z is the symmetry axis, and ϕ an angular measure about the axis. The symbols I and II indicate the type or symmetry (circular and axial, respectively) of the electromagnetic field in the vacuum regions (see Sec. II).

II. STATIC, CYLINDRICALLY SYMMETRIC SOLUTIONS OF THE EINSTEIN-MAXWELL FIELDS

Three physically independent types of solutions exist for the combined Einstein-Maxwell field with static cylindrical symmetry.[1,2] These are the following:

(I) an axial current producing a magnetic field whose lines are circles in the plane perpendicular to the axis and centered about the axis,

(II) an angular current producing a magnetic field parallel to the axis,

(III) an axial charge distribution producing a radial electric field.

The general solution for the first two types has been found but not for the third. Hence the models we consider will only involve physical types I and II. The general line element for type I can be written as[3]

$$ds^2 = -(\rho+\rho_0)^{2c^2-2c}[k+(\rho+\rho_0)^{2c}]^2 e^{2\gamma_0-2\psi_0}(dt^2-d\rho^2) + (\rho+\rho_0)^{2-2c}[k+(\rho+\rho_0)^{2c}]^2 e^{-2\psi_0}d\phi^2 + (\rho+\rho_0)^{2c}[k+(\rho+\rho_0)^{2c}]^{-2}e^{2\psi_0+2\mu_0}dz^2\,, \tag{2.1}$$

where ρ_0, γ_0, ψ_0, and μ_0 are constants. c and k are both non-negative, c is related to the mass per unit length of the source and k to the magnetic field strength as will be described. For model A and the external metric of model C, the line element (2.1) can be adjusted by an appropriate scaling of ρ, z, and t to have $\psi_0=\mu_0=0$. In these cases, we call $\gamma_0=a$ where a is a constant. Such a rescaling cannot be done simultaneously for both the interior and exterior regions in model C.

An examination of the motion of neutral test particles[2] (geodesics) in the region of type-I solutions external to the mass distribution shows that for small c, the gravitational mass per unit length is given to first order in c by

$$M_{\text{grav}} = 8\pi c\,. \tag{2.2}$$

The magnetic lines of force described by the type-I solution are circles in the plane perpendicular to the axis (ρ-ϕ plane) and centered about the axis. The nonvanishing components of the Maxwell tensor are

$$f_{\rho z} = \pm\frac{2ck^{1/2}(\rho+\rho_0)^{2c-1}}{[k+(\rho+\rho_0)^{2c}]^2}\,. \tag{2.3}$$

The corresponding physical component[4] of the magnetic field is given by

$$B_\phi \equiv f_{\rho z}(g^{11}g^{33})^{1/2} = \pm\frac{2ck^{1/2}e^a}{(\rho+\rho_0)^{(c+1)^2}[k+(\rho+\rho_0)^{2c}]^2} \tag{2.4}$$

The signs are chosen alike in Eqs. (2.3) and (2.4).

The general line element for type-II solutions is

$$ds^2 = -(\rho+\rho_0)^{2\delta+2\delta^2}[1+k(\rho+\rho_0)^{2+2\delta}]^2 e^{2\gamma_0-2\psi_0}(dt^2 - d\rho^2) + (\rho+\rho_0)^{2\delta}[1+k(\rho+\rho_0)^{2+2\delta}]^{-2}e^{-2\psi_0}d\phi^2 + (\rho+\rho_0)^{-2\delta}[1+k(\rho+\rho_0)^{2+2\delta}]^2 e^{2\psi_0+2\mu_0}dz^2 . \tag{2.5}$$

Again ρ_0, γ_0, ψ_0, and μ_0 are constants; δ and k are non-negative, δ is related to the mass per unit length of the source and k to the magnetic field strength. For model B and the external metric of model D, a rescaling of coordinates can be accomplished so that $\psi_0 = \mu_0 = 0$. We call $\gamma_0 = a$ as before.

Again an examination of the motion of neutral test particles shows that for small δ

$$M_{\text{grav}} = 8\pi\delta . \tag{2.6}$$

The magnetic field is parallel to the z axis. The nonvanishing components of the Maxwell tensor are

$$f_{\rho\phi} = \pm\frac{2\delta k^{1/2}(\rho+\rho_0)^{1+2\delta}}{[1+k(\rho+\rho_0)^{-2\delta}]^2} . \tag{2.7}$$

The corresponding physical component is given by

$$B_z \equiv f_{\rho\phi}(g^{11}g^{22})^{1/2} = \pm\frac{2(1+\delta)k^{1/2}e^a}{(\rho+\rho_0)^{\delta^2}[1+k(\rho+\rho_0)^{2+2\delta}]^2} . \tag{2.8}$$

Again, like signs go together.

The current source, I, of the fields is given by

$$I \equiv -\int_{V_3} \epsilon(L)L_\mu J^\mu d_3V = \oint \epsilon(N)\epsilon(M)F_{\mu\nu}M^\mu N^\nu d_2V . \tag{2.9}$$

d_2V is an invariant element of area, d_3V an invariant element of volume, V_2 is the two-surface bounding V_3. V_2, M, and N in order form a right-handed orthonormal tetrad, L is a unit vector orthogonal to V_3, $\epsilon(N)$ is +1 (−1) if N is spacelike (timelike). J^μ is the current 4-vector and $F_{\mu\nu}M^\mu N^\nu$ is the physical field (2.4) or (2.8). A surface current will be assumed and the volume element appropriately chosen to determine the current.

For a type-I field, V_3 spans the ρ, ϕ, and t directions and has unit coordinate length in the t direction. The current source, as calculated from (2.9), is

$$I_z = \pm 4\pi c k^{1/2} . \tag{2.10}$$

I_z is directed along the z axis. The ± sign is the same as the ± sign in Eq. (2.4) if the matter-free region is outside the current sheet. The signs are reversed if the matter-free region is inside the current sheet.

For a type-II field, V_3 spans the ρ, z, and t directions and has unit coordinate length along t. The integral (2.9) reduces to

$$I_\phi = \pm 2(1+\delta)k^{1/2} . \tag{2.11}$$

This is the total current per unit length on the surface of the cylinder bounding the type-II solution.

III. MODEL A: SOLID CENTRAL CYLINDER, CIRCULAR EXTERNAL MAGNETIC FIELD

Model A describes a solid central axial cylinder with an external type-I solution (Fig. 1). The stress-energy-momentum tensor is taken to be

$$T_\mu{}^\nu \equiv 2e^{2\psi-2\gamma}\begin{pmatrix} -d & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix} \equiv \begin{pmatrix} -D & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & P_2 & 0 \\ 0 & 0 & 0 & P_3 \end{pmatrix} \tag{3.1}$$

with an internal line element given by

$$ds^2 = -e^{2\gamma-2\psi}(dt^2 - d\rho^2) + \rho^2 e^{-2\psi}d\phi^2 + e^{2\psi+2\mu}dz^2 . \tag{3.2}$$

We define f, ν, and η by

$$f \equiv d - p_1 , \tag{3.3a}$$

$$\psi_\rho \equiv -\mu_\rho + \nu_\rho , \tag{3.3b}$$

$$\gamma_\rho \equiv -\mu_\rho + \eta_\rho , \tag{3.3c}$$

where the subscript ρ denotes differentation with respect to ρ. The Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\tfrac{1}{2}T_{\mu\nu}$$

reduce to

$$f = -\mu_{\rho\rho} - \mu_\rho{}^2 - 2\mu_\rho/\rho , \tag{3.4}$$

$$p_1 = \eta_\rho\mu_\rho + \eta_\rho/\rho - \nu_\rho{}^2 , \tag{3.5}$$

$$p_2 = \eta_{\rho\rho} + \nu_\rho{}^2 , \tag{3.6}$$

$$p_3 = \eta_{\rho\rho} + \nu_\rho{}^2 - 2\nu_\rho\mu_\rho - 2\nu_\rho/\rho - f . \tag{3.7}$$

For physically reasonable models, we impose the following conditions:

(1) The components D, P_1, P_2, and P_3 of the stress-energy-momentum tensor are finite at all points.

(2) The energy density $D \geq 0$ within the matter.

(3) For weak fields (small c and $k \ll c$), D is of

the order of the mass density and P_1, P_2, and P_3 are the order of the square of the mass density.

$$D=O(c), \quad P_i=O(c^2), \text{ for } c\ll 1, \quad k=O(c^2).$$

(4) Elementary flatness. If R is the ratio of the circumference of a circle (coordinate radius ρ) to its radius, $R\to 2\pi$ as $\rho\to 0$, for all circles centered on the axis and perpendicular to it.

The models we take for the distribution of matter are similar to those of Marder.[5] Assume

$$\mu=-\frac{\alpha}{n+1}\left(\frac{\rho}{\rho_2}\right)^{n+1}+\alpha_0, \tag{3.8}$$

$$\psi=-\mu-\frac{\beta}{m+1}\left(\frac{\rho}{\rho_2}\right)^{m+1}+\beta_0, \tag{3.9}$$

$$\gamma=-\mu+\frac{\epsilon}{q+1}\left(\frac{\rho}{\rho_2}\right)^{q+1}+\epsilon_0, \tag{3.10}$$

where α, β, ϵ, α_0, β_0, ϵ_0, n, m, and q are constants. To satisfy condition (1) above means to choose $n,m,q>1$. The first and second fundamental forms are to be continuous across the boundary. For the same coordinate grid across the boundary and for a boundary which corresponds to a fixed value of the coordinate ρ, this is equivalent to requiring that

$$(g_{A\mu\nu})_{\rho=\rho_2}=(g_{B\mu\nu})_{\rho=\rho_2} \tag{3.11}$$

and

$$(g_{A\mu\nu,\sigma})_{\rho=\rho_2}=(g_{B\mu\nu,\sigma})_{\rho=\rho_2}, \tag{3.12}$$

where $g_{A\mu\nu}$ is the metric in one region and $g_{B\mu\nu}$ is the metric in the other region, the regions being separated at $\rho=\rho_2$. Applying this to the external metric [obtained from (2.1)] and the internal metric [obtained from (3.2)] with (3.8) to (3.10),

$$\alpha=\frac{\rho_0}{\rho_2+\rho_0}, \tag{3.13}$$

$$\alpha_0=\ln\left(1+\frac{\rho_0}{\rho_2}\right)+\frac{1}{n+1}\frac{\rho_0}{\rho_0+\rho_2}, \tag{3.14}$$

$$\beta=\frac{c\rho_2}{\rho_2+\rho_0}\,\frac{1-k(\rho_2+\rho_0)^{-2c}}{1+k(\rho_2+\rho_0)^{-2c}}, \tag{3.15}$$

$$\beta_0=\ln\left(\frac{(\rho_2+\rho_0)^c}{k+(\rho_2+\rho_0)^{2c}}\right)+\frac{c\rho_2}{(m+1)(\rho_2+\rho_0)}\,\frac{1-k(\rho_2+\rho_0)^{-2c}}{1+k(\rho_2+\rho_0)^{-2c}}, \tag{3.16}$$

$$\epsilon=\frac{c^2\rho_2}{\rho_2+\rho_0}, \tag{3.17}$$

$$\epsilon_0=c^2\left(\ln(\rho_2+\rho_0)-\frac{1}{q+1}\frac{\rho_2}{\rho_0+\rho_2}\right)+a. \tag{3.18}$$

The replacements $\psi_0=\mu_0=0$ and $\gamma_0=a$ have been made outside the cylinder. The condition of elementary flatness requires that

$$R=2\pi(e^{-\gamma})_{\rho=0}=2\pi \tag{3.19a}$$

or

$$\gamma_{\rho=0}=0\Rightarrow\alpha_0=\epsilon_0. \tag{3.19b}$$

This gives

$$a=\frac{1}{\rho_2+\rho_0}\left(\frac{\rho_2c^2}{q+1}+\frac{\rho_0}{n+1}\right)-\ln[\rho_2(\rho_2+\rho_0)^{c^2-1}]. \tag{3.20}$$

The tensor components d, p_1, p_2, and p_3 are given from (3.3a), (3.5), and (3.6) as

$$\rho_2^2d=\alpha\left(\frac{\rho}{\rho_2}\right)^n\left[(n+2)\frac{\rho_2}{\rho}-\alpha\left(\frac{\rho}{\rho_2}\right)^n\right]+\rho_2^2p_1, \tag{3.21}$$

$$\rho_2^2p_1=\epsilon\left(\frac{\rho}{\rho_2}\right)^{q-1}-\epsilon\alpha\left(\frac{\rho}{\rho_2}\right)^{q+n}-\beta^2\left(\frac{\rho}{\rho_2}\right)^{2m}, \tag{3.22}$$

$$\rho_2^2p_2=\epsilon q\left(\frac{\rho}{\rho_2}\right)^{q-1}+\beta^2\left(\frac{\rho}{\rho_2}\right)^{2m}, \tag{3.23}$$

$$\rho_2^2p_3=\epsilon q\left(\frac{\rho}{\rho_2}\right)^{q-1}+\beta^2\left(\frac{\rho}{\rho_2}\right)^{2m}-2\beta\alpha\left(\frac{\rho}{\rho_2}\right)^{2m}+\left\{2\beta(m+1)\left(\frac{\rho}{\rho_2}\right)^{m-1}-\alpha\left(\frac{\rho}{\rho_2}\right)^n\left[(n+2)\frac{\rho_2}{\rho}-\alpha\left(\frac{\rho}{\rho_2}\right)^n\right]\right\}. \tag{3.24}$$

Condition (3) for a physically reasonable model is satisfied automatically for P_1 and P_2 and is satisfied for P_3 if $m=n$ and if

$$2\beta(n+1)=\alpha(n+2). \tag{3.25}$$

Thus for small c and $k=O(c^2)$,

$$\rho_0=2\frac{n+1}{n+2}c\rho_2 \tag{3.26}$$

and $D=O(c)$. As c or k increases, it would be necessary only to require (3.26) not (3.25). Assumming (3.26), condition 2 for a physically reasonable model is satisfied if $q\geq n>1$. Hence all conditions (1) to (4) can be satisfied as well as the boundary conditions.

One can think of the proper mass per unit proper length as being represented by

$$M=-2\pi\int_0^{\rho_2}T_0{}^0\rho e^{\gamma-2\psi}d\rho$$
$$=4\pi\int_0^{\rho_2}de^{-\gamma}\rho d\rho. \tag{3.27}$$

Let $\rho=\chi\rho_2$,

$$M=4\pi\int_0^1\rho_2{}^2de^{-\gamma}\chi d\chi.$$

$\rho_2{}^2d$ is given by (3.21). Hence

$$M=8\pi\left\{c+c^2\left[\frac{1}{2(q+1)}-\frac{1}{4n+2}\left(\frac{1-k\rho_2^{-2c}}{1+k\rho_2^{-2c}}\right)-\frac{(n+1)(n+3)}{(n+2)^2}\right]+O(c^3)\right\}e^{-\alpha_0-\epsilon_0}. \tag{3.28}$$

For any $\rho_2>0$,

$$M=8\pi c+O(c^2). \tag{3.29}$$

The proper mass contains only contributions from the matter. It is independent of ρ_2 and k to first order in c and is the same as the gravitational mass [Eq. (2.2)] which was determined by the motion of a test particle for $\rho>\rho_2$; however, it has higher-order contributions which depend upon the details of the model which the motion of a test particle at $\rho>\rho_2$ can not detect.

The model corresponds to a stressed wire with a surface current [given by (2.10)] and an external circular magnetic field. It is not difficult to construct models when the surface current is replaced by current distributions within the matter,[6] but we shall not pursue this point. For small central mass, the physical field can be found in terms of the current. Using Eqs. (2.4) and (2.10), for large ρ,

$$B_\phi\simeq I_z/2\pi\rho. \tag{3.30}$$

In the limit $\rho_2\to 0$, it should be possible to recover the line element given in Ref. 1, which is given by (2.1) with $\rho_0=0$, $\phi_0=\mu_0=0$, $\gamma_0=a$ (an arbitrary constant). If this limit is taken, Eq. (3.20) shows that

$$a\sim -c^2\ln\rho_2+\text{const}. \tag{3.31}$$

This choice of a is needed also to keep ϵ_0 and hence M finite as the limit is taken. It can readily be shown that the coordinates ρ, z, t, and ϕ can be rescaled to absorb a; the infinite value of a in the limit means an infinite rescaling of ϕ. In this way we can recover the general external line element for metrics of type I with a singularity on the axis.

IV. MODEL B: SOLID CENTRAL CYLINDER, AXIAL EXTERNAL MAGNETIC FIELD

Model B describes a solid central axial cylinder with an external type-II solution (Fig. 1). The stress-energy-momentum tensor will again be taken as in (3.1) and the internal line element will be described by (3.2) with (3.8), (3.9), and (3.10). Hence Eqs. (3.1)–(3.12) and (3.21)–(3.24) hold for model B. The boundary conditions on the continuity of the components of the metric tensor and its derivative at the boundary, (3.11) and (3.12), give

$$\alpha=\frac{\rho_0}{\rho_0+\rho_2}, \tag{4.1}$$

$$\alpha_0=\ln\left(1+\frac{\rho_0}{\rho_2}\right)+\frac{1}{n+1}\frac{\rho_0}{\rho_2+\rho_0}, \tag{4.2}$$

$$\beta=\frac{\delta\rho_2}{\rho_2+\rho_0}-2k(1+\delta)\frac{\rho_2(\rho_2+\rho_0)^{1+2\delta}}{1+k(\rho_2+\rho_0)^{2+2\delta}}, \tag{4.3}$$

$$\beta_0=\ln\left(\frac{1+k(\rho_2+\rho_0)^{2+2\delta}}{(\rho+\rho_0)^{\delta}}\right)+\frac{\delta\rho_2}{(m+1)(\rho_2+\rho_0)}-\frac{2k(1+\delta)(\rho_2+\rho_0)^{1+2\delta}\rho_2}{(m+1)[1+k(\rho_2+\rho_0)^{2+2\delta}]}, \tag{4.4}$$

$$\epsilon=\delta^2\frac{\rho_2}{\rho_2+\rho_0}+\frac{4(1+\delta)\rho_2k(\rho_2+\rho_0)^{1+2\delta}}{1+k(\rho_2+\rho_0)^{2+2\delta}}, \tag{4.5}$$

$$\epsilon_0=\ln\{(\rho_2+\rho_0)^{\delta^2}[1+k(\rho_2+\rho_0)^{2+2\delta}]^2\}-\frac{\delta^2\rho_2}{(q+1)(\rho_2+\rho_0)}-\frac{4(1+\delta)k\rho_2(\rho_2+\rho_0)^{1+2\delta}}{(q+1)[1+k(\rho_2+\rho_0)^{2+2\delta}]}+a. \tag{4.6}$$

From the condition for elementary flatness, $\alpha_0=\epsilon_0$,

$$a=\ln\left(\frac{(\rho_2+\rho_0)^{1-\delta^2}}{\rho_2[1+k(\rho_2+\rho_0)^{2+2\delta}]^2}\right)+\frac{\rho_0}{(n+1)(\rho_2+\rho_0)}-\frac{\delta^2\rho_2}{(q+1)(\rho_2+\rho_0)}+\frac{4(1+\delta)k(\rho_2+\rho_0)^{1+2\delta}}{1+k(\rho_2+\rho_0)^{2+2\delta}}\frac{\rho_2}{q+1}. \tag{4.7}$$

The other three conditions for a physically reasonable model are satisfied if $n=m$, $q\geq n>1$, and

$$\rho_0=2\frac{n+1}{n+2}\delta\rho_2. \tag{4.8}$$

The conditions on n, m, and q, and the relation between ρ_0 and ρ_2 are the same for models A and B. Since the investigation has been carried only to first order in δ and $k=O(\delta^2)$, this equivalence is not surprising. Calculating the proper mass according to Eq. (3.27) for finite ρ_2 yields

$$M=8\pi\delta+O(\delta^2). \tag{4.9}$$

Again the limit $\rho_2\to 0$ requires $a\sim -\delta^2\ln\rho_2+a_1$ and a rescaling of ρ, z, t, and ϕ outside the matter region. When $\rho\to\infty$,

$$B_z\sim I_\phi/k^2\rho^4. \tag{4.10}$$

Model B represents a cylinder with a surface current much like that of a solenoid. There is an external magnetic field but not an internal field. In the ordinary classical Maxwell theory of an infinite solenoid, one thinks of a uniform internal field but a vanishing external field. However, by adding a constant magnetic field to the system that bucks and cancels the internal field, one has a consistent solution of the field equations for a solenoid with an external magnetic field and a vanishing internal field. This picture seems more consistent with the Einstein-Maxwell model we have just described.

An interesting limiting case for model B exists for which $\delta=0$,

$$\rho_0=\alpha=\alpha_0=\epsilon_0=0\,, \tag{4.11}$$

$$\beta=-\tfrac{1}{2}\epsilon=-\frac{2k\rho_2^{\,2}}{1+k\rho_2^{\,2}}\,, \tag{4.12}$$

$$\beta_0=-\tfrac{1}{2}a=\ln(1+k\rho_2^{\,2})-\frac{2k\rho_2^{\,2}}{(m+1)(1+k\rho_2^{\,2})}\,, \tag{4.13}$$

$$B_z=\pm\frac{2k^{1/2}e^a}{(1+k\rho^2)^2}\,, \tag{4.14}$$

$$I_\phi=\pm 2k^{1/2}\,. \tag{4.15}$$

The exterior line element [Eq. (2.5)] for $\delta=0$ and $\rho_0=0$ is

$$ds^2=-(1+k\rho^2)^2e^{2a}(dt^2-d\rho^2)+(1+k\rho^2)^{-2}d\phi^2+(1+k\rho^2)^2dz^2\,. \tag{4.16}$$

If this is the line element over all space and $a=0$, it is a singularity-free completely magnetic universe,[2,7] i.e., a universe satisfying the Einstein-Maxwell theory and consisting only of a magnetic field being held together gravitationally. Taking model B with $\delta=0$ to the limit $\rho_2\to 0$ recovers the line element for the completely magnetic universe. I_ϕ remains constant as $\rho_2\to 0$. It is interesting that I_ϕ is needed for finite ρ_2 independently of its size; but, for $\rho_2=0$, there is no singularity in the magnetic field configuration and hence no need for I_ϕ.

V. MODEL C: TUBULAR MATTER SOURCE, AXIAL INTERIOR MAGNETIC FIELD, CIRCULAR EXTERIOR MAGNETIC FIELD

Model C (Fig. 1) is a tubular cylinder of matter with an exterior type-I solution (circular magnetic field) and an interior type-II solution (axial magnetic field). Special cases, of course, consist of the exterior or interior magnetic fields or both vanishing. Equations (3.1) to (3.7) describe the stress-energy-momentum tensor of the matter. The exterior line element is the simplified form of (2.1),

$$ds^2=-(\rho+\rho_0)^{2c^2-2c}[\kappa+(\rho+\rho_0)^{2c}]^2e^{2a}(dt^2-d\rho^2)+(\rho+\rho_0)^{2-2c}[\kappa+(\rho+\rho_0)^{2c}]^2d\phi^2+(\rho+\rho_0)^{2c}[\kappa+(\rho+\rho_0)^{2c}]^{-2}dz^2\,. \tag{5.1}$$

The interior line element is

$$ds^2=-(1+k\rho^2)^2e^{-2\psi_0}(dt^2-d\rho^2)+\rho^2(1+k\rho^2)^{-2}e^{-2\psi_0}d\phi^2+(1+k\rho^2)^2e^{2\psi_0+2\mu_0}dz^2\,. \tag{5.2}$$

κ is the parameter concerned with the exterior magnetic field; k is the parameter related to the interior magnetic field. Equation (5.2) is the most general line element of type II which has no singularity along the axis.

Let (3.2) be the line element within the tubular matter, with

$$\mu=-\frac{\alpha}{n+1}(\rho_2-\rho_1)x^{n+1}+\alpha_0\,, \tag{5.3}$$

$$\psi=-\mu-\frac{\beta_1}{n+1}(\rho_2-\rho_1)x^{n+1}-\frac{\beta_2}{\bar n+1}y^{\bar n+1}+\beta_0\,, \tag{5.4}$$

$$\gamma=-\mu+\frac{\epsilon_1}{q+1}(\rho_2-\rho_1)x^{q+1}+\frac{\epsilon_2}{\bar q+1}(\rho_2-\rho_1)y^{\bar q+1}+\epsilon_0\,, \tag{5.5}$$

$$x\equiv\frac{\rho-\rho_1}{\rho_2-\rho_1}\,,\quad y\equiv\frac{\rho_2-\rho}{\rho_2-\rho_1}\,. \tag{5.6}$$

As in the previous models, we shall want

$$\rho_0=2\lambda c\rho_2\,,\quad \lambda\equiv\frac{n+1}{n+2}\,. \tag{5.7}$$

The boundary conditions (3.11) and (3.12) applied at both ρ_1 and ρ_2 yield the values of all remaining constants, α, α_0, β_0, ρ_1, ρ_2, ϵ_0, ϵ_1, ϵ_2, and a,

$$\beta_2=-\tfrac{1}{2}\epsilon_2=\frac{2k\rho_1}{1+k\rho_1^{\,2}}\,, \tag{5.8}$$

$$\alpha=\frac{1}{\rho_2}\frac{2\lambda c}{1+2\lambda c}\,, \tag{5.9}$$

$$\beta_1=\frac{1}{\rho_2}\frac{c}{1+2\lambda c}\frac{(1+2\lambda c)^{2c}-\kappa\rho_2^{\,-2c}}{(1+2\lambda c)^{2c}+\kappa\rho_2^{\,-2c}}\,, \tag{5.10}$$

$$\epsilon_1=\frac{c^2}{\rho_2(1+2\lambda c)}\,, \tag{5.11}$$

$$\mu_0=\alpha_0=\ln(1+2\lambda c)+\frac{\alpha}{n+1}(\rho_2-\rho_1)\,, \tag{5.12}$$

$$\epsilon_0=2\ln(1+k\rho_1^{\,2})+\mu_0-\frac{\epsilon_2}{\bar q+1}(\rho_2-\rho_1)\,, \tag{5.13}$$

$$\psi_0=\beta_0-\mu_0-\frac{\beta_2}{\bar n+1}\,, \tag{5.14}$$

$$\beta_0=\ln\frac{\rho_2^{\,c}(1+2\lambda c)^c}{\kappa+\rho_2^{\,2c}(1+2\lambda c)^{2c}}+\frac{\beta_1}{n+1}(\rho_2-\rho_1)\,, \tag{5.15}$$

$$a=-\ln[\rho_2^{\,c^2+1}(1+2\lambda c)^{c^2+2}]+\epsilon_0+\frac{\epsilon_1}{q+1}(\rho_2-\rho_1)\,. \tag{5.16}$$

The energy density, D, is non-negative if $2\bar n$ is sufficiently greater than $\bar q$ and if q is sufficiently greater than $n > 1$. This can be seen by examining the form of d and realizing that both x and y vary between 0 and 1.

$$d = \frac{\alpha n}{\rho_2 - \rho_1} x^{n-1} + \frac{2\alpha}{\rho} x^n - \alpha^2 x^{2n} - \alpha\epsilon_1 x^{n+q}$$
$$+ \alpha\epsilon_2 x^n y^{\bar{q}} - \beta_1^2 x^{2n} - \beta_2^2 y^{2\bar{n}} + 2\beta_1\beta_2 x^n y^{\bar{n}} + \frac{\epsilon_1}{\rho} x^q - \frac{\epsilon_2}{\rho} y^{\bar{q}} . \tag{5.17}$$

Applying Eq. (3.27) to determine the proper mass per unit length to first order in c, assuming that $\kappa\rho_2^{-2c} = O(c^2)$ and $\kappa\rho_1^2 = O(c^2)$, we get

$$M = 8\pi c \left[1 - \frac{\rho_1}{(n+2)\rho_2}\right]. \tag{5.18}$$

Our definition of M does not include any contributions from the magnetic fields inside or outside the matter region. It is evident that the mass calculated this way does not correspond to the gravitational mass, $8\pi c$ [Eq. (2.2)], obtained by considering the geodesic motion of neutral test particles. All the conditions for a physically reasonable model can be satisfied by judicious choice of the exponents q, $\bar{q}$, n, and $\bar{n}$.

The surface current (at $\rho = \rho_1$) producing the interior field is angular and given by

$$I_\phi = 2k^{1/2}, \tag{5.19}$$

while the surface current (at $\rho = \rho_2$) producing the exterior field is axial and given by

$$I_z = 4\pi c k^{1/2}. \tag{5.20}$$

We shall examine two limiting cases, $c = \kappa = 0$, and $\rho_1, \rho_2 \to \infty$ with $\rho_2 - \rho_1$ finite. If $c = \kappa = 0$, the external metric is everywhere locally flat, but there is still a magnetic field within the cylinder. The gravitational mass of the tube (Eq. 2.2) vanishes but not its proper mass [Eq. (5.18)]. A calculation to first order in $k\rho_1^2$ will yield

$$M = \frac{16\pi k\rho_1^2}{\bar{q}+1}\left(\frac{\rho_2}{\rho_1} - 1\right). \tag{5.21}$$

As before this is the contribution from the matter between ρ_1 and ρ_2.

In this universe, any observer outside the tube of matter would see a flat space. However, if he went around the tube back to his starting point, he would find that the ratio of the circumference to the radius is not 2π. The universe is conical. This gives us a model of a source for the external metric discussed by Dowker.[8]

The other limiting case is $\rho_1, \rho_2 \to \infty$ with $\rho_2 - \rho_1$ remaining finite. The matter density and pressures remain well defined, the current on the inner surface remains constant [Eq. (5.19)], and the internal magnetic field remains. In this limit, the universe becomes the magnetic universe [Eq. (4.17) with $a = 0$]. The constants ψ_0 and μ_0 of Eq. (5.2) can be removed by a rescaling of coordinates. Hence the magnetic universe arises by this limiting process as well as by a limiting case of model B.

VI. MODEL D: TUBULAR MATTER SOURCE, AXIAL INTERIOR AND AXIAL EXTERIOR MAGNETIC FIELD

Model D (Fig. 1) is a tubular cylinder of matter with an exterior type-II solution (axial magnetic field) as well as an interior type-II solution. When the exterior magnetic field is zero, model D is the same as model C. When the exterior magnetic field is not zero, the line element is the same as the exterior line element of model B.

The external line element is described by Eq. (2.5) with $\psi_0 = \mu_0 = 0$, $\gamma_0 = a$; and the interior line element is described by Eq. (5.2). κ is the exterior field parameter and k the interior. The metric within the tubular cylinder of matter is assumed to have the form of Eqs. (5.3) to (5.6). Again

$$\rho_0 = 2\lambda\delta\rho_2, \quad \lambda \equiv \frac{n+1}{n+2}. \tag{6.1}$$

Matching boundary conditions will give all the constants as in model C.

$$\alpha = \frac{2\lambda\delta}{1+2\gamma\delta}\frac{1}{\rho_2}, \tag{6.2}$$

$$\beta_2 = -\tfrac{1}{2}\epsilon_2 = \frac{2k\rho_1}{1+k\rho_1^2}, \tag{6.3}$$

$$\beta_1 = \frac{\delta}{1+2\lambda\delta}\frac{1}{\rho_2} - \frac{2\kappa(1+\delta)(1+2\lambda\delta)^{2+2\delta}\rho_2^{2+2\delta}}{1+\kappa(1+2\lambda\delta)^{2+2\delta}\rho_2^{2+2\delta}}\frac{1}{\rho_2}, \tag{6.4}$$

$$\epsilon_1 = \frac{\delta^2}{1+2\lambda\delta}\frac{1}{\rho_2} + \frac{4\kappa(1+\delta)(1+2\lambda\delta)^{2+2\delta}\rho_2^{2+2\delta}}{1+\kappa(1+2\lambda\delta)^{2+2\delta}\rho_2^{2+2\delta}}\frac{1}{\rho_2}, \tag{6.5}$$

$$\mu_0 = \alpha_0 = \ln(1+2\lambda\delta) + \frac{\alpha}{n+1}(\rho_2 - \rho_1), \tag{6.6}$$

$$\epsilon_0 = 2\ln(1+k\rho_1^2) + \mu_0 - \frac{\epsilon_2}{\bar{q}+1}(\rho_2 - \rho_1), \tag{6.7}$$

$$\psi_0 = \beta_0 - \mu_0 - \frac{\beta_2}{\bar{n}+1}, \tag{6.8}$$

$$\beta_0 = \ln\Big(\rho_2^{-\delta}(1+2\lambda\delta)^{-1-\delta}[1+\kappa\rho_2^{2+2\delta}(1+2\lambda\delta)^{2+2\delta}] + \frac{\beta_1}{n+1}(\rho_2 - \rho_1)\Big), \tag{6.9}$$

$$a = -\ln\{\rho_2^{\delta^2-1}(1+2\lambda\delta)^{\delta^2}[1+k\rho_2^{2+2\delta}(1+2\lambda\delta)^{2+2\delta}]^2\} + \epsilon_0 + \frac{\epsilon_1}{q+1}(\rho_2 - \rho_1). \tag{6.10}$$

The physical requirements can all be satisfied by choosing $2\bar{n}$ sufficiently greater than $\bar{q}$ and q sufficiently greater than $n>1$. The proper mass per unit length to the first order in δ is given by Eq. (5.18) with c replaced by δ. The surface current at $\rho=\rho_1$ is

$$I_\phi = 2k^{1/2} \tag{6.11}$$

and the surface current at $\rho=\rho_2$ is

$$I_\phi = 2(1+\delta)^{1/2}. \tag{6.12}$$

Limiting cases have behaviors which are the obvious analogs of the corresponding limiting cases discussed for models B and C.

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[1]See *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), Chap. 9. See also a paper by L. Witten, in *Colloquium on the Theory of Relativity*, Centre de Recherches Mathematiques, Free University (Universitaire, Louvain, Belgium, 1960), p. 59.

[2]J. L. Safko and L. Witten, J. Math. Phys. 12, 257 (1971).

[3]We use the signature $(-+++)$. A comma denotes ordinary differentiation. $16\pi G=1$, G is the gravitational constant, $c=1$. The coordinate ranges are $0\leq\rho<\infty$, $-\infty\leq z$, $t\leq\infty$, $0\leq\phi<2\pi$.

[4]The physical components are defined as follows: Let $\lambda_{(\alpha)}^{\mu}$ be an orthonormal tetrad with $\lambda_{(0)}^{\mu}$ timelike and pointing to the future. The physical components of the tensor $t_{\mu\nu}$ are the invariants $f_{(\alpha\beta)}=f_{\mu\nu}\lambda_{(\alpha)}^{\mu}\lambda_{(\beta)}^{\nu}$. We have chosen

$$\lambda_{(0)}=((-g^{00})^{1/2},\ 0,0,0),\ \lambda_{(1)}=(0,(g^{11})^{1/2},0,0),$$

$$\lambda_{(2)}=(0,0,(g^{22})^{1/2},0),\ \lambda_{(3)}=(0,0,0,(g^{33})^{1/2}).$$

[5]L. Marder, Proc. Roy. Soc. (London) A244, 524 (1958); A246, 133 (1958).

[6]To deal with currents distributed within the matter, let $T_{\mu\nu(\text{matter})}=T_{\mu\nu}-T_{\mu\nu(\text{EM})}$, where $T_{\mu\nu(\text{EM})}$ is constructed from $f_{\mu\nu}$ within the matter. It is necessary to assume the $f_{\mu\nu}$ and J^μ within the matter consistent with (2.9). $T_{\mu\nu(\text{matter})}$ is then determined. For example, for "skin" currents choose

$$f_{\rho z}\simeq f_{\rho z}(\rho_2)\exp[-\lambda(\rho_2-\rho)],\ \rho\leq\rho_2,\ \lambda>>1.$$

[7]M. A. Melvin, Phys. Letters 8, 65 (1964); K. S. Thorne, Phys. Rev. 138, B251 (1965).

[8]J. S. Dowker, Nuovo Cimento 52B, 129 (1967). See also the discussion in D. Wisnivesky and Y. Aharonov, Ann. Phys. (N.Y.) 45, 479 (1967).

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Classical and Quantum Mechanics in Auxiliary Algebras *

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A canonical algebra is defined as the algebraic structure common to classical and quantum mechanics. It is a linear space equipped with two bilinear operations: an associative product, and a Lie product which is a derivation with respect to the associative product. In addition, the Lie products of the generators of this algebra are constants. The problem of the algorithmic formulation of a canonical algebra in terms of an auxiliary associative algebra is solved. The first part of the problem consists of finding all auxiliary algebras which can algorithmically support a canonical structure. The second part consists of finding all canonical algebras which can be formulated in terms of a given auxiliary structure.

I. INTRODUCTION

It has been shown by Groenewold[1] and Moyal[2] that quantum mechanics can be formulated in the language of classical mechanics, i.e., in the phase-space language. Koopman[3] has investigated the problem of formulating the Lie structure of classical mechanics in the Heisenberg algebra of quantum mechanics. The purpose of this paper is to study the general problem suggested by these results, namely, the problem of algorithmically formulating the algebraic structure of classical or quantum mechanics in an auxiliary algebra.

Conceptually, classical and quantum mechanics involve two algebras: an associative algebra of observables, and a Lie algebra of generators of the automorphisms of the algebra of observables. It is a peculiar feature of both theories that these two algebras have a common underlying set. We are thus led to define a *canonical algebra* as a lin-