The special choice of putting one of these parameters equal to zero leads immediately to the well-known Schwarzschild solution as in Einstein's theory. Quite generally, however, these two parameters may be fixed to give better agreement with experiment [in Dicke's theory[2] there is the arbitrary parameter ω with "$\omega \geq 6$" in his Eq. (36)]. For details of these solutions we refer the reader to Sen's paper.[3]

We have shown that our theory brings a very close and a natural connection between gravitation of the Lyra type, scaling, and scale invariance. The "switching off" of the gravitational coupling constant forces α to the value one and hence a meaning to scaling cannot be given in such a case. We have also maintained the spin content of Einstein's theory.

ACKNOWLEDGMENTS

The author would like to thank the Theoretical Physics Institute of the University of Alberta for the award of a post-doctoral fellowship. Useful discussions with Professor Y. Takahashi and Professor D. K. Sen are gratefully acknowledged.

[1]J. Schwinger, *Particles and Sources* (Gordon and Breach, New York, 1969).

[2]C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).

[3]D. K. Sen and K. A. Dunn, J. Math. Phys. 12, 578 (1971).

[4]D. K. Sen, Can. Math. Bull. 3, 255 (1960); J. R. Vanstone, *ibid.* 3, 263 (1960).

[5]G. Lyra, Math. Z. 54, 52 (1951); E. Scheibe, *ibid.* 57, 65 (1952); D. K. Sen, Z. Physik 149, 311 (1957).

[6]H. Weyl, Sitzber. Preuss. Akad. Wiss., 465 (1918).

[7]See, for example, the review of B. Zumino, in *Lectures on Elementary Particles and Quantum Field Theory,* Brandeis Lectures, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1970), Vol. 2.

[8]J. Schwinger, Phys. Rev. 173, 1264 (1968). We use the notation of this paper and that of Ref. 9.

[9]J. Schwinger, Phys. Rev. 130, 800 (1962).

[10]Y. Takahashi, Phys. Rev. D 3, 622 (1971).

[11]A. Salam and J. Strathdee, Phys. Rev. 184, 1760 (1969).

PHYSICAL REVIEW D VOLUME 5, NUMBER 12 15 JUNE 1972

Space-Time Code. III*

David Finkelstein

Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

(Received 3 June 1971)

The quantum (q) concept of space-time dynamics described previously is extended from free to interacting systems. The idea is developed that the world is not a plenum of q objects but a plexus of q processes. The q mode of description by Hilbert-space vectors with a special rule of interpretation *(stators)* ordinarily used for physical objects is used instead for physical processes acting upon objects, including free propagation. The geometrical object proposed as stator for a q network of processes is a natural generalization of a tensor, a Feynman-diagram amplitude, and a chain in a complex, called a *plexor*. The notation, algebra, and geometry of plexors is illustrated. A strong Mach principle is a self-evident feature of such a q-process theory of space-time, in which it is meaningless to speak of empty space-time or of space-time relations between noninteracting systems.

INTRODUCTION

There has to be a better way to unite relativity and quantum theory than the present quantum field theory. It is important that a theory account for the remarkable relativistic causal structure and quantum logical structure of the world, but a theory should also account for the no less remarkable *existence* of the world. In quantum field theory the existence of the world is an as yet unverified conjecture. In a theory in close correspondence with the actual microstructure of the world, the existence of the world would be a tautology as automatic in the theory as it is in reality.

The decision to make existence prior to everything else leads one to seriously consider finite models of the world, to build the world from finite elements through finite means of assembly, as a digital computer is assembled from binary digits.

Besetting these attempts, and any other fundamental theory, is the ever-present pitfall of numerology, to be avoided only by stating in advance the operational meaning of each construct and computation. By numerology I mean any scheme that incorporates so little of the rich structure of the world that the numbers it produces can be assigned *ex post facto* meanings to take advantage of fortuitous coincidences. Here we avoid numerology by fixing the operational meanings of the relativistic causal order C and the quantum logical relations $\subset$ and $\perp$, which we take to have their ordinary senses in the cq and c domains.[1] It is a nontrivial property of the world that all the concepts of physics can be expressed in the language of these two structures. For example, the photon is identifiable by its mass and spin, quantum numbers of the invariance group of the causal relation; charge is the coupling to the photon. In a theory which includes these basic (and incidentally dimensionless) concepts there is little ambiguity about the meaning of derived constructs, and little doubt about when the theory agrees with experiment, when it disagrees.

Within finite models a certain quarrel between the philosophies of relativity and quantum mechanics becomes sharper than was possible in a model whose existence itself was at stake.

According to relativity, the world is a collection of *processes* (events) with an unexpectedly unified causal or chronological structure. Then an object is secondary; is a long causal sequence of processes, a world line. According to quantum mechanics the world is a collection of *objects* (particles) with an unexpectedly unified logical or class-theoretical structure. Then a process is secondary; is a mapping of the objects or of their initial to their final conditions.

Which are we to build out of our quanta – beings or becomings, essences or existences? In one paper along these lines the attempt was to build objects; in a later paper, processes.[1] The results favor a process model, an existential physics. Let us suppose that the primitive system is an elementary process, not an object, and that we assemble these primitive processes into chromosomelike code sequences to build simple objects, and braid and cross-link these strands to make more complex objects and their interactions.[2] The idea of the quantum jump comes into its own, and reigns supreme, even over space and time. The primitive quantum processes or chronons of which world lines are made can be thought of as acts of emission or creation. Their duals, antichronons, represent acts of absorption or annihilation. A free process, one without interaction, typically consists of the one followed by the other. In order to account for the four-dimensional world it suffices to suppose that the chronon is binary, is a spin-$\frac{1}{2}$ system.[3]

The germ of such a q-process world is present in Feynman's cq conception of elementary-particle processes. The amplitude for a Feynman diagram is closely related to the wave function for a process, and when we use Feynman amplitudes we come close to applying quantum logic to processes rather than objects. We shall discuss this relation between the q and cq processes again at the end of this paper (Sec. IX).

A still more serious clash between the principles of relativity and quantum theory happens in finite models. The basic quantum cannot be Lorentz-invariant, since there is no nontrivial finite-dimensional unitary representation of the Lorentz group. However, we have seen[1] that Lorentz invariance can emerge in the cq limit, hence in the S matrix. Exact rotational invariance is built in as $SU(2, C)$ invariance.

I. THE QUANTUM PLEXUS

So I think the world is a plexus, not a plenum; a quantum network, not a quantum field. Here I give a language for a quantum (q) plexus, for a network not of classical (c) currents but of q processes. The binary q dynamics I have already described[1] is a special case.

The *quantum* brings in old semantic questions. To delineate my position I find helpful the new term *stator*, that which states, for the old concept of a vector in Hilbert space with a certain kind of rule of interpretation or meaning. The stator differs from a state vector or an electric field vector only in the definite stipulation of this particular meaning, which makes ill-formed such name phrases as "*the* stator of the system" and therefore "the evolution of the stator of the system" and "the collapse of the stator of the system;" much as "the adjective of the cat," "the evolution of the adjective of the cat," or "the collapse of the adjective of the cat" are not quite right in common speech. Apparently the semantics of many quantum theorists seems to permit them to speak of *the* ψ vector, ket, or wave function of a system as existing, evolving, or collapsing. It is to get clear of any possible confusion with such usage that I have adopted the special term stator. A system does not have a stator. An energy, a momentum, an angular momentum, yes. A stator, no (Sec. II).

The *plexus* leads to syntactic problems. How do we describe a network of q processes without a pre-existent space-time plenum? I have a simple suggestion for the stator of a q plexus that I call a *plexor*. A plexor may be represented by a tensor whose indices are not arranged in a line but laid on the cells of an (oriented-cell) complex. If the complex is of dimension n the plexor is called an n-plexor. The plexor is an elementary generalization of a tensor, a Feynman-diagram amplitude, and a chain in a complex, and can be approached from each of these sides:

Tensors are 0-plexors or specially simple 1-plexors, those whose complex is a cell decomposition of a line; general 1-plexors describe q interactions; and there are yet more general n-plexors whose utility I do not quite foresee.

Plexors are direct descendants of *Feynman-diagram amplitudes*. The Feynman amplitudes have a rich c substructure, the relativistic space-time continuum and the c momentum space which is its Fourier image, while plexors are pure q entities. But the lines in the complex still represent propagation processes and the vertices still represent interaction processes.

A *chain* is a limiting case of a plexor in which we first go from q theory to a c limit and then from finite to infinitesimal quantities. The first limit replaces noncommutative by commutative operators, the second replaces multiplicative by additive combinations (Secs. IV–VII).

A previous model, a q binary dynamics,[1] is an example of a pure q theory whose stators are merely tensors. The generalization from free to interaction processes leads from tensors to plexors. I suggest world stators are plexors. I show here how space-time geometrical relations among processes first arise after a plexor has been assigned to them, and are drawn from that plexor. In the process theory "empty space-time" is a nonsense phrase. Since a system does not have a stator, the world does not have a plexor, and therefore I expect the world too does not have a preassigned space-time geometry. But a stator for a system approximates a physical reality for some ensembles of many similar such systems and likewise for a plexor and a space-time geometry. My starting point is not exactly Mach's, but I seem to swing unexpectedly close to his principle.

II. STATORS

A physical theory has to do with physical facts and with mathematical or linguistic symbols for them. The link between them, meaning, may at first be gotten across by common speech. Ultimately, if we are pressed to define our terms again and again, or if there is a block to understanding, we are reduced to physical demonstration, pointing, touching.

To communicate a physical theory it suffices to convey a *classification scheme*, i.e., to point out and name the classes in which the system of the theory may enjoy membership, and, as the laws of the theory, the logical relations of inclusion and exclusion among these classes. A class, to which corresponds a yes-no question, "Is the system of that class?" and a proposition, "The system is of that class," may in critical cases be described *operationally* by exhibiting how we control membership in that class. Let us call a device regulating the system in advance a *channel*[4]; one determining membership after the fact, a *cochannel*. The operational expression of a class inclusion $A\subset B$ is that whenever an A channel is connected to a B cochannel the outcome is affirmative; of a class exclusion $A\perp B$, negative.

A *quantity* (with values in a set V) is defined when for each set of values (in V) is given a class, membership in which is equivalent to the quantity taking on a value in that set. To define a quantity operationally, we show how to control the values of the quantity. We may say *the quantity exists*, or is possessed by the system, if the classes $P(v)$, as v ranges over V, are exhaustive (existence) and mutually exclusive (uniqueness, implied by *the*). (We need not consider here the technical modifications of continuous spectra.[5])

In c physics, the empirical class calculus of the system is represented by the class calculus of an auxiliary set we assign to the system, its phase space. It then follows that each of the quantities possessed by the c system corresponds to a function on the phase-space (into the set of values) in such a way that a c eigenvalue principle and expectation-value formula hold. A proposition concerning the c system is represented by a c projection, a real quantity taking on only the values 0, 1. We may agree to express any information about the system by such a projection. Then when we change our information about the system, we must replace the projection by a new one. This is just a consequence of the meaning we give the projection.

To each point p of phase space corresponds a projection $P(p)$ and a class. Since these classes are exhaustive and mutually exclusive, the point in phase space exists, or is possessed by the system. On the other hand, the totality of all projection in phase space, while exhaustive, is not mutually exclusive. Therefore we cannot say the general projection exists or is possessed by the system.

In q physics, according to von Neumann and many others, the empirical class calculus of the system is represented by the subspace calculus of an auxiliary Hilbert space we assign to the system. It then follows that each of the complex quantities possessed by the system corresponds to a normal operator on the Hilbert space in such a way that the eigenvalue principle and the expectation-value formulas of quantum physics hold. A proposition concerning the system is represented by a q projection, a Hermitian operator with eigenvalues 0, 1.

By a quantum or a q system I mean such a system.

By a *stator* I mean a vector in the Hilbert space we associate with a q system with the rule of interpretation that classes correspond to subspaces, class inclusion to subspace inclusion, and class exclusion to subspace orthogonality. Each stator ψ determines a projection $P(\psi)$ and thus a class of the system.

We may agree to express information about the q system by a projection. Then when we change information about the system we change the projection. This is just a consequence of the meaning we give the projection. If our projections are determined by stators, we will have replaced one stator by another.

Stator projections are exhaustive but not mutually exclusive. Therefore we cannot say the stator ψ exists or is possessed by the system.

For each stator ψ the proposition "The system is in the class $P(\psi)$" or "$P(\psi)$ is 1" is meaningful and might be abbreviated "ψ!" Stators are names of qualities, adjectives. In this respect they resemble cells of c phase space more than points.

The question "What is *the* stator of the system?" is not of the form "What is the Q of the system?" for any quantity Q possessed by the system. It is ill-formed, and so also are references to the evolution or the collapse of *the* stator of a system.

References to the (1) evolution and (2) collapse of state vectors usually should be replaced by references to (1) the laws of evolution of the quantities of the system and (2) the rules of interpretation that assign particular stators to particular facts, classes, or controls.

There is the peculiar notion that the system does not always possess a position, energy, and so on, but only when we look for one. I think it this notion that led some to foist a ψ vector upon the deprived system by way of compensation. I call this notion peculiar because every time we look for a position, energy, and so on, we find one, and in common sense this would be reasonable proof that these quantities are always there, like the trees in the unseen woods. What ever led anyone to give up this basic rule of interpretation?

What seems to be responsible for abandoning this part of common sense is the simultaneous truth of the following three statements about position classes P_x, momentum classes P_p, the universal class I, and the null class $\varnothing$:

(a) There is always some position x: $\bigcup_x P_x = \mathrm{I}$.
(b) There is always some momentum p: $\bigcup_p P_p = \mathrm{I}$.
(c) Having a position x *and* a momentum p is impossible: $P_x \cap P_p = \varnothing$.

Carefully read, all three describe operational realities and are certainly noncontradictory. But there is a clear operational violation of the distributive law, or there would be the contradiction

$$\varnothing = \bigcup_{x,p} P_x \cap P_p = (\bigcup_x P_x) \cap (\bigcup_p P_p) = \mathrm{I} \cap \mathrm{I} = \mathrm{I}.$$

If the operational basis of the distributive law is not explicit then it is not easy to abandon this law as we learn more about the world. Then we are apt to infer from statement (c) that (a) or (b) must go.

To seek a theory without entities like stators that are replaced as we learn about the world is to seek a theory without rules of interpretation, without communication. To accept but attribute reality to the replaceable entities in the theory is to confuse the level of names with the level of things. To suppose the world must conform to the patterns of thought that served man in the past, and in particular to those we have called logic, is to ignore the lessons of relativity.

III. A STATOR NOTATION

The best notation for the theory of a simple pure q system is that of Dirac, who designates vectors by $|\,\rangle$, dual vectors by $\langle\,|$, inner products by $\langle\,|\,\rangle$, and outer products by $|\,\rangle\langle\,|$. But the best notation for the cq dynamics of interactions is that of Feynman, a network notation. Here I mix the Dirac and Feynman notations to make a network for pure q dynamics of interactions.

We hold on to the sign of Dirac for a stator, $|\,\rangle$, but in order later to make networks of stators we first set $|\,\rangle$ free from the printer's line. $|\,\rangle$ can now jump about as in Fig. 1. So $\langle\,|$ also must designate a stator for us, not a dual stator. Instead, thinking of $|\,\rangle$ as the head of an arrow, we depict a dual stator as the tail of an arrow: $\rangle\,|$. This too may be written anywhere in any way. $|\,\langle$ too is a dual stator. Evidently $\langle\,|$ and $|\,\langle$ are not bra and ket but *arr* and *row*. If an *arr* is a column then a *row* is a row.

The inner product of $|\,\rangle$ and $\rangle\,|$ is a complex number that should be $|\,\rangle\,|$ because a complex number needs no "polarity." The outer product of $|\,\rangle$ and $\rangle\,|$ is an operator and should be written $\rangle\,|\,\rangle$, a complete arrow, to indicate how it acts upon stators $|\,\rangle$ and upon dual stators $\rangle\,|$. The general linear operator is $\rangle\,\rangle$. The effect of a linear operator $\langle A \langle$ on a vector $\langle B|$ is $\langle A \langle B|$ or $|B\rangle A\rangle$. These symbols too may be written anywhere in any way.

Now we consider compound systems. If $|1\rangle$ and $|2\rangle$ are stators for systems No. 1 and 2, then in the Dirac linear script an outer-product stator describing the pair system 1×2 is written $|1\rangle|2\rangle$ and its conjugate is $\langle 1|\langle 2|$ or $\langle 2|\langle 1|$. This makes it impossible to bring together the factors which should be together in the inner product $|1\rangle|2\rangle\langle 1|\langle 2|$ (or $|1\rangle|2\rangle\langle 2|\langle 1|$). In the arrow notation we designate a product stator describing the compound system by

$$\begin{matrix}|1\rangle \\ |2\rangle\end{matrix} \quad \text{or} \quad \begin{matrix}\langle 1| \\ \langle 2|\,.\end{matrix}$$

The dual product vectors are written

$$\begin{matrix}\rangle 1| \\ \rangle 2|\end{matrix} \quad \text{or} \quad \begin{matrix}|1\langle \\ |2\langle\,.\end{matrix}$$

The inner products are then the juxtapositions

$$\begin{matrix}|1\rangle 1| \\ |2\rangle 2|\end{matrix} \quad \text{or} \quad \begin{matrix}|1\langle 1| \\ |2\langle 2|,\end{matrix}$$

and the outer products are

$$\begin{matrix}\rangle 1|1\rangle \\ \rangle 2|2\rangle\end{matrix} \quad \text{or} \quad \begin{matrix}\langle 1|1\langle \\ \langle 2|2\langle\,.\end{matrix}$$

For general (nonproduct) vectors or operators of the product system, we drop the multiplication sign. Such vectors are drawn

$$\left|\begin{matrix}1\rangle \\ 2\rangle\,.\end{matrix}\right.$$

Such operators are shown as

$$\begin{matrix}\rangle \\ \rangle\end{matrix}\left|\begin{matrix}\rangle \\ \rangle\,.\end{matrix}\right.$$

Now we take up complex systems. The product stator describing a *sequence* of similar systems is

$$\begin{matrix}|\,\rangle \\ \vdots \\ |\,\rangle\,.\end{matrix}$$

The product dual stator of the sequence is drawn

$$\begin{matrix}|\,\langle \\ \vdots \\ |\,\langle\,.\end{matrix}$$

For general stators and dual stators the bars are united:

$$\left|\begin{matrix}\rangle \\ \vdots \\ \rangle\,.\end{matrix}\right.$$

I think of these drawings as the two halves of a zipper. In the outer product the zipper is open. When we form an inner product we close the zipper. The outer and inner products, with a partial inner product between them, look like

$$\left|\begin{matrix}\rangle & \rangle \\ \rangle & \rangle \\ \rangle & \rangle \\ \rangle & \rangle\end{matrix}\right|, \qquad \left|\begin{matrix}\rangle| & \\ \rangle| & \\ \rangle & \rangle| \\ \rangle & \rangle|\end{matrix}\right., \qquad \left|\begin{matrix}\rangle \\ \rangle \\ \rangle \\ \rangle\end{matrix}\right|.$$

It is natural to regard the dual stator $\rangle\,|$ as stator of another kind of system, an antisystem. The linear operator $\rangle\,|\,\rangle$ is then stator of a system-antisystem pair. The inner product $|\,\rangle\,|$ is the amplitude for a system-antisystem annihilation leaving no system at all. Thus complex numbers too may be regarded as stators, for the vacuum.

So far no concepts peculiar to dynamics have been introduced. The notation expresses concepts of q logic alone. The order of factors suggested in the product is spurious; a cyclic rendering like Fig. 2 is an equally valid picture.

Now we take up q kinematics and chronology.

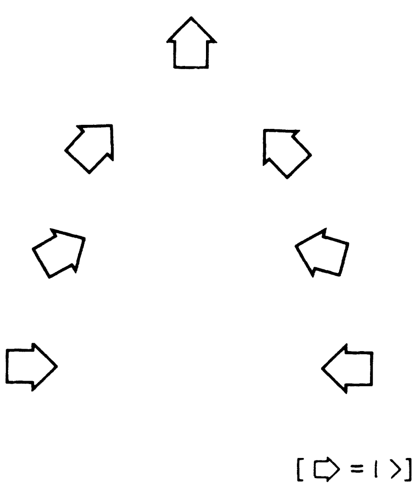

FIG. 1. Somersault of a stator symbol.

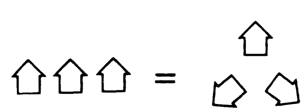

FIG. 2. Unordered tensor product.

IV. CAUSAL NETWORKS

A q dynamics is a system of (kinematic) processes Π, a binary composition $\Pi_2 \odot \Pi_1$, and a class D of dynamical (allowed) processes. In one dynamics, that of the single free "nuon," the process is a q sequence of binary decisions, and systems are neither created nor destroyed. Such processes can be represented, informally at first, by the pictures (with labels shown)

$$\rightarrow \quad \rightarrow\rightarrow \quad \rightarrow\rightarrow\rightarrow \quad \cdots,$$

$$\chi \quad \chi\odot\chi \quad \chi\odot\chi\odot\chi \quad \cdots,$$

showing one, two, three, $\cdots$ chronons χ in sequence. I suppose the above diagrams also describe the kinematics of many free nuons, except that in them the generator $\rightarrow$ is to be replaced at every appearance by one of the *parallel products* (still informal)

$$\rightarrow \quad \substack{\rightarrow\\ \rightarrow} \quad \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \quad \cdots,$$

$$\chi \quad \chi\obar\chi \quad \chi\obar\chi\obar\chi \quad \cdots,$$

which advance an assembly of 1, 2, 3, ... nuons forward one chronon. Using the sequential product $\odot$ and the parallel $\obar$ we can make series-parallel circuits. It is natural to describe interactions by networks.

We suppose the interactions are the result of processes of creation and annihilation represented, still informally, by pictures such as Fig. 3, and $\odot$ products thereof. We can call these processes, Y, X, etc. after their shape. Processes of this generalized kind no longer will fall within the scheme $[\Pi, \odot, D]$ because the product is not unique. If Π_2 is the X process and Π_1 is $\chi\odot\chi$, then two products $\Pi_2 \odot \Pi = \Pi_3$ are defined by joining lines out of Π_1 into Π_2 in two possible ways.

By a *network* we mean a collection of points and a collection of arrows between them. All the pictures we have drawn here are causal networks, in which the arrows represent the causal relations; and the points events. We now bring together the causal and quantum concepts.

V. PLEXORS

We often put into the tensor product as element of structure a concept of order that is irrelevant and ignored. We go from tensors to plexors by generalizing this order, so we must first make it fully explicit. Consider the tensor product of three linear spaces L, M, N:

$$T = LMN$$
$$= (L(MN)).$$

There is a first factor N, a second factor M, and a third factor L. If the three factors are replicas of L we may even number them to show this order:

$$T = L_3 L_2 L_1.$$

Sometimes this order is relevant. In the q theory of three hard-core particles on a line, the order of the factors may reflect the order of the particles. In the nuon model, the order of factors represents the chronological order of elementary processes.

More often the order of factors is irrelevant. When we multiply a position stator ψ and a spin stator χ to describe a spinning particle, we do not really want to call the position stator the first factor or the second factor either. It is enough that the two factors can be told apart. Only the linear notation leads us to arbitrarily put one factor first, and perhaps to feel that some factor has to come first, whether we like it or not. It is important to realize that a product is possible in which the factors are assigned no order at all.

It is not easy to write a product without introducing some extraneous order, but the mathematical concept is clear. A vector in the usual tensor product LMN is a formal sum of *ordered sequences* (ψ, φ, χ) of three vectors ψ, φ, χ, one from each of the factors L, M, N in that order. We may instead consider a formal sum of *unordered sets* of three vectors ψ, φ, χ, with ψ in L, φ in M, and χ in N. If the spaces L, M, N do not overlap then the three vectors ψ, φ, χ in the set cannot be confused. If they overlap confusion can be avoided without introducing an extraneous order. Individuality does not imply order. Instead of numbering three replicas of a linear space L as L_1, L_2, L_3 we can distinguish them with unordered labels like $L_\Diamond$, $L_\Box$, $L_\bigcirc$. If we write or read in the usual linear way, one factor is going to appear first, but this linear order is an artifact. If we cannot shut this order out of our minds, we can have the three factors read out simultaneously in three different voices, or printed on the same space with three colors of ink.

To make the order in the tensor product explicit, I think of the product $L_3 L_2 L_1$ as formed in two steps. First we define an order through a network like

$$\rightarrow\rightarrow\rightarrow.$$

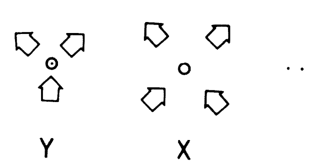

FIG. 3. Y and X process.

Then we put vectors ψ, φ, χ on each arrow in the diagram. (Putting them on the arrows seems arbitrary but is more useful in my immediate application than putting the vectors on the vertices in the diagram. When we generalize there will be vectors everywhere.)

$$\underset{\rightarrow}{\psi}\ \underset{\rightarrow}{\varphi}\ \underset{\rightarrow}{\chi}$$

This is the vector in the ordered tensor product usually written $\chi\varphi\psi$. Because the order is explicit the same vector may be written

$$\underset{\leftarrow}{\chi}\ \underset{\leftarrow}{\varphi}\ \underset{\leftarrow}{\psi} \quad \text{or} \quad \begin{matrix}\downarrow\psi\\ \downarrow\varphi\,.\\ \downarrow\chi\end{matrix}$$

We ignore the accidental visual order and follow the arrows. With this agreement, the totally unordered kind of product can be written without recourse to unusual symbols or colored inks as

$$\psi\uparrow \quad \varphi\uparrow \quad \chi\uparrow$$

if the vectors ψ, φ, χ are from different spaces. If the vectors are from the same space we label them to tell them apart, and since now the only order that counts is the explicit one of the arrows we can use letters a, b, c or even numbers, 1, 2, 3 as marks of individuation without paying attention to their order.

This unordered tensor product has the same dimension as the ordered tensor product LMN. There are 1-1 linear-space isomorphisms between it and LMN, though no natural one. All this product lacks is a certain element of structure, the order. We no longer have the right to say that one of the factor spaces L or M or N is first, one second, and one third. We can only address them by their names.

We have dissolved the irrelevant order. Now we can establish a more relevant order.[2] The relevant order may be of a more general kind, such as a partial order (for example a Y) or a network (for example an A). These are not linear orders but they are still one-dimensional structures. The higher-dimensional structure of an oriented-cell complex of dimension n is a still more general kind of order. For now let us stop at that degree of generality; and I am not even sure $n>1$ will be at all useful. It is the causal order we wish to represent finally.

A product of vectors in a simple or linear order is an ordered tensor; a plexor will be a product of vectors in a more general order. For clarity I parallel their definitions.

An ordered tensor space

$$T=L_1L_2\cdots L_n=\prod_{m=1}^{n}L_m$$

is defined by a *sequence* of linear spaces $L_1, L_2, \ldots, L_n$. A plexor space

$$P=\prod_{\sigma\in K}L_\sigma$$

is defined by a *complex* of linear spaces $\{L_\sigma\}$, that is, an oriented-cell complex $K=\{\sigma\}$ with a distinct linear space L_σ given for each cell σ in K.

An ordered product tensor $t=\psi_1\psi_2\cdots\psi_n$ in $L_1L_2\cdots L_n$ is defined by a *sequence* of vectors $\psi_1, \psi_2, \ldots, \psi_n$ with ψ_i in L_i.

A product plexor (over a complex K)

$$\pi=\prod_{\sigma\in K}\psi_\sigma$$

is defined by giving a *complex* of vectors ψ_σ, that is, a vector ψ_σ in L_σ for each cell σ of K.

The general ordered tensor of $L_1L_2\cdots L_n$ (plexor of $\prod_\sigma L_\sigma$) is a linear combination of ordered product tensors (plexors) with identifications expressing linearity of the product in each of the factor vectors.

Let us confine ourselves to the case where all the linear spaces L_σ belonging to cells of the same dimension n are replicas of one linear space L_n. A vector in L_σ can be regarded as a formal product of the cell σ of dimension n and a vector in L_n. For the sake of the printer I also designate the plexor space

$$\prod_{\sigma\in K}L_\sigma$$

simply by L^K or $L(K)$ for short. If the underlying complex K is a line complex ($\cdot$ or $\rightarrow$ or $\rightarrow\rightarrow$ or $\cdots$) the plexors over K are called *linear*. Ordinary tensors are either 0-plexors (no order) or linear plexors (linear order).

VI. PLEXOR NOTATION

A notation for plexors follows naturally from that for stators. First we draw the cell complex K. Then we show the orientation of K with arrs. Then for many purposes we are done. We do not need the base symbol of the tensor, its proper name, because context reveals what object we are talking about. We have already drawn the stators on the complex, for the arrs that give the orientation can be read as stator symbols as well. We may infer the linear spaces L_σ from which the stators are taken from the context. Usually all stators range freely over their linear spaces independently so no labels are needed. When some stators are fixed or related to others, we add labels to show this.

Linear plexors, for example, I write as one of the possibilities

$\cdot, \; |\rangle, \; |\rangle|\rangle, \; |\rangle|\rangle|\rangle, \; \ldots,$

the generic one of which I write as

$|\rangle \cdots |\rangle .$

Here the linear space attached to the vertices is C, the complex plane, which serves as an identity in this calculus and can usually be ignored. Therefore I have not shown the vertices, only the lines joining them. The linear space on the lines has to be given separately. Each arr $|\rangle$ stands for a vector in this linear space.

When two consecutive processes described by a plexor are identical (always in the Bose-Einstein sense) the plexor is to be symmetric with respect to the exchange of the corresponding stators. I show this symmetry by a colon regarded as a degenerate equality sign:

$|\rangle : |\rangle : \cdots |\rangle .$

The free-nuon process is described by a linear plexor of this symmetric kind.

VII. PLEXOR ALGEBRA

Plexors over a single complex are added and multiplied by complex numbers just like tensors.

When the linear spaces L_σ entering into a plexor product are provided with conjugation or * operations, assigning to each vector ψ a dual vector ψ^* depending antilinearly on ψ, we provide the plexor product with a * operation as well. We set $(L(K))^* = L^*(K^*)$, where the spaces L_σ^* are the duals to the spaces L_σ, and K^* is the complex K with all orientations reversed. For example we now conjugate the ordered tensor product of two stators thus:

$$(|x\rangle\,|y\rangle)^* = |x\langle\,|y\langle = \rangle y|\,\rangle x| ,$$

where we have used the stator notation already set up:

$$|x\rangle^* = |x\langle ,$$

$$|y\rangle^* = |y\langle| .$$

Spaces, plexors, and so on provided with such a * operation I call * spaces, * plexors, and so on. Stators are * vectors, for example.

We may multiply two product plexors $P = \prod \psi$ and $P' = \prod \psi'$ of $L(K)$ and $L'(K')$ by forming the disjoint union of their complexes:

$$K'' = K + K' ,$$

keeping fixed the vectors associated with the cells

$$\psi''_\sigma = \psi_\sigma \quad \text{for } \sigma \text{ in } K$$

$$= \psi'_\sigma \quad \text{for } \sigma \text{ in } K' .$$

The general plexors are then multiplied by linearity. We call this plexor (and its plexor space) the direct product of the given plexors (and their plexor spaces), and write it as

$$L''(K'') = L(K)L'(K') = L'(K')L(K) .$$

There are many other plexor products as well. Every way of connecting the two complexes K, K' into one K'' defines a product of plexors over K, K'. For example, linear plexors can be multiplied in *sequence*,

$$|a\rangle\,|b\rangle \odot |m\rangle\,|n\rangle = |a\rangle\,|b\rangle\,|m\rangle\,|n\rangle ,$$

a noncommutative product for which I reserve the multiplication sign $\odot$, or in *parallel*,

$$|a\rangle\,|b\rangle \oplus |m\rangle\,|n\rangle = \begin{matrix} |a\rangle\,|b\rangle \\ |m\rangle\,|n\rangle \end{matrix} ,$$

a commutative product for which we use the multiplication sign $\oplus$. The parallel product is merely the direct product specialized to linear plexors.

Rather than spell out the meaningful intrinsic relations among plexors one by one, we may attempt to characterize them all at once by their invariance transformations, the morphisms of the category of plexor spaces, or the *plexor morphisms*.

A plexor morphism m: $L(K) \to L'(K')$ is a cell-complex morphism m: $K \to K'$, and for each cell σ in K, a linear transformation (preserving inner products in the case of * plexors) m: $L_\sigma \to L'_{\sigma'}$, where $\sigma' = m\sigma$. The cell-complex morphism m maps each point (0-cell) σ_0 of K into a point σ_0' of K', $\sigma_0' = m\sigma_0$, with points forming a +-oriented cell of K mapping into points forming a +-oriented cell of K'.

This definition of plexor morphism is tentative, and may be modified as we see more closely what structure is useful to express physical laws. I give one such relation to reality next, to show the intertwining of the quantum and the plexus in quantum geometry.

VIII. PLEXOR GEOMETRY

Let us go to free dynamics for an example of how a description of a dynamical process leads to a space-time structure. Now the linear space underlying our plexors is the two-dimensional * space whose vectors are two-component spinors describing a basic dichotomy or binary process. I picture the c analog of the process as that undergone by a man in the game of checkers: forward to the right or forward to the left.[3]

A linear plexor like

$$|1\rangle\;|2\rangle\;|3\rangle \tag{1}$$

describes three binary processes or chronons in sequence. Each has a Pauli algebra of 2×2 matrices for its quantities, and these algebras may be taken as commuting within the algebra of quantities of the triple sequence, provided we indicate the causal order explicitly as in $\sigma^\mu(1)$, $\sigma^\mu(2)$, $\sigma^\mu(3)$. ($\mu=0,1,2,3$ with $\sigma^0=1$ labels the usual Hermitian basis for the Pauli algebra.)

A class of chronons is defined by a projection operator n in the Pauli algebra of L, $n^2=n=n^*$. The σ^μ form a basis for the n's: $n=n_\mu\sigma^\mu$. We identify these n's with (null) directions. Each such one-chronon projection n has a many-process counterpart, a coordinate $x(n)$ giving the number of chronons of the kind n. The four space-time coordinates x^μ for a process are only a convenient choice of basis operators for the expression of the coordinate $x(n)$ for all n:

$$x(n)=x(n_\mu\sigma^\mu)=\sum n_\mu x(\sigma^\mu)$$
$$=\sum n_\mu x^\mu.$$

When we express a single-chronon projection n in terms of the operators σ^μ we get its coordinates n_μ, real numbers. When we express the many-chronon coordinate operator $x(n)$ in terms of the n_μ, we get the usual coordinates x^μ of the composite process. From the plexor (1) we infer, for example,

$$x^\mu=\sigma^\mu(1)+\sigma^\mu(2)+\sigma^\mu(3)$$

for the coordinate operators of the compound process. This is taken as describing the space-time displacement undergone in the process. The time of the process according to this is the number of chronons in it. For this reason I have called the elementary q process in this model a chronon.

For two free nuons, a plexor like

$$|1\rangle\ |2\rangle\ |3\rangle$$
$$|4\rangle\ |5\rangle\ |6\rangle$$

is the stator of a kinematically possible process. From the order structure of this plexor we see there are two nuons ν, ν' and that they undergo space-time displacements x^μ, x'^μ, respectively, given by

$$x^\mu=\sigma^\mu(1)+\sigma^\mu(2)+\sigma^\mu(3)\,,$$
$$x'^\mu=\sigma^\mu(4)+\sigma^\mu(5)+\sigma^\mu(6)\,.$$

Nothing can be said about the coordinate of ν relative to ν'. The two nuons might as well be in separate universes. Unless they are linked by interactions, there is no meaning to their relative coordinate.

For two interacting nuons, a plexor like Fig. 4 is a kinematically possible stator. We read from it that there are two nuons ν_1, ν_2 in the initial and final states of this process, that the initial relative coordinate is

$$x^\mu(\nu_1\to\nu_2)=\sigma^\mu(1)-\sigma^\mu(4)\,,$$

that the final relative coordinate (with an arbitrary labeling of one of the final nuons as ν_1', the other ν_2') is

$$x^\mu(\nu_1'\to\nu_2')=-\sigma^\mu(6)-\sigma^\mu(5)+\sigma^\mu(2)+\sigma^\mu(3)\,,$$

and that one set of coordinates of a final nuon relative to an initial one is

$$x^\mu(\nu_1\to\nu_2')=\sigma^\mu(1)+\sigma^\mu(2)+\sigma^\mu(3)\,.$$

When the stators $|1\rangle,\ldots,|6\rangle$ are known, these operators all acquire expectation values, and we can speak of c-number coordinates.

The familiar causal structure of the c space-time of special relativity emerges in the c limit from the fact that the coordinates of long, linear plexors always have expectation values within the null cone. I have already given a dynamical law for nuons that makes these expectation values lie on the null cone, and gives the free nuon mass zero. It is that in each segment of a plexor of the form $|\,\rangle\ \ |\,\rangle$ the two chronons involved are to be identical in the sense of Bose-Einstein statistics. This may be suggestively regarded as a degenerate case of a law akin to Kirchhoff's: The chronons going toward a vertex (the middle vertex here) in some sense add up to those going away from it,

$$|\,\rangle : |\,\rangle\,.$$

Graham Frye and I are presently studying extensions of this law.

IX. THE IDEA OF THE PROCESS

Looking at a cq process in ordinary quantum mechanics helps us understand the q concept and establish its correspondence limit of classical time ($\tau\to0$). Imagine that classical time is not

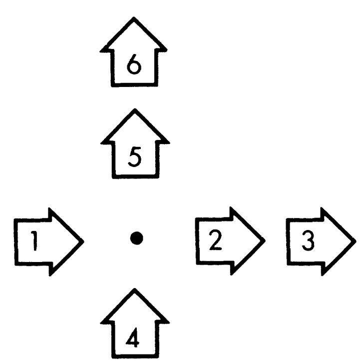

FIG. 4. Plexor of an interaction.

quantized but merely atomized,

$$t=n\Delta t.$$

Then energy is not a useful concept and we take up instead the unitary operator $U=\langle U\langle$ that advances the system $\langle\ |$'s from t to $t+\Delta t$. Call the system S.

There are two ways to think of a unitary operator like U: (i) U is a mapping of stators. This is old. (ii) U is a stator in its own right. This is the way important for the present work. The symbol $\langle U\langle$ shows U as a stator of a pair made of an S (described by a $\langle\ |$) and a dual S^T (described by a dual $|\ \langle$). The system of which U is a stator I call the primitive process of S. In this simple example, the primitive process P is merely a pair,

$$P=SS^T.$$

The coordinates of such a process are simply those of the system, those of the dual system, and their algebraic combinations.

Similar statements hold for the process of a c system as well, though there the distinction between S and S^T is vacuous and the primitive process is simply an ordered pair of systems, the initial one and the final one. Now the transformation formula for stators

$$\langle 2|=\langle U\langle 1|$$

acquires a purely logical meaning. When a pair comes from a channel with stator U, and one member of the pair passes into the cochannel with stator $\langle 1|$, the second member acts as if it came from the channel with stator $\langle 2|$ given by this formula, a partial inner product of a pair stator and a one-system stator. This is the way we would compute the stator of a proton produced from the ground state of the deuteron by a measurement upon the neutron.

The composite process π in the absence of interactions, or briefly the free process, is simply a sequence of primitive processes of unspecified length. Its stators are therefore objects of the kind

$$\langle\pi|=U\odot U\odot\cdots\odot U,$$

the tensor (outer, direct, or uncontracted) product of several replicas of U. This is a stator of several pairs. If we represent U by a matrix $|m\langle U\langle n|$, this induces a representation of the stator $|\pi\rangle$ as a product of matrices

$$|m_1\langle U\langle n_1|\ |m_2\langle U\langle n_2|\cdots|m_p\langle U\langle n_p|.$$

If successive indices are set equal,

$$n_1=m_2,\ n_2=m_3,\ \ldots,$$

the resulting product

$$|m_1\langle U\langle m_2|\ |m_2\langle U\langle m_3|\cdots|m_p\langle U\langle n_p|$$

is the Feynman amplitude for the process defined by the sequence of eigenvalues $m_1,\ m_2,\ldots,m_p,\ n_p$. Thus the Feynman amplitude represents a part of the geometrical object I take as central here, the process stator. Setting successive indices equal destroys the transformation properties. The process stator is a geometric object under the unitary group of S, although the Feynman amplitude is not.

I now use this conceptual apparatus to demand of the q theory that its process stators give correct Feynman amplitudes in the cq correspondence limit $t\to 0$. To construct the Feynman amplitude we require the plexor structure and not merely the stator. We must equate causally consecutive indices of the primitive processes. Then the formation of the S matrix from the Feynman amplitudes is to be carried out as usual.

---

*Supported in part by the National Science Foundation and the Young Men's Philanthropic League.

[1]The physical basis for the formal developments of this paper is given in David Finkelstein, Phys. Rev. D 5, 320 (1972), and works cited therein.

[2]The homology used here recalls David Bohm's insistence on homology as a language for the formulation of basic theory. Bohm's influence on this work, both personal and through George Yevick, is gratefully acknowledged.

[3]This chronon is practically the *Urobjekt* of C. F. von Weizsäcker. See his article in *Beyond Quantum Theory*, edited by E. Bastin (Cambridge, 1970). I am indebted to him for stimulating and helpful discussions of these ideas.

[4]John M. Blatt and Victor F. Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952).

[5]Quantum logic is treated also in the beautiful work of V. S. Varadarajan, *Geometry of Quantum Theory* (Van Nostrand, Princeton, N. J., 1968).