PHYSICAL REVIEW D VOLUME 5, NUMBER 12 15 JUNE 1972

Anomalous Motion of Radiating Particles in Strong Fields

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(Received 15 July 1971; revised manuscript received 20 March 1972)

The anomalous motion of radiatively damped particles in a uniform magnetic field is analyzed. A particularly simple method of solution is presented, and some possible astrophysical consequences of the motion are briefly discussed.

I. INTRODUCTION

The radiative damping force acting on a particle moving in an electromagnetic field is usually negligible compared to the Lorentz force. For a relativistic particle, the ratio of the forces is $\gamma^2\omega/\omega_0$, where γ is the Lorentz factor, $\omega = eB/mc^2$ is the Larmor frequency, and $\omega_0 = 3mc^3/2e^2 \simeq 10^{23}\ \text{sec}^{-1}$.[1] For an electron, $\gamma^2\omega/\omega_0 \simeq 10^{-15}\gamma^2 B$, with B in gauss. However, the recent discoveries of pulsars, with fields that may be as strong as 10^{12} G, and the possibility that the pulsars may be a significant source of high-energy particles,[2,3] presents a situation where radiative reaction may have a significant effect on the motion of particles (i.e., $\gamma^2 B > 10^{15}$ G).

Shen[4] has considered radiative damping with respect to the spectrum of synchrotron radiation emitted; we will be concerned here solely with the specific motion of the particles, and the possible significance to astrophysical problems. We consider the basic problem of the motion of a radiatively damped highly relativistic particle moving in a uniform magnetic field. (Gunn and Ostriker[5] have also investigated radiatively damped motion for the case of a pulsar's far-field radiation zone.) We introduce an extremely simple method of solution, which may also be useful in solving other similar problems. This does not seem to have been presented before.

II. MOTION IN A UNIFORM MAGNETIC FIELD

The general covariant equation of motion of a particle in an electromagnetic field with radiative damping is[1]

$$\dot{U}_i = \frac{e}{mc}F_{ik}U^k + \frac{1}{\omega_0}\left(\ddot{U}_i - \frac{1}{c^2}U_i\dot{U}^j\dot{U}_j\right), \tag{1}$$

where the dot indicates differentiation with respect to proper time τ, and F_{ik} is the electromagnetic field tensor with $i = 1$ to 4.

For $\vec{B} = (0, 0, B)$, $\vec{E} = 0$, the x and y ($i = 1$ and 2) equations become

$$\dot{U}_1 = \omega U_2 + \omega_0^{-1}(\ddot{U}_1 - U_1\dot{U}^j\dot{U}_j), \tag{2}$$

$$\dot{U}_2 = -\omega U_1 + \omega_0^{-1}(\ddot{U}_2 - U_2\dot{U}^j\dot{U}_j), \tag{3}$$

where $\omega \equiv eB/mc$. $\dot{U}^j\dot{U}_j$ is a scalar; therefore we can go into the rest frame of the particle $[U_j = (0, 0, 0, c)]$ and immediately find to lowest order in $\gamma\omega/\omega_0$, which is sufficient for our analysis,

$$\dot{U}^j\dot{U}_j = \gamma^2 c^2\omega^2 \sin^2\phi, \tag{4}$$

where ϕ is the angle between the three-velocity of the particle ($\vec{v} = d\vec{x}/dt$) and $\vec{B}$.

We can readily find solutions for both $\dot{\gamma}$ and the motion along z: We assume that v_z = constant and that γ has the form $\gamma = \gamma_0/(1+\alpha t)$. With the aid of Eqs. (1) and (4), we find

$$\alpha = \frac{\gamma_0\omega^2(\sin^2\phi_0)}{\omega_0}$$

and (5)

$$\gamma = \frac{\gamma_0}{1+\gamma_0\omega^2(\sin^2\phi_0)t/\omega_0}.$$

To solve for the motion in the x-y plane, we multiply Eq. (3) by i, add it to Eq. (2), and, defining $U = U_1 + iU_2$, we have

$$\dot{U} = -i\omega U + \omega_0^{-1}(\ddot{U} - \dot{U}^j\dot{U}_j U). \tag{6}$$

We try a solution for v ($U_i = dx_i/d\tau = \gamma dx_i/dt \equiv \gamma v_i$, $\dot{U}_i = \cdots$) of the form

$$v = e^{-i(1+At)Dt}, \tag{7}$$

where A and D are constants to be determined. Inserting Eq. (7) into Eq. (6) and separating the real and imaginary parts, we find for the imaginary part

$$-\gamma(2ADt + D) + \omega - \omega_0^{-1}[-3\gamma\dot{\gamma}(2ADt + D) - \gamma^2 2AD] = 0, \tag{8}$$

where a dot now indicates differentiation with respect to t. Let

$$D = \frac{\omega}{\gamma_0}, \tag{9}$$

and, using Eq. (5), Eq. (8) gives, to lowest order,

$$A=\frac{\gamma_0\omega^2\sin^2\phi_0}{2\omega_0}\,. \qquad (10)$$

Combining Eqs. (7), (9), and (10), the complete solution for the motion of a radiating particle in a constant uniform field, $\vec{B}=(0,0,B)$, can be written as

$$v_x=v_0\sin\phi_0\cos G(t)\,, \qquad (11)$$

$$v_y=v_0\sin\phi_0\sin G(t)\,, \qquad (12)$$

$$v_z=v_0\cos\phi_0\,, \qquad (13)$$

where

$$G=\left(1+\frac{\gamma_0\omega^2\sin^2\phi_0}{2\omega_0}t\right)\left(\frac{\omega}{\gamma_0}t\right). \qquad (14)$$

We wish to solve Eqs. (11) and (12) for the motion of a strongly radiating particle in the plane perpendicular to $\vec{B}$. If we let $V=v_0\sin\phi_0$, and

$$z=\sqrt{AD}\,(t+1/2A)\,,$$

Eq. (11) becomes

$$x(t)=V\int_0^t\cos(1+At)Dt\,dt$$

$$=\frac{V}{\sqrt{AD}}\int_{\sqrt{D/4A}}^{\sqrt{AD}(t+1/2A)}\cos(z^2-D/4A)\,dz$$

$$=\frac{V}{\sqrt{AD}}(\tfrac{1}{2}\pi)^{1/2}[(\cos\beta)\bar{C}(t)+(\sin\beta)\bar{S}(t)]\,, \qquad (15)$$

where $\beta=D/4A$, and

$$\bar{S}(t)=S(AD[t+1/2A]^2)-S(\beta)\,,$$
$$\bar{C}(t)=C(AD[t+1/2A]^2)-C(\beta)\,.$$

$C(z)$ and $S(z)$ are the tabulated Fresnel integrals.[6] Similarly,

$$y(t)=\frac{V}{\sqrt{AD}}(\tfrac{1}{2}\pi)^{1/2}[(\cos\beta)\bar{S}(t)-(\sin\beta)\bar{C}(t)]\,. \qquad (16)$$

A qualitative picture of Eqs. (15) and (16) is readily envisioned if radiative reaction is considered analogous to a friction force: The particle spirals around within its original Larmor radius, moving closer and closer to its asymptotic limit

$$\bar{R}=[x^2(\infty)+y^2(\infty)]^{1/2}\,.$$

TABLE I. Representative values of $\bar{R}$ and $v_{\rm rad}$, the average velocity of the net spatial displacement of an electron. $v_{\rm rad}$ is calculated for the time that the radiative reaction effects are significant.

B (G)	γ_0	$\bar{R}$ (cm)	$v_{\rm rad}/c$
10^8	10^6	1.18×10^{-1}	0.19
10^8	10^4	7.08×10^{-2}	0.12
10^6	10^7	1.18×10^{2}	0.19
10^6	10^5	7.08×10^{1}	0.12
10^4	10^7	1.13×10^{5}	0.18
10^4	10^6	7.08×10^{4}	0.12
10^2	10^8	1.13×10^{8}	0.18
10^2	10^7	7.08×10^{7}	0.12
1	10^8	7.08×10^{10}	0.12

(For many physical problems of interest, the Larmor radius can be quite large compared to the dimensions of the field's source.)

A computer analysis of Eqs. (15) and (16), necessitated by the presence of the Fresnel integrals, reveals that, in general, radiative damping becomes negligible (at $t=T$) after two or three circuits of the particle. However, the *net* spatial displacement during that time is appreciable ($>0.6\bar{R}$). [We note that the particle is of course still quite relativistic after radiative damping becomes negligible: $\gamma(T)>10^{-3}\gamma_0$.]

It is also interesting that, regardless of the values of B and γ, the average velocity of the net spatial displacement, $[v_x^2(T)+v_y^2(T)]^{1/2}$, always lies in the same narrow range of $0.1c$–$0.2c$. (See Table I).

In astrophysics we of course do not have the neat case of constant uniform fields. Real magnetic fields curve and twist and are sometimes time-dependent; so too for any electric fields that might be present. However, while the detailed solutions of motion in these fields are usually quite involved, we can appreciate some general effects of radiative reaction from the present analysis.

For very high-energy particles and fields $<10^4$ G, the Larmor radius of a particle can be quite large (Fig. 1). In this case, the radiative reaction force will considerably alter the motion, entrapment, and escape of particles from that calculated classically for a static or time-dependent magnetic (and induced electric) field.[7,8] The exact modifications of the motion depend on the initial conditions for the particles, but the qualitative nature of the motion is clear from the preceding discussions. (The analogous condition for the importance of radiative reaction for protons is $\gamma^2B>10^{21}$ G, so that the motion of only the very high-energy tail of the cosmic particles might be altered.)

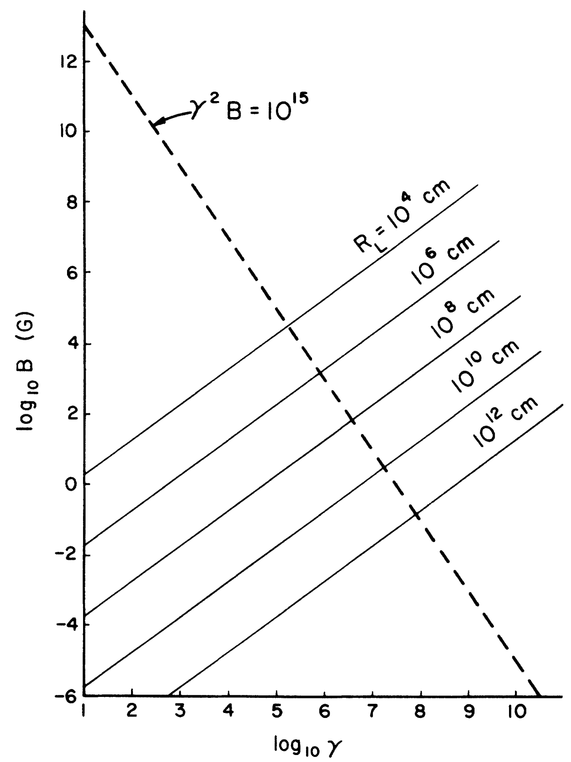

FIG. 1. $\gamma^2 B = 10^{15}$ G and various values of the Larmor radius of an electron, R_L, plotted on a $\log_{10}\gamma$ vs $\log_{10} B$ scale. Radiative reaction effects become important in the range around and to the right of the dashed line.

Considering specific models of pulsars, it is sometimes assumed that the pulsar magnetosphere is co-rotating.[9] If we consider a dipole magnetic field for example, with high-energy particles within the velocity of light circle, it is clear that there would be regions where some of these particles would have a net velocity of approximately $0.15\,c$ in addition to the drift velocity, which is generally $\ll c$. Instead of co-rotating on average, the particles could readily escape their local region. In addition, very high-energy electrons may have a net displacement to other latitudes, instead of neatly streaming out on average along the open field lines in the wind zone. These effects are of course again dependent upon the initial conditions for the particles, and would be applicable to pulsars other than the Crab pulsar, whose high field makes any Larmor radius quite small.

ACKNOWLEDGMENTS

I would like to thank Professor Leo Sartori for his helpful comments and suggestions. I also want to acknowledge the fine facilities and hospitality of the Institute for Space Studies.

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