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Backscattering Caused by the Expansion of the Universe*

Leonard Parker
University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201†
and Princeton University, Princeton, New Jersey 08540‡
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We show that in a Robertson-Walker space-time, expansion-induced backscattering of electromagnetic waves is rigorously absent, contrary to earlier expectations. The absence of backscattering is a direct consequence of the conformal invariance of the equations of motion, and therefore extends to neutrinos and gravitational waves. Any non-null observation of backscattering caused by the expansion would indicate that there exists expansion anisotropy.

I. INTRODUCTION

In 1939, Schrödinger[1] investigated a peculiar and potentially very significant phenomenon in the closed isotropic expanding universe. He found that a wave traveling in one direction could give rise to a backscattered wave traveling in the opposite direction. That is, the homogeneous expanding space could effectively reflect the wave, in a manner analogous to the reflection of light in a medium with a time-dependent index of refraction. For simplicity, Schrödinger investigated the phenomenon using the following scalar wave equation[2]:

$$(\Box - m^2)\phi = 0\,. \qquad (1)$$

He showed that the backscattering is very slight when the expansion parameter varies slowly, but he did not reach a definite conclusion as to the possibility of observing the effect.

From an observational viewpoint, it is clearly most important to determine the magnitude of the backscattering for electromagnetic waves. In our investigation of that problem, we generalize Schrödinger's work by dealing directly with the electromagnetic field as well as the scalar field, and by considering the open, flat, and closed Robertson-Walker space-times, rather than just the closed model. We show that the backscattering is rigorously absent in the case of light, so that there is no possibility of observation, unless the expansion is anisotropic.[3] The fundamental reason for that conclusion is that the equations governing the propagation of light in a given background metric are conformally invariant.[4]

A conformal transformation is a transformation of the metric, such that

$$g_{jk} \rightarrow \tilde{g}_{jk} = \Omega^{-2} g_{jk}\,, \qquad (2)$$

where Ω is a scalar function of the coordinates. The transformation corresponds to a stretching of the interval at each point ($ds = \Omega d\tilde{s}$). A field equation is conformally invariant if, under the conformal transformation (2), together with a transformation of the field (involving multiplication by a suitable power of Ω), the transformed equation has the same form as the original equation. The simplest generally covariant equations governing the massless fields of nonzero spin are all conformally invariant,[4] so that the method we use can be applied to neutrinos and gravitational waves, as well as light.[5]

The backscattering is the classical analog of the gravitationally induced pair creation investigated by this author.[5-8] In contrast to the usual kind of backscattering, the backscattered wave receives its energy from the expansion, and thus does not deplete the forward-traveling wave. In fact, the amplitude of the forward wave is somewhat increased, corresponding to the simultaneous production of a pair of oppositely directed waves along the line of motion. The term expansion-induced backscattering refers to that process. Schrödinger[1] was evidently aware of the relationship between backscattering and pair creation, although he did not investigate the pair creation further. The present paper is a generalization and application of some of the results first obtained in Ref. 6 (see also Ref. 5) for the Robertson-Walker metric with flat 3-space. Our considerations will be restricted to massless fields.

II. SCALAR WAVES

It is instructive to first consider scalar waves, as did Schrödinger. However, we use the wave equation

$$(\Box+\tfrac{1}{6}R)\phi=0\,, \tag{3}$$

where R denotes the scalar curvature. We use Eq. (3) in the present context because, unlike Eq. (1), it shares the property of conformal invariance with Maxwell's equations, and our object is to obtain results which are valid for light as well as scalar waves. We do not mean to imply by our use of Eq. (3) in the present context that physical scalar waves necessarily satisfy that equation.[9] Under the conformal transformation (2), Eq. (3) leads to[4]

$$(\tilde{\Box}+\tfrac{1}{6}\tilde{R})\tilde{\phi}=0\,, \tag{4}$$

where $\tilde{\Box}$ and $\tilde{R}$ are the generalized covariant Laplacian and the scalar curvature in the conformally transformed space-time, and

$$\tilde{\phi}=\Omega\phi\,. \tag{5}$$

The Robertson-Walker line element is

$$ds^2=a(\tau)^2\{-d\tau^2+(1+\tfrac{1}{4}\epsilon u^2)^{-2}\times[du^2+u^2(d\theta^2+\sin^2\theta\, d\varphi^2)]\}\,, \tag{6}$$

where $\epsilon=+1$, 0, or -1. We have used the convenient coordinate τ instead of the cosmic time t, which satisfies $dt=a(\tau)d\tau$. We now make the conformal transformation (2), with

$$\Omega=a(\tau). \tag{7}$$

Then

$$d\tilde{s}^2=-d\tau^2+(1+\tfrac{1}{4}\epsilon u^2)^{-2}[du^2+u^2(d\theta^2+\sin^2\theta\, d\varphi^2)]\,, \tag{8}$$

and

$$\tilde{R}=-6\epsilon\,. \tag{9}$$

The wave equation (4) becomes

$$-\frac{\partial^2}{\partial\tau^2}\tilde{\phi}+{}^{(3)}\tilde{\Delta}\tilde{\phi}-\epsilon\tilde{\phi}=0\,, \tag{10}$$

where

$$\tilde{\phi}=a(\tau)\phi\,, \tag{11}$$

and ${}^{(3)}\tilde{\Delta}$ is the covariant Laplacian operator, formed from the spatial part of the line element (8), and applied to $\tilde{\phi}$ as a scalar under spatial transformations. The complete set of scalar eigenfunctions satisfying

$${}^{(3)}\tilde{\Delta}Q(u,\theta,\varphi)=-\lambda Q(u,\theta,\varphi) \tag{12}$$

are well known.[10] For $\epsilon=+1$, the values of λ are (n^2-1) with $n=1, 2, 3, \ldots$; for $\epsilon=-1$, λ can have any real value greater than 1; and for $\epsilon=0$, λ is non-negative and Q is the usual Fourier component. Thus, one finds that any solution of (10) can be expanded in terms of proper modes of the form

$$\tilde{\phi}^{(\pm)}=e^{\pm ik\tau}Q(u,\theta,\varphi)\,, \tag{13}$$

where $k=(\lambda+\epsilon)^{1/2}$, and λ is the eigenvalue corresponding to Q. The corresponding solutions of Eq. (3) are

$$\phi^{(\pm)}=a(\tau)^{-1}\tilde{\phi}^{(\pm)}\,. \tag{14}$$

During any period when $a(\tau)$ is constant, τ is equal to $a^{-1}t$ to within an additive constant, so that $\phi^{(+)}$ and $\phi^{(-)}$ correspond to oppositely directed waves of frequency $(2\pi)^{-1}k/a$. For the case $\epsilon=0$, they are ordinary plane waves moving in opposite directions.[11] If only one of the waves, say $\phi^{(+)}$, is present during an initial period when $a(\tau)$ is constant, then in an intermediate period during which $a(\tau)$ varies arbitrarily the opposite wave, $\phi^{(-)}$, will never appear because (16) is an exact solution; and if $a(\tau)$ finally comes to rest at another constant value, only the wave $\phi^{(+)}$ traveling in the original direction will be present. Since a backscattered wave does not appear as the result of any change in $a(\tau)$, we conclude that precisely no backscattering occurs for waves satisfying the conformally invariant scalar wave equation in an arbitrarily expanding (or contracting) Robertson-Walker space-time.[12] Thus, for example, a wave packet formed from a set of $\phi^{(+)}$ with nearby frequencies will not develop a backscattered wave packet as a result of changes in $a(\tau)$.

III. ELECTROMAGNETIC WAVES

Light waves can be treated analogously to scalar waves. The fully covariant Maxwell equations in vacuum can be written as

$$\partial_k[(-g)^{1/2}F^{jk}]=0 \tag{15}$$

and

$$\partial_{[j}F_{kl]}=0\,, \tag{16}$$

where ∂_j denotes the ordinary derivative. Under the conformal transformation (2) one obtains the equations

$$\partial_k[(-\tilde{g})^{1/2}\tilde{F}^{jk}]=0 \tag{17}$$

and

$$\partial_{[j}\tilde{F}_{kl]}=0\,, \tag{18}$$

with

$$\tilde{F}_{jk}=F_{jk} \tag{19}$$

or

$$\tilde{F}^{jk}=\Omega^4F^{jk}\,. \tag{20}$$

Defining the 4-vector potential $\tilde{A}_j$ by

$$\tilde{F}_{jk}=\partial_j\tilde{A}_k-\partial_k\tilde{A}_j \tag{21}$$

and

$$\tilde{\nabla}^2\tilde{A}_j=0\,, \tag{22}$$

one finds that[13]

$$\tilde{\Box}\tilde{A}_l+\tilde{R}_l{}^m\tilde{A}_m=0\,, \tag{23}$$

where $\tilde{R}_{lm}$ is the contracted Riemann tensor in the conformally transformed space-time.

We again consider the Robertson-Walker line element, and make the conformal transformation (7) to the static line element (8). Since only electromagnetic waves in vacuum are under consideration, we set

$$\tilde{A}_0=0\,. \tag{24}$$

Making use of

$$\tilde{R}_\alpha{}^\beta=2\delta_\alpha{}^\beta\epsilon\quad(\alpha,\beta=1,2,3)\,, \tag{25}$$

one finds that with the metric of (8), Eq. (23) becomes

$$-\frac{\partial^2}{\partial\tau^2}\tilde{A}_\alpha+{}^{(3)}\tilde{\Delta}\tilde{A}_\alpha+2\epsilon\tilde{A}_\alpha=0\,, \tag{26}$$

where ${}^{(3)}\tilde{\Delta}$ is applied to $\tilde{A}_\alpha$ as a vector under spatial transformations. The vector eigenfunctions S_α of ${}^{(3)}\tilde{\Delta}$ have been investigated.[10] They satisfy the equations

$$\tilde{\nabla}^\alpha S_\alpha=0 \tag{27}$$

and

$${}^{(3)}\tilde{\Delta}S_\alpha=-\mu S_\alpha\,. \tag{28}$$

For the case $\epsilon=+1$, μ takes the values (n^2-2) with $n=2, 3, \ldots,$ while for the other two cases, μ ranges over a continuum of non-negative values. Therefore, the proper modes have the form

$$\tilde{A}_\alpha^{(\pm)}=e^{\pm ik\tau}S_\alpha(u,\theta,\varphi)\,, \tag{29}$$

where $k=(\mu-2\epsilon)^{1/2}$ is real.

As in the scalar case, $\tilde{A}_\alpha^{(+)}$ and $\tilde{A}_\alpha^{(-)}$ correspond to waves traveling in opposite directions, and reduce to plane waves when $\epsilon=0$. The differentiations involved in forming $F_{jk}^{(\pm)}=\tilde{F}_{jk}^{(\pm)}$ from $\tilde{A}_j^{(\pm)}=(0,\tilde{A}_\alpha^{(\pm)})$ clearly do not alter the directions of the waves. Thus

$$F_{jk}^{(\pm)}=\partial_j\tilde{A}_k^{(\pm)}-\partial_k\tilde{A}_j^{(\pm)} \tag{30}$$

are the proper modes of Eqs. (15) and (16), with $F_{jk}^{(+)}$ and $F_{jk}^{(-)}$ corresponding to oppositely directed waves. If only $F_{jk}^{(+)}$, say, is present during an initial period when $a(\tau)$ is constant, then an admixture of $F_{jk}^{(-)}$ will never appear as the result of an intermediate period during which $a(\tau)$ undergoes arbitrary variations. Therefore, as for the scalar waves considered earlier, backscattering is rigorously absent for light waves in the expanding isotropic universe.

IV. CONCLUSIONS

The method we have used is quite straightforward, and may be extended to any conformally invariant wave equation as follows. In the static metric (8), any acceptable wave equation must possess a set of proper modes $\tilde{\xi}^{(\pm)}$ of definite frequency [i.e., solutions with time dependence $\exp(\pm ik\tau)$]. Under the conformal transformation to the general Robertson-Walker metric (6), the above proper modes, multiplied by suitable factors of $a(\tau)$, become exact solutions, $\xi^{(\pm)}$, of the wave equation in the general metric. During any period when $a(\tau)$ is constant, $\xi^{(+)}$ and $\xi^{(-)}$ correspond to oppositely directed waves of definite frequency. Because they are exact solutions even when $a(\tau)$ is not constant, no mixing of $\xi^{(+)}$ and $\xi^{(-)}$ occurs as the result of any change in $a(\tau)$. Hence, backscattering of waves is rigorously absent.[14]

In particular, the above reasoning applies to the wave equations for massless fields of arbitrary nonzero spin s

$$\nabla^{\nu_1\dot{\sigma}}\xi_{\nu_1\nu_2\cdots\nu_{2s}}=0\quad(\nu_j,\sigma=1,2)\,, \tag{31}$$

which are all conformally invariant.[4] For $s=\frac{1}{2}$, one obtains the two-component neutrino equation, while the vacuum solutions of Maxwell's and Einstein's field equations are simply related to the solutions of (31) for $s=1$ and $s=2$, respectively. Therefore, we conclude that no backscattering occurs for neutrinos and gravitational waves, as well as light.[15]

For a line element with anisotropic expansion rates, one generally cannot make the conformal transformation to a static metric. Therefore, the oppositely directed proper modes of definite frequency, which are present whenever the expansion parameters are constant, will be mixed as a result of variations of the expansion parameters. If expansion-induced backscattering could be observationally separated from other effects,[16] then at least in principle it could serve as a measure of the expansion anisotropy. For example, the frequency dependence of the backscattering might serve to distinguish it from other similar effects. Any non-null result would then be an indication of expansion anisotropy.[17]

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†Permanent address.

‡Address for the current academic year.

[1]E. Schrödinger, Physica 6, 899 (1939).

[2]We use units with $\hbar = c = 1$, and metric signature +2. The generalized Laplacian is $\Box = g^{jk}\nabla_j\nabla_k$, where ∇_j denotes the covariant derivative.

[3]We consider only backscattering induced by the expansion.

[4]R. Penrose, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964), p. 565.

[5]For the Robertson-Walker metric with flat 3-space, the extension to arbitrary nonzero spin is given (in the context of pair creation) in L. Parker, Phys. Rev. 183, 1057 (1969), Sec. H.

[6]L. Parker, Ph.D. thesis, Harvard University, 1966 (unpublished).

[7]L. Parker, Phys. Rev. Letters 21, 562 (1968).

[8]L. Parker, Phys. Rev. D 3, 346 (1971).

[9]Some consequences of the scalar wave equations are discussed in L. Parker, Phys. Rev. Letters 28, 705 (1972).

[10]E. Lifshitz, J. Phys. (USSR) 10, 116 (1946).

[11]The waves are discussed in the case $\epsilon = +1$ for a particular set of eigenfunctions, Q, in E. Schrödinger, *Expanding Universes* (Cambridge Univ. Press, London, 1956), pp. 80–86.

[12]For scalar waves satisfying Eq. (1) with $m = 0$, a mixing of the oppositely directed waves does generally occur as the result of a change in $a(\tau)$. That is why Schrödinger did not obtain a null result for vanishing m.

[13]J. L. Synge, *Relativity: The General Theory* (North Holland, Amsterdam, 1960), p. 357.

[14]The same method can be applied to any conformally static space-time, to demonstrate the absence of backscattering induced by the time dependence of the original metric. Of course, if the static space is inhomogeneous, then scattering of a different type (as from obstacles or a space-dependent index of refraction) can occur.

[15]An alternative proof of the rigorous absence of backscattering could be based on the conformal flatness of the Robertson-Walker metric, which is demonstrated in, for example, L. C. Shepley and A. H. Taub, Commun. Math. Phys. 5, 237 (1967). One would deal with the wave equation in the conformally equivalent Minkowski space-time, without introducing more complicated proper modes. However, the method we have used can be generalized to any conformally static metric by using the proper modes of the static space-time (see footnote 14).

[16]Such effects might be scattering from dust and plasma, and the backscattering from local gravitational fields of matter studied in K. Nordtvedt, Jr., Phys. Rev. 186, 1352 (1969).

[17]For a recent discussion of expansion anisotropy and the cosmic black-body radiation, see for example, S. N. Rasband, Astrophys. J. 170, 1 (1971) and references cited therein.