vector as it is transported from place to place. This results in a Riemannian geometry which appears quite different from the one used in general relativity with modified derivatives and a metric tensor which is not covariantly constant. Having paid this price, we end with an unambiguous theory with very simple field equations directly related to the curvature tensor as in general relativity, in which the scalar field plays a rather elegant geometrical role.

[1]C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).
[2]R. H. Dicke and H. M. Goldenberg, Phys. Rev. Letters 18, 313 (1967).
[3]A. P. Ingersoll and E. A. Spiegel, Astrophys. J. 163, 375 (1971).
[4]An excellent review article with an extensive bibliography on the oblateness of the sun especially in its relation to a rotating solar core and the resulting abundances at the surface is R. H. Dicke, Ann. Rev. Astron. Astrophys. 8, 297 (1970).
[5]I. I. Shapiro, M. E. Ash, R. P. Ingalls, W. B. Smith, D. B. Campbell, R. B. Dyce, R. F. Jurgens, and G. H. Pettengill, Phys. Rev. Letters 26, 1132 (1971).
[6]H. Weyl, Sitzber. Preuss. Akad. Wiss. Berlin, 465 (1918), reprinted in H. A. Lorentz, A. Einstein, and R. Minkowski, *Das Relativitätsprinzip* (Leipzig, 1918).
[7]H. Weyl, Ann. Physik 59, 101 (1919).
[8]H. Weyl, *Space, Time, and Matter* (Dover, New York, 1950).
[9]R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1965).
[10]R. H. Dicke, Phys. Rev. 125, 2163 (1962).
[11]A. Einstein, remarks found in original Appendix to Ref. 6 but not in reprinted version.
[12]A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. 39, 65 (1938); A. Einstein and L. Infeld, *ibid.* 41, 455 (1940); Can. J. Math. 1, 209 (1949); L. Infeld and A. Schild, Rev. Mod. Phys. 21, 408 (1949).

PHYSICAL REVIEW D VOLUME 5, NUMBER 2 15 JANUARY 1972

Gravitational Lagrangian

Moshe Carmeli and Stuart I. Fickler
General Physics Research Laboratory, Aerospace Research Laboratories, Wright-Patterson AFB, Ohio 45433
(Received 23 April 1971)

The method of recasting the Newman-Penrose formalism for the gravitational field equations into a Yang-Mills–type theory is reviewed. The free-field gravitational Lagrangian density structured along the lines of the free-field Yang-Mills Lagrangian density by Kibble is generalized to give the complete set of gravitational field equations one obtains in the Newman-Penrose formalism.

I. INTRODUCTION

In Yang-Mills[1] theory one assumes that at each space-time point there exists a 2-dimensional internal space. Under an isotopic gauge transformation $S(x)$, the 2×2 matrix potential and matrix field then transform according to

$$B'_\mu = S^{-1}B_\mu S - S^{-1}\partial_\mu S\,,\qquad F'_{\mu\nu} = S^{-1}F_{\mu\nu}S\,, \tag{1.1}$$

where

$$F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + [B_\mu, B_\nu]\,, \tag{1.2}$$

and $[B_\mu, B_\nu] = B_\mu B_\nu - B_\nu B_\mu$. The action principle applied to the Lagrangian density

$$\mathfrak{L} = -\tfrac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\,, \tag{1.3}$$

where Tr denotes trace, then gives rise to the free gauge field equation

$$\partial_\beta F^{\alpha\beta} - [B_\beta, F^{\alpha\beta}] = 0\,. \tag{1.4}$$

In introducing the gravitational field from a generalized Poincaré invariance, Kibble[2] has extended the above Lagrangian density into

$$-\tfrac{1}{4}(-g)^{1/2}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\,, \tag{1.5}$$

describing the free gravitational field. Here $g = \det g_{\mu\nu}$ and $g_{\mu\nu}$ is the geometrical metric.

More recently Carmeli,[3] who has shown[4,5] that the Newman-Penrose (NP) formalism[6] for the gravitational field equations can be cast into a Yang-Mills–type theory by use of the group SL(2, C), used a first-order form of the Lagrangian density (1.5) to obtain the vacuum NP equations.

The question arises as to whether one can generalize the Lagrangian density (1.5) into one which gives the complete NP equations.[7] In addition to the obvious interest in connection with classical gravitational radiation theory, such a Lagrangian density could be useful in quantum gravidynamics since, as has recently been pointed out by Fickler and Russo,[8] certain methods used for gauge fields can directly be generalized into gravidynamics when cast as a gauge theory. The purpose of this paper is to present a Lagrangian density that gives the full gravitational field equations obtained in the NP formalism.

In Sec. II we briefly review previous work so as to establish our notation. In Sec. III we present a Lagrangian density from which equations of motion are derived using Hamilton's principle. The resultant equations of motion are then written explicitly in Sec. IV using the standard notation. Section V is devoted to concluding remarks.

II. GRAVITATIONAL-FIELD DYNAMICAL VARIABLES

In this section we briefly review the formalism of Newman and Penrose using Carmeli's notation. For more details the reader is referred to Refs. 4 and 5, and to the original work, Ref. 6.

At each point of space-time, one introduces two 2-component spinors $\zeta_a{}^A$, where $a=0,1$, to define a spin frame. Latin italic capitals are used for spinor indices taking the values 0, 1. The two spinors $\zeta_a{}^A$ are normalized such as $\zeta_a{}^B\epsilon_{BA}\zeta_b{}^A=\epsilon_{ab}$, where the ϵ's are the skew-symmetric Levi-Civita symbols defined by $\epsilon_{01}=-\epsilon_{10}=1$. Raising or lowering a spinor index is done by means of these symbols with the convention that $\xi^A=\epsilon^{AB}\xi_B$ and $\xi_A=\xi^B\epsilon_{BA}$ for a spinor ξ^A.

An arbitrary spinor $S^{AB'}$ can now be written in terms of the spin frame, $S^{AB'}=S^{ab'}\zeta_a{}^A\bar{\zeta}_{b'}{}^{B'}$, where $S^{ab'}$ are some quantities called dyad components of the spinor $S^{AB'}$. Here prime indices refer to the complex conjugate and lower-case indices are used for dyad components, taking the values 0, 1. The latter indices behave the same way algebraically as ordinary spinor indices except when covariant differentiation is applied in which case no term involving an affine connection appears for them. By the same token the quantity $\nabla_\mu\xi^A$, obtained by taking the covariant derivative of a spinor ξ^A, can also be written in terms of the spin frame as $\nabla_\mu\xi^A=B^b{}_\mu\zeta_b{}^A$, where $B^b{}_\mu$, with $b=0,1$, are two vectors. (Greek letters are used for space-time indices running over 0, 1, 2, 3, and the metric is taken to be $+---$.) In particular, the last formula applies to the two spinors $\zeta_a{}^A$. This gives

$$\nabla_\mu\zeta_a{}^A=B_a{}^b{}_\mu\zeta_b{}^A, \tag{2.1}$$

where again, $B_a{}^b{}_\mu$, with $a,b=0,1$, are some vectors. In the above formulas the covariant derivative[9] of a spinor is one for which the spinor affine connection is fixed by the requirement that the covariant derivatives of $\tilde{\sigma}^\mu{}_{AB'}$, ϵ_{AB}, and $\epsilon_{A'B'}$ shall all vanish. Here the quantities $\tilde{\sigma}^\mu{}_{AB'}$ define four 2×2 Hermitian matrices by means of which one makes the correspondence between tensors and spinors. (In flat space, when Cartesian coordinates are used, they may be chosen as the unit matrix and the three Pauli matrices.) Using matrix notation, Eq. (2.1) becomes

$$\nabla_\mu\zeta=B_\mu\zeta, \tag{2.2}$$

where B_μ and ζ are 2×2 matrices whose elements are $B_a{}^b{}_\mu$ and $\zeta_a{}^A$, respectively. The normalization condition that the two spinors $\zeta_a{}^A$ have to satisfy then implies that the matrix B_μ be traceless and the matrix ζ be unimodular.

The four Hermitian matrices $\tilde{\sigma}^\mu$ are not vectors. One defines another set of Hermitian matrices related to $\tilde{\sigma}^\mu$ by

$$\sigma^\mu=\zeta\tilde{\sigma}^\mu\zeta^\dagger, \tag{2.3}$$

where $\zeta^\dagger$ is the Hermitian conjugate of ζ. The elements of the new matrices, however, define a null tetrad of vectors. Accordingly, $\sigma^\mu{}_{00'}$ and $\sigma^\mu{}_{11'}$ are real null vectors whereas $\sigma^\mu{}_{01'}$ and $\sigma^\mu{}_{10'}$ are complex null vectors, conjugate to each other, by the Hermiticity requirements of σ^μ. These four null vectors,[10] in addition, satisfy the orthogonality relation of the form $\sigma^\mu{}_{ab'}\sigma_{\mu cd'}=\epsilon_{ac}\epsilon_{b'd'}$. The geometrical metric can be obtained as $g^{\mu\nu}=\sigma^\mu{}_{ab'}\sigma^{\nu ab'}=\tilde{\sigma}^\mu{}_{AB'}\tilde{\sigma}^{\nu AB'}$. Contrary to the matrices $\tilde{\sigma}^\mu$ whose covariant derivatives vanish by the definition of the covariant derivative, that of the matrices σ^μ do not. One has, using Eq. (2.2),

$$\nabla_\alpha\sigma^\mu=B_\alpha\sigma^\mu+\sigma^\mu B_\alpha^\dagger. \tag{2.4}$$

The commutator of the covariant derivatives, $\nabla_\nu\nabla_\mu-\nabla_\mu\nabla_\nu$, when applied on ζ, gives $F_{\mu\nu}\zeta$, where

$$F_{\mu\nu}=\partial_\nu B_\mu-\partial_\mu B_\nu+[B_\mu,B_\nu] \tag{2.5}$$

is a 2×2 complex traceless matrix whose elements are skew-symmetric tensors. In (2.5) the commutator $[B_\mu,B_\nu]=B_\mu B_\nu-B_\nu B_\mu$. Hence the relation between the F and B matrices is identical to that of Eq. (1.2) for the Yang-Mills field.[11] Moreover, under a change of the spin frame $\zeta=S\zeta'$, where S is a 2×2 unimodular complex matrix whose elements $S_a{}^b$ are functions of space-time, one easily finds that B_μ and $F_{\mu\nu}$ transform into

$$\begin{aligned}B'_\mu&=S^{-1}B_\mu S-S^{-1}\partial_\mu S,\\F'_{\mu\nu}&=S^{-1}F_{\mu\nu}S,\end{aligned} \tag{2.6}$$

identically to those of Eqs. (1.1) for the Yang-Mills field when subjected to an isotopic gauge transformation.

From the matrices B and F one might obtain all the dynamical variables of the gravitational field used in the NP formalism by defining two new sets of matrices

$$B_{ab'} = \sigma^{\mu}{}_{ab'} B_{\mu}, \quad F_{ab'cd'} = \sigma^{\mu}{}_{ab'} \sigma^{\nu}{}_{cd'} F_{\mu\nu}, \tag{2.7}$$

which are again traceless. The B's now describe the spin coefficients of general relativity according to the scheme

$$B_{00'} = \begin{pmatrix} \epsilon & -\kappa \\ \pi & -\epsilon \end{pmatrix}, \quad B_{01'} = \begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix},$$
$$B_{10'} = \begin{pmatrix} \alpha & -\rho \\ \lambda & -\alpha \end{pmatrix}, \quad B_{11'} = \begin{pmatrix} \gamma & -\tau \\ \nu & -\gamma \end{pmatrix}. \tag{2.8}$$

The six F's represent the curvature tensor,

$$F_{01'00'} = \begin{pmatrix} \Psi_1 & -\Psi_0 \\ \Psi_2 + 2\Lambda & -\Psi_1 \end{pmatrix},$$
$$F_{11'00'} = \begin{pmatrix} \Psi_2 + \Phi_{11} - \Lambda & -\Psi_1 - \Phi_{01} \\ \Psi_3 + \Phi_{21} & -\Psi_2 - \Phi_{11} + \Lambda \end{pmatrix},$$
$$F_{11'10'} = \begin{pmatrix} \Psi_3 & -\Psi_2 - 2\Lambda \\ \Psi_4 & -\Psi_3 \end{pmatrix},$$
$$F_{10'01'} = \begin{pmatrix} -\Psi_2 + \Phi_{11} + \Lambda & \Psi_1 - \Phi_{01} \\ -\Psi_3 + \Phi_{21} & \Psi_2 - \Phi_{11} - \Lambda \end{pmatrix}, \tag{2.9}$$
$$F_{10'00'} = \begin{pmatrix} \Phi_{10} & -\Phi_{00} \\ \Phi_{20} & -\Phi_{10} \end{pmatrix},$$
$$F_{11'01'} = \begin{pmatrix} \Phi_{12} & -\Phi_{02} \\ \Phi_{22} & -\Phi_{12} \end{pmatrix}.$$

In these equations the five Ψ's describe the ten (real) components of the Weyl tensor, the nine Φ's describe the trace-free part of the Ricci tensor, and $\Lambda = \frac{1}{24} R$, where R is the Ricci scalar.

III. ACTION PRINCIPLE

We now write down the action functional

$$J = \int \mathfrak{L}\, d^4x, \tag{3.1}$$

where the Lagrangian density $\mathfrak{L}$ is defined by

$$\mathfrak{L} = -\tfrac{1}{2}(-g)^{1/2}\, \mathrm{Tr}\{H^{\mu\nu}(-\tfrac{1}{2}F_{\mu\nu} + \partial_\nu B_\mu - \partial_\mu B_\nu + [B_\mu, B_\nu])\} + \text{c.c.} \tag{3.2}$$

Here $H^{\mu\nu} = \tilde{\sigma}^{\mu CB'} \tilde{\sigma}^{\nu AD'} \tilde{\sigma}^{\alpha}{}_{AB'} \tilde{\sigma}^{\beta}{}_{CD'} F_{\alpha\beta}$, Tr denotes trace, c.c. denotes complex conjugate, added so that the Lagrangian density becomes real, and $g = \det g_{\mu\nu}$. The matrix elements of B_μ and $F_{\mu\nu}$ are considered to be the independent field variables, and Eq. (2.5) is assumed to be unknown. The metric tensor $g_{\mu\nu}$ as well as $\tilde{\sigma}^{\mu}$ are introduced in $\mathfrak{L}$ as auxiliary quantities in order to accomplish invariance.

Application of the usual procedure of variational calculus then leads to Eq. (2.5) and to the following equation of motion:

$$\partial_\nu((-g)^{1/2} H^{\mu\nu}) - [B_\nu, (-g)^{1/2} H^{\mu\nu}] = 0. \tag{3.3}$$

Equation (3.3) gives the dynamical equation of motion which the Riemann tensor has to satisfy, and accordingly we have a full description of the dynamical system. Equation (2.5) gives the Riemann tensor in terms of the spin coefficients, whereas (2.4) gives the spin coefficients in terms of the tetrad of null vectors.

IV. THE NP EQUATIONS

To recover the NP equations out of (3.3), (2.5), and (2.4) one has merely to rewrite these equations in terms of the NP field variables, using (2.7), (2.8), and (2.9). One obtains the following sets of equations:

$$\partial^{cd'} F_{cb'ad'} - \{(B^{pd'})^{c}{}_{p} + (B^{\dagger q'c})_{q'}{}^{d'}\} F_{cb'ad'} - \{\delta^{f'}_{b'} (B^{cd'})^{e}{}_{a} + \delta^{e}_{a} (B^{\dagger d'c})_{b'}{}^{f'}\} F_{cf'ed'} - [B^{cd'}, F_{cb'ad'}] = 0, \tag{4.1}$$

$$F_{ab'cd'} = \partial_{cd'} B_{ab'} - \partial_{ab'} B_{cd'} - (B_{cd'})_{a}{}^{f} B_{fb'} - (B^{\dagger}_{d'c})^{f'}{}_{b'} B_{af'} + (B_{ab'})_{c}{}^{f} B_{fd'} + (B^{\dagger}_{b'a})^{f'}{}_{d'} B_{cf'} + [B_{ab'}, B_{cd'}], \tag{4.2}$$

$$\partial_{ab'} \sigma^{\mu}{}_{cd'} - \partial_{cd'} \sigma^{\mu}{}_{ab'} = (B_{ab'} \sigma^{\mu})_{cd'} + (\sigma^{\mu} B^{\dagger}_{b'a})_{cd'} - (B_{cd'} \sigma^{\mu})_{ab'} - (\sigma^{\mu} B^{\dagger}_{d'c})_{ab'}. \tag{4.3}$$

Here, dyad indices have been raised and lowered by means of the Levi-Civita symbols, and the differentiation operator $\partial_{ab'} = \sigma^{\mu}{}_{ab'} \partial_{\mu}$.[12]

Equations (4.1) and (4.2) are the Newman-Penrose equations.[7] Equation (4.3) is the metric equation.

V. REMARKS

We have seen that the equations of motion (2.5) and (3.3), obtained from the Lagrangian density (3.2) using Hamilton's principle, lead to the field equations (4.1) and (4.2) obtained in the NP formalism.[7] The metric equation (4.3) has also been obtained from (2.4).

The Lagrangian density (3.2) is a natural generalization to the free-field Lagrangian density used by Kibble[2] and by Carmeli,[3] and reduces to the latter in that case. This can easily be seen since the expression in braces in $\mathfrak{L}$ can be written as $\frac{1}{2}H^{\mu\nu}\times F_{\mu\nu}$, and by (2.7) this is equal to $\frac{1}{2}F^{cb'ad'}F_{ab'cd'}$. In empty space (i.e., when all Φ's and Λ are assumed to be zero) this last expression can be seen, by Eq. (2.9), to be equal to $\frac{1}{2}F^{ab'cd'}F_{ab'cd'}$, or equal to $\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, thus giving the expression[13]

$$-\tfrac{1}{2}(-g)^{1/2}\,\mathrm{Tr}\{F^{\mu\nu}(-\tfrac{1}{2}F_{\mu\nu}+\partial_\nu B_\mu-\partial_\mu B_\nu+[B_\mu,B_\nu])\}+\text{c.c.}$$

for the Lagrangian density (3.2) in free space, which is the Lagrangian density used by Carmeli.[3]

---

[1]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[2]T. W. B. Kibble, J. Math. Phys. 2, 212 (1961).

[3]M. Carmeli, Nucl. Phys. (to be published).

[4]M. Carmeli, Lett. Nuovo Cimento 4, 40 (1970).

[5]M. Carmeli, J. Math. Phys. 11, 2728 (1970).

[6]E. T. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).

[7]These equations are given in F. A. E. Pirani, *Lectures on General Relativity* (Prentice-Hall, Englewood Cliffs, New Jersey, 1964), p. 350.

[8]S. I. Fickler and M. Russo, Phys. Rev. D 3, 1782 (1971).

[9]The covariant derivative $\nabla_\mu\xi_A$ of the spinor ξ_A is $\nabla_\mu\xi_A=\partial_\mu\xi_A-\xi_B\Gamma^B{}_{A\mu}$, where $\Gamma^B{}_{A\mu}$ is the spinor affine connection. The corresponding quantity $\bar{\Gamma}^{B'}{}_{A'\mu}$ deals with the spinor $\xi_{A'}$. Throughout this paper there will be no need to know the explicit form of any affinities.

[10]Some authors denote these vectors as follows: $\sigma^\mu{}_{00'}=l^\mu$, $\sigma^\mu{}_{01'}=m^\mu$, $\sigma^\mu{}_{10'}=\bar{m}^\mu$, and $\sigma^\mu{}_{11'}=n^\mu$.

[11]It will be noted that in the Yang-Mills theory it is the spin affinities which are considered as potentials whereas here the vectors B_μ are defined by Eq. (2.1). Obviously spin affinities are not space-time vectors in the Riemannian space whereas the B's are.

[12]In Ref. 7 the four operators $\partial_{ab'}$ are denoted as follows: $\partial_{00'}=D$, $\partial_{01'}=\delta$, $\partial_{10'}=\bar{\delta}$, and $\partial_{11'}=\Delta$.

[13]The similarity of this expression, which can be written as a second-order Lagrangian density of the form $-\frac{1}{4}(-g)^{1/2}\,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$ +c.c., to that given by Eq. (1.5) of Kibble is obvious. The difference between them is due only to the group structure, which is SL(2,C) in the present case and is the Poincaré group in Kibble's case. This fact explains why we here need to add the complex conjugate term in order to make the Lagrangian density real.

PHYSICAL REVIEW D VOLUME 5, NUMBER 2 15 JANUARY 1972

Models of Static, Cylindrically Symmetric Solutions of the Einstein-Maxwell Field

John L. Safko
University of South Carolina, Columbia, South Carolina 29208

and

Louis Witten
University of Cincinnati, Cincinnati, Ohio 45221

(Received 6 May 1971)

Models of static, cylindrically symmetric solutions of the combined Einstein-Maxwell field equations are given. These models consist of extended distributions of matter with surface electric currents and magnetic fields outside the matter. The electric currents serve as sources of the magnetic fields; the distribution of matter as well as the magnetic fields serve as sources of the gravitational field. The magnetic lines of force may be parallel to the axis or circular and centered on the axis. The matter distribution is cylindrically symmetric and may be contained within a central cylinder or a tube centered about the axis. All ordinary physical and geometric requirements are satisfied by the models.

I. INTRODUCTION

The static, cylindrically symmetric source-free solutions of the combined Einstein-Maxwell gravitational and electromagnetic fields are fairly well understood.[1,2] In most cases these solutions are singular along the axis and not singular anywhere else. The singularity along the axis is interpreted as the source of the fields. To avoid singularities, it is necessary to introduce a distribution of the matter region over a finite portion of space. Such is the purpose of this paper.

Four models are discussed (Fig. 1). The first two consist of a cylinder of matter centered along the axis with different external magnetic fields. The other two consist of tubes of matter with dif-