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Two-Variable Expansions and the $K \to 3\pi$ Decays

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Previously suggested two-variable expansions of three-body decay amplitudes in terms of harmonic functions of an O(4) group are discussed and applied to analyze the Dalitz-plot distribution of over 3.2 million $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$ decay events. Among the general features of the O(4) expansions we wish to stress that they are written in the c.m. system of two of the final particles, the angular momentum of which is displayed explicitly, and that each term in the expansion has a good behavior at the threshold, pseudothreshold, and at the boundary of the physical region. We analyze the recent data of Ford *et al.* on charged $K \to 3\pi$ decays, using both O(4) expansion and the standard power-series expansion in terms of the Dalitz-Fabri variables. In both cases it is perfectly adequate to keep four terms in the corresponding expansion. The χ^2 fit is marginally better for the O(4) expansion. We conclude that the $K \to 3\pi$ Dalitz plot has too little structure in it to provide a real test of the advantages or disadvantages of different treatments. It is thus most desirable to apply the O(4) expansions to Dalitz plots of other processes, like $\eta \to 3\pi$ or $\bar{p}n \to \pi\pi\pi$. No conclusive evidence is found for *CP* violation. However, the "linear" term in the O(4) expansion of the difference between the squared matrix elements for K^+ and K^- decays does differ from zero by more than two standard deviations. The effect is stable with regard to the number of terms kept in the expansions. An important distinctive feature of the O(4) expansions is their intimate relation to two-variable O(3, 1) expansions of physical scattering amplitudes.

I. INTRODUCTION

In a previous publication[1] (to be referred to as I), we presented a general formalism for performing harmonic analysis on Dalitz plots, i.e., for analyzing Dalitz-plot distributions for three-body decays, involving particles of spin zero. The main purpose of this paper is to apply the formalism to $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$ decays, using the new data of Ford *et al.*,[2,3] and also to discuss some further features of our approach.

The formalism presented in I consists of two-variable expansions of decay amplitudes in terms of basis functions of irreducible representations of the group O(4). It is actually an extension and modification of an approach developed in a series of previous articles,[4-11] devoted mainly to two-body scattering. The purpose of the whole approach is to develop a reaction theory based on two-variable expansions of relativistic amplitudes and thus to display the entire dependence on the kinematic parameters $s=(p_1+p_2)^2$ and $t=(p_1-p_3)^2$ explicitly in certain special functions, whereas the entire dynamics of the process under consideration is summarized in the expansion coefficients, which we call the Lorentz amplitudes. The motivation is thus twofold, theoretical (the incorporation of general principles, the formulation of dynamical hypotheses) and phenomenological – the fitting of larger bodies of data than can be fitted by single-variable expansions. In this article the phenomenological aspect is stressed.

For scattering, the two-variable expansions are obtained, making use of little else than Lorentz invariance. Indeed, consider the reaction

$$1+2 \to 3+4 \tag{1}$$

and let the particles have arbitrary masses but zero spins. The scattering amplitude $f(s,t)$ can be considered to be a function $M(p_1, \ldots, p_4)$ of the momenta $p_1, \ldots, p_4$, each on its own mass shell. Lorentz invariance and conservation laws naturally restrict the form of $M(p_1, p_2, p_3, p_4)$. As usual, a frame of reference is chosen by standardizing some of the vectors p_i. This can always be done in such a manner that the scattering amplitude becomes a function of the coordinates of one of the momenta only (and of the four masses). Thus, we write $f(s,t)=M(p_1,p_2,p_3,p_4)=F(p)=F(\alpha,\beta)$. Here the momentum p (say p_3, with $p_3^2=m_3^2$) is characterized by some curvilinear coordinates α, β, and ϕ, where α and β are functions of s and t and the amplitude does not depend on ϕ. We thus obtain a mapping of the physical region of a scattering channel onto the entire upper sheet of the two-sheeted hyperboloid $p^2=m^2$. The scattering amplitude as a function on a homogeneous manifold can now be quite naturally expanded[5, 10, 11] in terms of the basis functions of a certain set of irreducible representations of the corresponding group of motions, namely the homogeneous Lorentz group O(3, 1). Here let us just mention some properties of the O(3, 1) two-variable expansions of scattering amplitudes.

(1) The actual form of the two-variable expansions depends on three interrelated choices – that of the frame of reference, of the coordinates on the hyperboloid, and of the specific basis of the representations of O(3, 1).

(2) The O(3, 1) expansions incorporate the fixed-s or fixed-t little-group expansions[11–15] in the following manner. If we choose the c.m. frame of reference by standardizing a timelike vector $p_1+p_2=(\sqrt{s},0,0,0)$, spherical coordinates on the hyperboloid, and a basis corresponding to the group reduction $O(3,1)\supset O(3)\supset O(2)$, then we obtain the O(3) little-group expansion of partial-wave analysis, supplemented by an integral expansion for the partial-wave amplitude $a_l(s)$. If we choose the Breit frame of reference by standardizing a spacelike vector $p_1-p_3=(0,0,0,\sqrt{-t})$ (for $t<0$), hyperbolic coordinates, and a basis corresponding to the reduction $O(3,1)\supset O(2,1)\supset O(2)$, then we obtain the O(2, 1) expansion of Regge-pole theory, supplemented by an integral representation of the Reggeized partial-wave amplitude $a(l,t)$ (the Froissart-Gribov amplitude[15]). If we choose the "light velocity system" by constructing and standardizing a lightlike vector $K=p_4(m_2/m_4)e^{-A}-p_2=(\omega,0,0,\omega)$ where $\cosh A=(m_2^2+m_4^2-t)/2m_2m_4$ and ω is arbitrary,[5, 10, 11] and choose "horospheric" coordinates on the hyperboloid, and a basis corresponding to the group reduction $O(3,1)\supset E_2\supset O(2)$, then for $t=0$ we obtain the Euclidean-group little-group expansion for non-equal-mass scattering. For $t\neq 0$ we obtain a generalized Euclidean-group expansion, supplemented by an integral expansion of the corresponding partial-wave amplitude (related to the impact-parameter approximation). Finally, the Toller O(3, 1) little-group expansion for elastic forward scattering can be obtained as a special limiting case of our expansions.

(3) The $O(3,1)\supset O(2,1)\supset O(2)$ expansion is particularly appropriate for considering behavior in the complex angular momentum plane. Indeed singularities in the l plane will occur when the integral representations for $a(l,t)$ diverge.[5]

(4) The assumption of Mandelstam analyticity and crossing symmetry for the amplitude $f(s,t)$ gets reflected in simple analyticity properties of the Lorentz amplitudes.[7]

(5) If we choose a specific "symmetric" frame of reference, use elliptic coordinates on the hyperboloid, and an appropriate basis consisting of products of Lamé functions, we obtain explicitly crossing-symmetric expansions[8] (identical and convergent in two channels).

(6) All the above properties hold for amplitudes describing the reaction (1) in a physical scattering region. The four masses m_i are arbitrary, a generalization to nonzero spins is in progress. The O(3, 1) expansions involve at least one integral – over the four-dimensional angular momentum σ, completely specifying the representations of O(3, 1) (for spinless particles), and one further sum or integral over the representations of the subgroup O(3), O(2, 1), or E_2. The presence of integrals is due to the noncompactness of the group O(3, 1), or in other words, to the fact that the physical scattering regions are infinite.

(7) In I we have shown how the fact that the physical region for the decay

$$1\to 2+3+4$$

is finite implies that the variables α, β, and ϕ will lie on a certain section of the hyperboloid $p^2=m^2$, $p_0\geqslant m$ (a "cup" close to the vertex). This region was in turn mapped onto an O(4) sphere and the amplitude, as a function on this sphere, was expanded in terms of the irreducible representations of O(4). The expansions obtained are very similar to the $O(3,1)\supset O(3)\supset O(2)$ expansions for scattering amplitudes, except that the continuous variable σ is replaced by a discrete variable n. The expansions for decay amplitudes then involve double sums, instead of one integral and a sum (or two integrals). Not surprisingly, it is much simpler to treat decays than scattering. Consequently, our first application of the method to an actual treatment of data is to fit $K\to 3\pi$ Dalitz plots (all four particles are conveniently spinless).

(8) Scattering processes are usually analyzed using either specific models or single-variable expansions. As far as we know, the only

exception, not related to the work already mentioned,[4-11] are articles by Balachandran and coworkers and by Charap and Minton,[16] in which two-variable expansions with convenient properties with respect to crossing symmetry are suggested. However, for scattering these are applicable only in the nonphysical Euclidean region (where the three-momenta of all particles are imaginary or zero). Three-body decays, on the other hand, are customarily treated using two-variable expansions, so that our approach can be readily compared to that of other authors.[16-18]

In Sec. II of this article we reproduce the O(4) expansions as derived in I, discuss some further properties, and compare them to other treatments of three-body decays. In Sec. III we discuss the data on $K^{\pm} \to \pi^{\pm}\pi^{\pm}\pi^{\mp}$ decays and describe the fitting procedure. Section IV is devoted to the results of the numerical fitting of data by the expansions and finally in Sec. V we summarize the results, conclusions, and further outlook.

II. TWO-VARIABLE EXPANSIONS FOR DECAY AMPLITUDES

Consider the decay

$$1 \to 2 + 3 + 4, \tag{2}$$

in the center-of-mass-like system, introduced in I ($\vec{p}_1 = \vec{p}_2$, $\vec{p}_4 = -\vec{p}_3$). Assume that all four particles have zero spin and use the O(4) variables $0 \leq \alpha \leq \pi$, $0 \leq \theta \leq \pi$, satisfying

$$\cos\alpha = 1 - \frac{[(m_1+m_2)^2 - s][(m_1-m_2)^2 - s]}{2m_1^2R^2s} = 1 - \frac{|\vec{p}_1|^2}{m_1^2R^2}, \tag{3}$$

$$\cos\theta = \frac{2s(t-m_1^2-m_3^2) + (s+m_1^2-m_2^2)(s+m_3^2-m_4^2)}{\{[-s+(m_1+m_2)^2][-s+(m_1-m_2)^2][s-(m_3+m_4)^2][s-(m_3-m_4)^2]\}^{1/2}} = \frac{\vec{p}_2\cdot\vec{p}_3}{|\vec{p}_2||\vec{p}_3|}, \tag{4}$$

with

$$R^2 = \frac{[(m_1+m_2)^2-(m_3+m_4)^2][(m_1-m_2)^2-(m_3+m_4)^2]}{4m_1^2(m_3+m_4)^2}. \tag{5}$$

The O(4) expansion of the decay amplitude $f(s,t) = f(\alpha,\theta)$ is

$$f(\alpha,\theta) = \sum_{n=0}^{\infty}\sum_{l=0}^{n} a_{nl}N_{nl}\psi_{nl}(\alpha,\theta), \tag{6}$$

$$a_{nl} = 2\pi N_{nl}^{*}\int_0^{\pi}\sin^2\alpha\, d\alpha\int_0^{\pi}\sin\theta\, d\theta\, f(\alpha,\theta)\,\psi_{nl}(\alpha,\theta), \tag{7}$$

where

$$N_{nl} = e^{-il\pi/2}\frac{2^{l+1/2}\Gamma(l+1)}{2\pi}\left[(2l+1)\frac{(n+1)\Gamma(n-l+1)}{\Gamma(n+l+2)}\right]^{1/2} \tag{8}$$

and

$$\psi_{nl}(\alpha,\theta) = (\sin\alpha)^l C_{n-l}^{l+1}(\cos\alpha)P_l(\cos\theta) \tag{9}$$

[$C_{n-l}^{l+1}(\cos\alpha)$ is a Gegenbauer polynomial].

The functions $\phi_{nl}(\alpha,\theta) = N_{nl}\psi_{nl}(\alpha,\theta)$ form an orthonormal basis for the irreducible unitary representations of O(4) for which one Casimir operator is $J_1 = \vec{L}^2 + \vec{A}^2 = n(n+2)$ and the other is $J_2 = \vec{L}\cdot\vec{A} = 0$.

The representation theory of O(4) immediately provides an expansion formula for the square modulus of $f(\alpha,\theta)$, namely

$$|f(\alpha,\theta)|^2 = \sum_{N=0}^{\infty}\sum_{L=0}^{N} b_{NL}N_{NL}\psi_{NL}(\alpha,\theta), \tag{10}$$

with

$$b_{NL} = \frac{(N+1)^{1/2}}{\sqrt{2}\,\pi}\sum_{nl,\,n'l'}[(n+1)(n'+1)(2l+1)(2l'+1)]^{1/2}(l0l'0|L0)\begin{Bmatrix}\frac{1}{2}n & \frac{1}{2}n & l\\ \frac{1}{2}n' & \frac{1}{2}n' & l'\\ \frac{1}{2}N & \frac{1}{2}N & L\end{Bmatrix}e^{i\pi l'}a_{nl}a_{n'l'}^{*}. \tag{11}$$

Note that the coefficients a_{nl} are in general complex, whereas the b_{NL} are real for even L and pure imaginary for odd L. In (11), $(l0l'0|L0)$ is an O(3) Clebsch-Gordan coefficient and the expression in curly brackets is a 9-J symbol.[19]

Several properties of the variables α and θ and the O(4) expansions (6) and (10) sould be pointed out.

(1) If particles 3 and 4 are identical, then we must have

$$f(\alpha, \theta) = f(\alpha, \pi - \theta) , \tag{12}$$

i.e., the summation in (6) is over even l only. Notice that l and l' being even in (11) implies that L in (10) is also even.

(2) Let all three final particles be identical. Then the amplitude $f(s, t, u)$ should be completely symmetric with respect to permutations of s, t, and u. Let $M = m_1$ and $\mu = m_2 = m_3 = m_4$.

The condition $f(s, t, u) = f(s, u, t)$ is readily ensured by taking only even values of l in (6); however it is easy to check, e.g., by induction, that the condition

$$f(s, t, u) = f(t, s, u) \tag{13}$$

can never be satisfied by a finite number of terms, so that complete exact crossing symmetry can only be achieved by retaining infinitely many terms in the sum.

Alternatively, we can impose the permutation symmetry by writing the "s-channel" and "t-channel" expansions of the left- and right-hand sides of (12), calculating the partial-wave crossing matrices and finding their eigenvectors. Indeed, we have

$$\begin{aligned} f(s, t, u) &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} a_{nl} N_{nl} \psi_{nl}(\alpha_s, \theta_s) \\ &= \sum_{N=0}^{\infty} \sum_{L=0}^{N} a_{NL} N_{NL} \psi_{NL}(\alpha_t, \theta_t) , \end{aligned} \tag{14}$$

where (α_s, θ_s) are given by (3) and (4) and (α_t, θ_t) by the same formulas with s and t interchanged. We then have

$$a_{nl} = \sum_{NL} X_{nl}^{NL} a_{NL} \tag{15}$$

with

$$\begin{aligned} X_{nl}^{NL} = 2\pi N_{nl}^{*} N_{NL} \int_0^{\pi} \sin^2\alpha_s \, d\alpha_s \\ \times \int_0^{\pi} \sin\theta_s \, d\theta_s \, \psi_{nl}(\alpha_s, \theta_s) \psi_{NL}^{*}(\alpha_t, \theta_t) . \end{aligned} \tag{16}$$

The calculation of this crossing matrix X_{nl}^{NL} is a tedious procedure; what is more, the matrix is not quasidiagonal and does not have any other helpful properties, so that the eigenvalue problem (16) can only be solved approximately.

We found the most advantageous way of approximately enforcing the crossing symmetry (13) to be the following: Write a truncated expansion

$$f(s, t, u) = \sum_{n=0}^{N} \sum_{l=0}^{n} a_{nl} N_{nl} \psi_{nl}(\alpha, \theta) , \tag{17}$$

then expand each term into a power series about the center point $s_0 = t_0 = u_0 = \frac{1}{3}M^2 + \mu^2$. We can always make the first few terms in the resulting expansion

$$f(s, t, u) = \sum_{i, k} b_{ik} (s - s_0)^i (t - t_0)^k$$

crossing symmetric, by imposing the symmetry properties on the coefficients a_{nl} in (17). The higher powers in $(s - s_0)$ and $(t - t_0)$ will not be symmetric; they will however be small.

(3) As was mentioned in I the expansion (6) does provide the correct threshold and pseudothreshold behavior in the s variable. Indeed, we have

$$\sin\alpha = \frac{\{[(m_1+m_2)^2 - s][(m_1 - m_2)^2 - s][s - (m_3+m_4)^2][(m_1^2 - m_2^2)^2 - (m_3+m_4)^2 s]\}^{1/2}}{2m_1^2 (m_3+m_4) R^2 s} , \tag{18}$$

so the presence of the factor $(\sin\alpha)^l$ in each term [see (9)] ensures that all partial waves except the s wave $(l=0)$ vanish correctly for $s = (m_1 - m_2)^2$ and for $s = (m_3 + m_4)^2$ [and also for the scattering threshold $s = (m_1 + m_2)^2$]. The t and u threshold and pseudothreshold behavior is, however, not incorporated explicitly. To illustrate this, consider the case of three equal-mass final particles, e.g., the threshold $t = (2\mu)^2$ on the boundary of the physical region corresponds to

$$\cos\theta_0 = -1, \quad \cos\alpha_0 = \frac{M^2 - 5\mu^2}{M^2 - \mu^2} ,$$

a point at which the basis functions (9) do not vanish. This is not surprising since the displayed angular momentum l is associated with a chosen pair of final particles, namely 3 and 4 (so that it corresponds to s-channel angular momentum for scattering).

(4) The equation for the boundary of the physical

decay region is very simple, namely,

$$\sin\theta = 0 . \tag{19}$$

(5) The transformation from the standard Dalitz-Fabri variables to the O(4) variables is continuous on the Dalitz plot and the Jacobian of the transformation is reasonably simple, so the expression for an element of phase space is also reasonable. Indeed, consider the decay $1 \to 2+3+4$ in the rest frame of particle 1 and let the kinetic energies of the produced particles in this frame be T_2, T_3, and T_4. The Dalitz-Fabri variables are

$$x = \sqrt{3}\,\frac{T_3 - T_4}{Q}, \quad y = \frac{3T_2 - Q}{Q}, \tag{20}$$

where

$$Q = T_2 + T_3 + T_4 = m_1 - m_2 - m_3 - m_4 .$$

We also have

$$s = (p_1 - p_2)^2 = (m_1 - m_2)^2 - \tfrac{2}{3} m_1 Q (1+y),$$

$$t = (p_1 - p_3)^2 = (m_1 - m_3)^2 - \tfrac{1}{3} m_1 Q (2 + \sqrt{3}x - y), \tag{21}$$

$$u = (p_1 - p_4)^2 = (m_1 - m_4)^2 - \tfrac{1}{3} m_1 Q (2 - \sqrt{3}x - y).$$

An element of phase space can be written as

$$\begin{aligned} dxdy &= \frac{3\sqrt{3}}{2m_1^2 Q^2} ds\, dt \\ &= \frac{3\sqrt{3}m_1^2}{Q^2} R^4 \frac{s^2}{(m_1^2 - m_2^2)^2 - s^2} [s - (m_3 - m_4)^2]^{1/2} \\ &\quad \times \left[-s + \frac{(m_1^2 - m_2^2)^2}{(m_3 + m_4)^2}\right]^{-1/2} \sin^2\alpha\, d\alpha \sin\theta\, d\theta \\ &= \mu(s) \sin^2\alpha \sin\theta\, d\alpha\, d\theta \\ &= g(\alpha) \sin^2\alpha \sin\theta\, d\alpha\, d\theta . \end{aligned} \tag{22}$$

Thus, the element of phase space differs from the invariant measure on the O(4) sphere simply by a factor $\mu(s) = g(\alpha)$, depending on the variable s only, increasing monotonously for $(m_3 + m_4)^2 \le s \le (m_1 - m_2)^2$, and having physically meaningful singularities lying outside the physical decay region. Let us note that the coordinate curves α = const and θ = const are shown on Fig. 3 of I.

(6) An advantage of expanding amplitudes in terms of a set of functions orthogonal over the Dalitz plot is, as was pointed out by Lee[18] and in I, that the expansion coefficients will be statistically independent (i.e., their errors will not be correlated) if the absolute statistical error varies properly over the physical region. Statistical independence is important because it guarantees the stability of the parameters against truncation of the expansion.

In the case of the two-variable O(4) expansion of the matrix element squared, the solution is found by minimizing

$$\chi^2 = \sum_{k=0}^{K} \left(\frac{|F(\alpha_k, \theta_k)|^2 - \sum_{n,l} b_{nl}\phi_{nl}(\alpha_k, \theta_k)}{\Delta(\alpha_k, \theta_k)} \right)^2 , \tag{23}$$

where there are K bins of data, and (α_k, θ_k) is some "center" point for the kth bin, and where $F(\alpha_k, \theta_k)$ is the experimental value of the amplitude [and $\Delta(\alpha_k, \theta_k)$ is the corresponding statistical error]. In the limit that there are an infinite number of events $F(\alpha, \theta)$ [and $\Delta(\alpha, \theta)$] becomes a continous function. The number of bins can thus be increased without bound.

If the data are presented in $dxdy$ bins, then we have

$$\chi^2 = \int_0^\pi \int_0^\pi \left(\frac{|F(\alpha, \theta)|^2 - \sum_{n,l} b_{nl}\phi_{nl}(\alpha, \theta)}{\Delta(\alpha, \theta)} \right)^2 dxdy .$$

Using (22) the inverse error matrix can be written[20]

$$\begin{aligned} H_{nl,\, n'l'} &= \frac{1}{2} \frac{\partial^2 \chi^2}{\partial b_{nl} \partial b_{n'l'}} \\ &= \int_0^\pi \int_0^\pi \frac{\phi_{nl}(\alpha, \theta)\phi_{n'l'}(\alpha, \theta)}{|\Delta(\alpha, \theta)|^2} \\ &\qquad \times \mu(s) \sin^2\alpha\, d\alpha \sin\theta\, d\theta . \end{aligned} \tag{24}$$

Hence, if $\mu(s) = |\Delta(\alpha, \theta)|^2$, the orthogonality of the functions ϕ_{nl} assures that $H_{nl, n'l'}$ is diagonal:

$$H_{nl,\, n'l'} = \frac{e^{i\pi l}}{2\pi} \delta_{nn'} \delta_{ll'} .$$

Equivalently, if $\Delta(\alpha, \theta)$ is constant but the data are presented in $d(\cos\theta) d(2\alpha - \sin 2\alpha)$ bins, then the inverse error matrix is diagonal.

Lee's expansion,[18] on the other hand, yields statistical independence when using $dxdy$ bins and constant $\Delta(x, y)$. In any case since $\Delta(\alpha, \theta)$ is determined by both the matrix element and the experimental efficiency, it is generally not possible to arrange to satisfy any of the above conditions.

(7) Let us compare the O(4) expansions of three-body decay amplitudes with some of the other explicit two-variable expansions in the literature.

The first such treatment which comes to mind is the standard power-series expansion in terms of the Dalitz-Fabri variables x and y of Eq. (20). In I we wrote this expansion as

$$f(s, t) = \sum_{m,k} R_{km} x^m y^k \tag{25}$$

and gave expressions for the coefficients R_{km} in terms of the O(4) coefficients a_{nl} and vice versa.

Some numerical comparisons between these two expansions for the $K^{\pm}\to\pi^{\pm}\pi^{\pm}\pi^{\mp}$ decays are given in the following sections. Obvious advantages of (25) are its simplicity, the trivial expression $dxdy$ for an element of phase space, the fact that the linear approximation $R_{00}+R_{10}y$ is usually good, and the fact that it is easy to implement symmetry of $f(s,t,u)$ under permutations for each power $N=m+k$ separately and exactly:

$$f=R_{00}+R_{20}(x^2+y^2)+R_{30}(y^3-3yx^2)+R_{40}(x^2+y^2)^2+\cdots.$$

On the other hand, angular momentum is not displayed, the equation for the boundary of the physical region is complicated, no threshold or pseudothreshold conditions are incorporated, and the individual terms are not mutually orthogonal, so the coefficients R_{km} are not statistically independent.

An equivalent expansion was suggested by Weinberg[17] and is obtained by introducing polar coordinates on the Dalitz plot,

$$x=\rho\sin\phi,\quad y=\rho\cos\phi. \tag{26}$$

The expansion is

$$f(s,t)=\sum_{n,m} b_{nm}\left(\frac{Q}{m_1}\right)^n\rho^n\cos m\phi\,, \tag{27}$$

where it is assumed that the particles 3 and 4 are identical. The same comments as above apply.

Two-variable expansions in terms of orthogonal functions have been suggested by Balachandran *et al.*[16] and Lee.[18] Balachandran *et al.* expand three-body amplitudes for particles of arbitrary masses into a set of orthogonal polynomials. Their explicit form depends on the masses and they can be obtained in each specific case using the Gram-Schmidt orthogonalization method (see second of Refs. 16). The polynomials explicitly display the correct threshold and pseudothreshold behavior in one variable (say, s). Angular momentum (in one channel) is diagonalized and the polynomials have useful properties with respect to crossing symmetry. Except for the case when all four masses are equal, no general expression is given for the polynomials, they do not have any known group-theoretical meaning and the expansions, when generalized to scattering, involve amplitudes in nonphysical regions.

The approach of Lee[18] consists of mapping the Dalitz plot onto a circle in some new variables α and β and then expanding in terms of polynomials, orthogonal in this circle. The variables α and β are fixed by the condition that the boundary of the decay region should be given by the condition $\alpha=1$, that the transformation $(\rho,\phi)\to(\alpha,\beta)$ should be continuous over the Dalitz plot, and that an element of phase space be

$$\rho\,d\rho\,d\phi=\text{const}\;\alpha\,d\alpha\,d\beta\,. \tag{28}$$

The orthogonal polynomials are products of Jacobi polynomials and trigonometric functions and they can be interpreted as being basis functions for the representations of an SU(3) group. A further convenience is that the values of β and ϕ coincide along the sextant boundaries $\phi=\frac{1}{3}\pi,\frac{2}{3}\pi,\ldots,2\pi$. In general, the relation between the variables (α,β) and (x,y) is complicated and not given explicitly, the formalism is worked out for equal-mass final particles only, angular momentum and threshold behavior are not displayed, and it is hard to foresee a generalization to scattering.

Let us mention again that from our point of view the intimate connection between O(3, 1) expansions for scattering and O(4) ones for decays is a distinct advantage, which enables us to view the description of decay data as a "training ground" for treating scattering.

(8) The parameters (α,θ) of the O(4) expansion can be simply related to the parameters (x,y) of the conventional expansion when $m_2=m_3=m_4$ and the nonrelativistic limit $(m_1\to m_2+m_3+m_4)$ is taken:

$$\cos\alpha\to -y\,,$$
$$\sin\alpha\cos\theta\to -x\,.$$

This suggests the definitions

$$y'=-\cos\alpha\,,$$
$$x'=-\sin\alpha\cos\theta\,.$$

The element of phase space can then be written as $dxdy=\mu(s)dx'dy'$ [see (22)]. The O(4) expansion can then trivially be rewritten as

$$\begin{aligned} f(\alpha,\theta)&=\sum_{n,l} a_{nl}\phi_{nl}(\alpha,\theta)\\ &=\sum_{m,l} R'_{km}x'^{m}y'^{k}\\ &=\frac{1}{\sqrt{2}\pi}\big[a_{00}-a_{20}+\sqrt{2}a_{22}-2a_{10}y'\\ &\qquad+(4a_{20}-\sqrt{2}a_{22})y'^2-\sqrt{2}\,3a_{22}x'^2+\cdots\big]. \end{aligned}$$

The physical region is the interior of a circle in the x',y' plane. Thus, the O(4) expansion is equivalent (except for a rearrangement of coefficients) to mapping the decay region into a circle and then performing a conventional power-series expansion.

(9) Several comments on various symmetries of three-body decays are in order. First consider various space-time symmetries for $K\to3\pi$ and $\eta\to3\pi$ decays. *CPT* invariance, strictly speaking, simply tells us that the decay rates for particles

and antiparticles are equal, and also that separately the decay rates for 3π, 2π, and leptonic decays of particles and antiparticles must be equal. Thus, CPT itself has little bearing on Dalitz-plot distributions.

Consider now the implications of CP invariance. For charged K decays CP invariance implies[21] the equality of K^+ and K^- decay amplitudes. Thus

$$f_{K^+\to\pi^+\pi^+\pi^-}(s,t)=f_{K^-\to\pi^-\pi^-\pi^+}(s,t)\,, \tag{29}$$

$$f_{K^+\to\pi^+\pi^0\pi^0}(s,t)=f_{K^-\to\pi^-\pi^0\pi^0}(s,t)\,. \tag{30}$$

For the O(4) expansions CP invariance would thus imply the equality of all the amplitudes a_{nl} of (6) and (7) for each of the corresponding K^+ and K^- decays:

$$a_{nl}^+=a_{nl}^- \tag{31}$$

separately for $K^\pm\to\pi^\pm\pi^\pm\pi^\mp$ and $K^\pm\to\pi^\pm\pi^0\pi^0$. It is well known that the equality of the constant term $a_{00}^+=a_{00}^-$ is not a sensitive test of CP violation[21]; thus a comparison of higher terms, which we discuss in Sec. IV, is of interest.

For neutral K decays, CP invariance, besides forbidding the $K_L\to 2\pi$ decay, implies that $K_S\to\pi^0\pi^0\pi^0$ is forbidden (but $K_S^0\to\pi^+\pi^-\pi^0$ is only suppressed by a centrifugal barrier). Further, both for $K_L\to\pi^+\pi^-\pi^0$ and $\eta\to\pi^+\pi^-\pi^0$ decays CP invariance (or simply C invariance in the electromagnetic $\eta\to\pi^+\pi^-\pi^0$ decay) implies that the corresponding amplitudes should be symmetric under an interchange of π^+ and π^-. For the O(4) expansion (6) this implies that

$$a_{nl}(K_L\to\pi^+\pi^-\pi^0)=a_{nl}(\eta\to\pi^+\pi^-\pi^0)=0\,,\qquad l=\text{odd} \tag{32}$$

(if the charged pions are taken to be particles 3 and 4).

Further, consider the implications of the $\Delta I=\frac{1}{2}$ rule for $K\to 3\pi$ decays. It is well known[21] that an exact $\Delta I=\frac{1}{2}$ rule for $K\to 3\pi$ decays would lead to two relations between the amplitudes of the four otherwise independent $K\to 3\pi$ decays. Consider the amplitudes

$$\begin{aligned}&F^+(\omega_+\omega_+\omega_-)\,,\quad F^{+\prime}(\omega_0\omega_0'\omega_+)\,,\\ &F_L(\omega_+\omega_-\omega_0)\,,\quad F_L'(\omega_0,\omega_0',\omega_0'')\end{aligned} \tag{33}$$

for the decays $K^+\to\pi^+\pi^+\pi^-$, $K^+\to\pi^0\pi^0\pi^+$, $K_L\to\pi^+\pi^-\pi^0$, and $K_L\to\pi^0\pi^0\pi^0$, respectively (the ω's are the pion energies). The $\Delta I=\frac{1}{2}$ rule implies

$$F_L(\omega_+\omega_-\omega_0)=-\sqrt{2}\,F^{+\prime}(\omega_0,\omega_0',\omega_+)\,, \tag{34}$$

$$F^+(\omega_+\omega_+\omega_-)+F^{+\prime}(\omega_0,\omega_0',\omega_+)=-\sqrt{3}\,F_L'(\omega_0,\omega_0',\omega_0'')\,. \tag{35}$$

The implications of these rules for decay rates are well known. In view of the orthogonality relations of the O(4) expansion functions $\phi_{nl}(\alpha,\theta)$, we immediately find that the coefficients in the expansion (6) of the four amplitudes (33) satisfy the relations

$$a_{nl}^L=-\sqrt{2}\,a_{nl}^{+\prime} \tag{36}$$

and

$$a_{nl}^+ + a_{nl}^{+\prime}=-\sqrt{3}\,a_{nl}^{L\prime} \tag{37}$$

if the $\Delta I=\frac{1}{2}$ rule holds. Thus, if sufficient information on the Dalitz-plot distributions existed for all four decays, then the equalities (36) and (37) would provide very complete tests of the $\Delta I=\frac{1}{2}$ rule. Equation (37) is particularly difficult to test, since it involves the decay $K_L\to\pi^0\pi^0\pi^0$; however, additional information can be obtained, if we restrict ourselves to low values of n in (6) and make use of the fact that the amplitude $F_L'(\omega_0,\omega_0',\omega_0'')$ must be completely symmetric under permutations of the three pions. Thus, in the linear approximation $n\leq 1$ we would keep only the amplitudes a_{00} and a_{10} in (6). However, total symmetry implies $a_{10}^{L\prime}=0$. Thus, we find that in the linear approximation of (6) $\Delta I=\frac{1}{2}$ implies

$$a_{10}^+=-a_{10}^{+\prime}\,. \tag{38}$$

In the next approximation we keep $n\leq 2$. As was mentioned above, total symmetry can only be imposed approximately and indeed, if we keep $a_{00}^{L\prime}$, $a_{10}^{L\prime}$, $a_{20}^{L\prime}$, and $a_{22}^{L\prime}$ in (6) and expand each term about the center point of the Dalitz plot $c=s_0=t_0=u_0=\frac{1}{3}M^2+\mu^2$, then we can exclude the term linear in $(s-c)$ [since the term $(t-c)$ is absent] and symmetrize the quadratic terms $(s-c)^2$, $(t-c)^2$, and $(s-c)(t-c)$. This can be achieved by imposing two conditions of the type

$$\alpha_i a_{22}^{L\prime}+\beta_i a_{20}^{L\prime}+\gamma_i a_{10}^{L\prime}=0\,,\quad i=1,2 \tag{39}$$

where α_i, β_i, and γ_i are constants, depending on the K_L and π^0 masses only. In this approximation Eq. (37), i.e., the $\Delta I=\frac{1}{2}$ rule implies two constraints on the $K^+\to\pi^+\pi^+\pi^-$ and $K^+\to\pi^0\pi^0\pi^+$ O(4) amplitudes:

$$\alpha_i a_{22}^+ +\beta_i a_{20}^+ +\gamma_i a_{10}^+ = -(\alpha_i a_{22}^{+\prime}+\beta_i a_{20}^{+\prime}+\gamma_i a_{10}^{+\prime})\,,\qquad i=1,2\,. \tag{40}$$

(10) Many different theoretical approaches and models have been applied to describe the $K\to 3\pi$ decays. Roughly speaking, these fall into two classes, depending on whether they ascribe the structure in $K\to 3\pi$ decays essentially to the weak interactions themselves, or on the contrary, mainly to the final-state strong interactions of the pions. Among the first, let us mention the applications of current algebra not only to relate the $K\to 3\pi$ and

$K\to 2\pi$ decays, but also to obtain predictions of the $\Delta I=\frac{1}{2}$ rule and to predict values of the slope parameter [the parameter R_{01}/R_{00} in expansion (25)]. For details and references we refer to Márshak *et al.*[21] The final-state interactions of pions have been treated either by using a general dispersion-relation approach, or specific models (or both). Some of the models predict a definite form for the decay amplitude $f(s,t,u)$ which can then be used to explain Dalitz-plot distributions (see, e.g., Ref. 22). We plan to return to the problem of various models, describing three-body decays, to calculate the O(4) partial-wave amplitudes a_{nl} of (7) for these models, and to compare these with the values of a_{nl} obtained phenomenologically by fitting Dalitz-plot distributions. In particular we hope to analyze various applications of the Veneziano model and other dual models to $K\to 3\pi$ and $\eta\to 3\pi$ decays[23,24] from this point of view.

(11) Finally, let us note that expansion (6) also describes the spectrum of the odd pion. Indeed, the variable α is a function of s alone [Eq. (3)], thus of y alone [Eq. (21)]. Integrating expansion (6) over all values of z, we have

$$F(\alpha)=\frac{1}{2}\int_{-1}^{1} f(\alpha,\theta)dz=\frac{1}{\sqrt{2}\pi}\sum_{n=0}^{\infty} a_{n0}C_n^1(\cos\alpha)\,, \quad (41)$$

or alternatively

$$S(\alpha)=\frac{1}{2}\int_{-1}^{1} |f(\alpha,\theta)|^2 dz=\frac{1}{\sqrt{2}\pi}\sum_{N=0}^{\infty} b_{N0}C_N^1(\cos\alpha)\,, \quad (42)$$

with

$$b_{N0}=\frac{(N+1)^{1/2}}{\sqrt{2}\pi}\sum_{nn'l}(-1)^{(N-n-n')/2}\left(\frac{(2l+1)(n+1)(n'+1)}{N+1}\right)^{1/2}(l0l0\,|\,L0)W(\tfrac{1}{2}n,\tfrac{1}{2}n,\tfrac{1}{2}n',\tfrac{1}{2}n';l,\tfrac{1}{2}N)a_{nl}a^*_{n'l}\,, \quad (43)$$

where $W(a,b,c,d;e,f)$ is a Racah coefficient.

III. DATA AND FITTING PROCEDURE

Data were used from recent high-precision $K^\pm\to\pi^\pm\pi^\pm\pi^\mp$ experiments.[2,3] In each case the data are binned into 153 (0.1×0.1) squares on the x,y plane. This bin size is somewhat larger than the accuracy with which an event can be located on the Dalitz plot (<0.07). With each bin, i, is associated a number of events N_i and a statistical error $N_i^{1/2}$. The square of the matrix element $|F_i|^2$ (and its statistical uncertainty Δ_i) can be found if the experimental efficiency for each bin, E_i, is known:

$$|F_i|^2=N_i/E_i\,,\quad \Delta_i=N_i^{1/2}/E_i\,. \quad (44)$$

The conventional method is to fit the squared matrix element with a power-series expansion in the Dalitz-Fabri variables (20) by minimizing

$$\chi^2=\sum_i\left(\frac{|F_i|^2-\sum_{m,k}\tilde{R}_{km}x_i^m y_i^k}{\Delta_i}\right)^2, \quad (45)$$

where (x_i,y_i) is either the geometrical center of the ith bin or some weighted center.

Using the O(4) two-variable expansion we have fitted the data with both the expansion for the square of the matrix element (23) and the expansion for the matrix element itself,

$$\chi^2=\sum_i\left(\frac{|F_i|^2-\left|\sum_{nl}a_{nl}\phi_{nl}(\alpha_i,\theta_i)\right|^2}{\Delta_i}\right)^2. \quad (46)$$

Because there are two identical particles, the Dalitz plot is necessarily symmetric about $x=0$. Thus the fits are to half a Dalitz plot, and only even powers of x or even values of l can enter. We have compared two methods for handling the ambiguity in the definition of the "bin center" (α_i,θ_i) or (x_i,y_i):

(a) Numerical integration was used to average each expansion function [either $\phi_{nl}(\alpha_i,\theta_i)$ or $x_i^m y_i^k$] over the surface of the bin including the non-square-edge bins.

(b) The "bin center" was generated by a Monte Carlo simulation of the experiment.[3] This provides for each bin a single point which is not the geometric center. For the expansions (23) and (45) the two methods give identical results. For expansion (46) procedure (a) is not appropriate.

The data can be modified to partly remove the effects of final-state Coulomb interactions.[3,22,25] The longest-range part of the graphs with a photon exchanged by two pions gives rise to a correction factor[22,25] given nonrelativistically by

$$C=1-\frac{\pi}{2}\left(\frac{e_2e_3}{v_{23}}+\frac{e_2e_4}{v_{24}}+\frac{e_3e_4}{v_{34}}\right), \quad (47)$$

where e_i is the charge of the ith pion and v_{ij} is the relative velocity of pions i and j. When the number of events per bin is divided by C the effects of such graphs are removed. We expect the removal of these contributions to improve the convergence of the O(4) expansion since these long-range contributions would contribute to higher partial waves. We find, in fact, that for all the expansions, the removal of these long-range terms aids convergence. Hereafter, we shall refer only to Coulomb-corrected data.

Instead of analyzing τ^+ and τ^- decay separately, we have analyzed the sum and difference of the two charge states. This is appropriate because the difference is much smaller than the sum.

In Ref. 3 (with 654 512 events) the two charge states are not differentiated. Thus it provides only the sum of the matrix elements for the two charge states,

$$|F_i^s|^2 \equiv |F_i^+|^2 + |F_i^-|^2 , \tag{48}$$

allowing us to find

$$b_{nl}^s \equiv b_{nl}^+ + b_{nl}^- \quad \text{and} \quad \tilde{R}_{km}^s \equiv \tilde{R}_{km}^+ + \tilde{R}_{km}^- .$$

When expansion (46) is fitted to $|F_i^s|^2$ we call the resulting coefficients a_{nl}^s even though

$$a_{nl}^s \neq a_{nl}^+ + a_{nl}^-$$

in general.

In the published data[2] the two charge states were detected separately, giving 1 626 956 events for each charge. However, the experimental efficiency E_i was not determined. Thus for each bin, i, this experiment gives N_i^+ and N_i^-, but not E_i^+ or E_i^-. The experiment does guarantee that $E_i^+ = E_i^- \equiv E_i^\pm$. This allows us to calculate the difference in the squares of the matrix elements for the two charge states by

$$|F_i^d|^2 \equiv |F_i^+|^2 - |F_i^-|^2 = \frac{N_i^+}{E_i^+} - \frac{N_i^-}{E_i^-} = \frac{N_i^+ - N_i^-}{E_i^\pm}$$
$$= \frac{N_i^+ - N_i^-}{N_i^+ + N_i^-} |F_i^s|^2 , \tag{49}$$

and the statistical error in $|F_i^d|^2$ by

$$(\Delta_i^d)^2 = \frac{4|F_i^s|^4 N_i^+ N_i^-}{(N_i^+ + N_i^-)^3} + \left\{\Delta_i^s \frac{N_i^+ - N_i^-}{N_i^+ + N_i^-}\right\}^2 , \tag{50}$$

where the $N_i^\pm$'s are from the second experiment and $|F_i^s|^2$ and Δ_i^s are from the first experiment. Thus $|F_i^d|^2 \pm \Delta_i^d$ can be used to calculate $\tilde{R}_{km}^d$ and b_{nl}^d.

The minimization of χ^2 was done with the computer program MINUIT[26] which we have modified to include random starting points. Specifically, a random starting value (within a generous interval) was chosen for each of the (finite number of) coefficients in the truncated expansion (23), (45), or (46). From this point in multidimensional space χ^2 is minimized, eventually giving the minimum value χ_0^2 and the position of the minimum (best coefficient values). In each case this procedure is repeated a sufficient number of times ($\gg 20$), so that it is clear that the global minimum has been found. A small fraction of the tries result in local minima, but in no case does the corresponding χ^2 come close to χ^2 at the global minimum, so, in effect, there is always an essentially unique solution. This procedure is very reassuring for it eliminates almost entirely the dependence on choice of starting values.

IV. NUMERICAL RESULTS

Altogether we obtained best fits for five cases: expansions (23), (45), and (46) for the $\tau^+ + \tau^-$ data, and expansions (45) and (46) for the $\tau^+ - \tau^-$ data. In each we experimented with Coulomb corrections (Sec. III), dependence on "bin center" definition (Sec. III), and method of truncation. The two-variable O(4) sums were always truncated at some value N of n, the outer sum index. The inner index l already has a finite range. For any value of $N > 1$ the expansion with a_{nl}'s will depend on more real parameters than the b_{nl} series since the a_{nl}'s are in general complex. We have constrained a_{00} to be real. We truncated the conventional power-series expansion by requiring $k + m \leq K$. Various values of N (or K) were tried in each of the five cases in order to find the proper number of terms required and to test the stability against truncation.

To determine the proper number of terms in each truncated expansion it is useful to consider χ^2/NDF (number of degrees of freedom) as a function of the number of free parameters (Fig. 1). Generally, one expects χ^2/NDF to fall, reach a minimum, and then increase as the number of parameters grows large. The minimum reflects the ideal number of parameters above which one is fitting the noise in the data and below which the data are not being adequately fitted.

For 2–7 parameters the fits to the $\tau^+ + \tau^-$ data (Fig. 1) indicate that the 2- and 3-parameter fits are not adequate. The minima are at 6 and 7 parameters, but in the cases of the two expansions of the square of the matrix element, $|\text{M.E.}|^2$, the fits with 4 parameters are only marginally worse. Thus the evidence in favor of using 6 parameters in these cases is quite weak. Beyond seven parameters χ^2/NDF increases and the (overparametrized) fits are no longer unique. It is to be expected that for large numbers of parameters, the χ^2 surface will develop many local minima.

Table I gives the parameter values and statistical errors for the $\tau^+ + \tau^-$ data where the best number of parameters is 4 or 7. The values for the x, y parameters are consistent with those found by Remmel[3] and also with those found by Mainkar *et al.*[22] (from a much smaller set of data and thus with larger errors). The stability of expansions (23) and (45) is about equivalent.

CP conservation predicts that $\tau^+ - \tau^-$ is zero. Our fits to $\tau^+ - \tau^-$ data (bottom of Fig. 1) indicate very little change in χ^2/NDF as the number of pa-

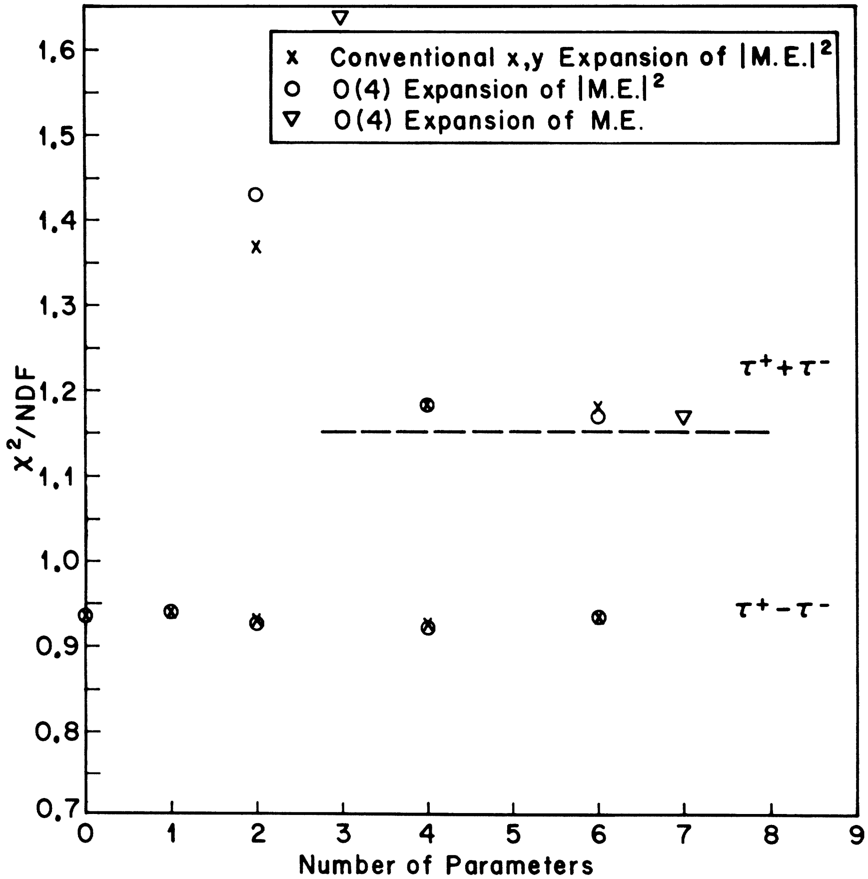

FIG. 1. The $\tau^+ + \tau^-$ data are fitted with 3 parametrizations and the $\tau^+ - \tau^-$ data are fitted with 2 parametrizations. The dashed line is merely to separate the $\tau^+ + \tau^-$ from the $\tau^+ - \tau^-$ fits and to provide a horizontal line against which to see the small decrease from 4 to 7 parameters. Other "less natural" truncations were tried (giving, e.g., 5 parameters) but these all failed to be as good as the ones shown.

rameters is varied. Even for zero parameters (all parameters zero) the fit is not appreciably worse then at the minimum (4 parameters). There is no strong evidence that any parameter (Table II) is nonzero. Thus we see no evidence of *CP* violation. It is, however, interesting to note that the constant term b_{00}^d is consistently equal to zero, within its errors for the 1-, 2-, 4-, and 6-parameter fit, whereas the "linear" term b_{10}^d is consistently greater than two standard deviations above the zero value.

V. SUMMARY AND CONCLUSIONS

As was mentioned in the Introduction, the standard theoretical and phenomenological methods of treating particle scattering usually involve single-variable expansions of amplitudes (for fixed energy or for fixed momentum transfer). On the other hand, the standard way of treating three-particle decays is to make use of two-variable expansions of the decay amplitude (or of the matrix element).

In this article and in I we have modified an approach to scattering, based on two-variable expansions of scattering amplitudes, so as to make it applicable to three-body decays. This enabled us to use the mentioned O(3, 1) expansions,[4-11] modified to be O(4)-group expansions for decays, to the actual treatment of $K \to 3\pi$ Dalitz-plot distributions, which have recently become available thanks to the high-precision experiments of Ford *et al.*[2,3] For decays, the two-variable expansions[4-11] can thus be compared with other treatments, both as to their general properties and as to their suitability for the description of a large body of data.

The general features of the O(4) expansions, which will also characterize O(3, 1) expansions for scattering, were compared with those of other expansions in paragraph 7 of Sec. II. In particular, we wish to emphasize the threshold and pseu-

TABLE I. Best-fit parameter values (and statistical errors) for $\tau^+ + \tau^-$. There are 153 bins. The number of degrees of freedom (NDF) is the number of bins less the number of free parameters. The data used were the Coulomb-corrected preliminary data of Remmel's thesis (Ref. 3). No corrections for possible systematic experimental errors were applied.

Conventional power-series expansion			
NDF	151	149	147
χ^2	206.53	176.18	172.85
χ^2/NDF	1.37	1.18	1.18
$\tilde{R}^s_{00}$	0.974 ± 0.002	0.973 ± 0.003	0.973 ± 0.003
$\tilde{R}^s_{10}$	0.271 ± 0.004	0.277 ± 0.004	0.286 ± 0.011
$\tilde{R}^s_{20}$		0.031 ± 0.009	0.031 ± 0.009
$\tilde{R}^s_{02}$		-0.028 ± 0.009	-0.031 ± 0.009
$\tilde{R}^s_{30}$			-0.006 ± 0.020
$\tilde{R}^s_{12}$			-0.051 ± 0.028

Two-variable O(4) expansion of the $\lvert$M.E.$\rvert^2$			
NDF	151	149	147
χ^2	215.34	175.88	172.33
χ^2/NDF	1.43	1.18	1.17
b^s_{00}	4.56 ± 0.01	4.54 ± 0.01	4.53 ± 0.01
b^s_{10}	-0.61 ± 0.01	-0.56 ± 0.01	-0.57 ± 0.01
b^s_{20}		-0.05 ± 0.01	-0.04 ± 0.01
b^s_{22}		-0.03 ± 0.01	-0.04 ± 0.01
b^s_{30}			-0.01 ± 0.01
b^s_{32}			0.02 ± 0.01

Two-variable O(4) expansion of the M.E.		
NDF	150	146
χ^2	246.49	170.58
χ^2/NDF	1.64	1.17
a^s_{00}	4.491 ± 0.004	4.457 ± 0.008
Rea^s_{10}	-0.307 ± 0.004	-0.278 ± 0.006
Ima^s_{10}	0 ± 0.05	-0.40 ± 0.03
Rea^s_{20}		-0.046 ± 0.005
Ima^s_{20}		0.114 ± 0.054
Rea^s_{22}		-0.025 ± 0.007
Ima^s_{22}		-0.176 ± 0.068

dothreshold behavior, the behavior on the boundary of the physical region, the orthogonality of the basis functions over the physical region, and the diagonalization of angular momentum. Other features, typical for the O(3, 1) expansions, but losing their meaning for decays (like the relation to Regge-pole theory), are discussed in the references.[4-11]

In Sec. IV we found that the O(4) expansions are perfectly adequate for fitting the 3.2 million $K^\pm \to 3\pi$ events, measured by Ford *et al.*[2,3] As can be seen from Fig. 1, an excellent fit to the data is obtained when the square of the matrix element is fitted using 4 parameters in the O(4) expansion. Table I shows that the fit is reasonably stable with respect to the truncation of the expansion, i.e., the parameters b_{NL}, once established for $N \leq N_0$, do not change much when we add further parameters with $N > N_0$. It is of course also obvious from Table I that the conventional expansion (25) into a power series in x and y fits the data equally well with the same number of parameters (i.e., we keep the terms: const, y, y^2, and x^2). According to our opinion this is due to the fact that there is very little phase space available for the $K \to 3\pi$ decays, so the Dalitz-plot distribution is very smooth and is thus easily fitted by a constant term plus any sort of correction containing at least three parameters (this agrees with the conclusions of other authors[22]).

When fitting the Dalitz-plot distribution for the difference between the K^+ and K^- decays into charged pions we, like Ford *et al.*,[2,3] find no real evidence for CP violation. However, as was mentioned in Sec. IV, a glance at Table II will convince us that the coefficient of the first nonconstant term b^d_{10} in the O(4) expansion of the difference $|F^+_i|^2 - |F^-_i|^2$ is consistently nonzero (in the 2-, 4-, and 6-parameter fits) and becomes more pronouncedly so as the number of parameters increases. It is well known, on the other hand, that the CPT theorem, together with the $\Delta I \leq \frac{3}{2}$ rule essentially implies that the constant term (in any expansion of the difference $|F^+_i|^2 - |F^-_i|^2$) is equal to zero, even if CP is violated. Several comments are in order here. The b^d_{10} coefficient, which differs quite significantly from zero, emphasizes the top and the bottom of the Dalitz plot (where $\cos\alpha \sim \pm 1$, i.e., $y \sim \pm 1$). Unfortunately the top lines in the Dalitz plot are precisely those for which the data are least reliable[2,27] and most influenced by possible experimental asymmetries (in the spectrometer magnet). Indeed if we omit the data from the top four rows of lines in our O(4) analysis, the effect in the b^d_{10} coefficient vanishes. From the point of view of discovering a CP violation this is somewhat dissappointing. Let us, however, note that from the point of view of the O(4) expansions themselves we find the situation encouraging. Indeed, the O(4) expansions, as opposed to, say, the x, y expansions, turned out to be sensitive to a numerical asymmetry in the treated Dalitz plots, be it dynamical (a CP violation) or purely instrumental.

The above observations together indicate that, notwithstanding the impressive amount of data already collected on $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$ decays[2,3] it is still very desirable to increase the number of observed events considerably, in the hope that the coefficient b^d_{10} will still remain at the same level but its error will be decreased (by a factor of 2 or so).

TABLE II. Best-fit parameter values (and statistical errors) for $\tau^+ - \tau^-$. All parameter values have been multiplied by 10^2.

			Conventional power-series expansion		
NDF	153	152	151	149	147
χ^2	143.15	143.09	140.70	137.79	136.99
χ^2/NDF	0.936	0.941	0.932	0.925	0.932
$\tilde{R}^d_{00}$		0.01 ± 0.05	-0.02 ± 0.06	0.03 ± 0.10	0.05 ± 0.11
$\tilde{R}^d_{10}$			-0.18 ± 0.12	-0.26 ± 0.12	0.02 ± 0.33
$\tilde{R}^d_{20}$				-0.37 ± 0.24	-0.45 ± 0.29
$\tilde{R}^d_{02}$				0.15 ± 0.28	0.11 ± 0.29
$\tilde{R}^d_{30}$					-0.50 ± 0.60
$\tilde{R}^d_{12}$					-0.55 ± 0.86
			Two-variable O(4) expansion of the $\lvert$M.E.$\rvert^2$		
NDF	153	152	151	149	147
χ^2	143.15	143.09	139.95	137.45	136.91
χ^2/NDF	0.936	0.941	0.927	0.922	0.931
b^d_{00}		0.06 ± 0.23	-0.27 ± 0.30	-0.34 ± 0.31	-0.39 ± 0.32
b^d_{10}			0.46 ± 0.26	0.76 ± 0.33	0.85 ± 0.36
b^d_{20}				-0.43 ± 0.28	-0.50 ± 0.36
b^d_{22}				0.13 ± 0.29	0.01 ± 0.35
b^d_{30}					0.09 ± 0.29
b^d_{32}					0.20 ± 0.31

As far as the O(4) expansions are concerned, we can only conclude that they fit the data just as well as any other expansion, model, or approximation used to fit $K \to 3\pi$ decays. This in itself is certainly encouraging enough to carry over the techniques to an O(3, 1) treatment of scattering. In the immediate future we plan to look at other processes, for which the physical region of allowed kinematic parameters s and t is also finite, but for which the Dalitz plots have a more interesting structure. Such are the $\eta \to 3\pi$ decays and nucleon-antinucleon annihilations into three pions (for fixed values of the initial energy). The generalization of the O(3, 1) and O(4) expansions to nonzero spins, necessary for the last application, is in progress. A satisfactory treatment of the $\bar{p}n \to \pi^-\pi^-\pi^+$ Dalitz plots is one of the major successes of the Veneziano model and its extensions.[23,24] We hope to compare our O(4) treatment with the predictions of the Veneziano model, in particular to calculate the O(4) partial-wave amplitudes in this model. A meaningful check of the $\Delta I = \frac{1}{2}$ rule predictions, using the O(4) expansion, has to be postponed until more data on other $K \to 3\pi$ expansions are available.

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