*Work supported in part by the U. S. Atomic Energy Commission.

†National Science Foundation Graduate Fellow.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 11 1 JUNE 1972

Spectral Representations and Their Application to Inclusive Processes*

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(Received 28 January 1972)

We give a representation for the product of two currents sandwiched between two-particle states, akin to the Deser-Gilbert-Sudarshan representation of commutators. In this case, arguments based on causality are not available for the two-particle matrix element. It is shown that in the process $pp \to \mu^+\mu^-$ + anything, the cross section cannot decrease rapidly in $q_\perp$ (the perpendicular momentum transfer) for $q_\perp{}^2 \ll q^2$ (q is the $\mu^+\mu^-$ momentum), unless it decreases extremely rapidly in q^2. An application of the free-quark model to the product of two pion currents yields Feynman scaling for the process $pp \to \pi$ + anything, when proper account is taken of the rapid decrease in $q_\perp$ of this cross section.

I. INTRODUCTION

The Deser-Gilbert-Sudarshan (DGS) representation[1,2] (a specialized version of the Jost-Lehmann-Dyson representation), based on causality and spectrum conditions, has been of great value in the investigation of deep-inelastic electroproduction and neutrino production.[3] For these semileptonic inclusive reactions, matrix elements of the type

$$\langle p|[J_\mu(x), J_\nu(0)]|p\rangle \tag{1}$$

(where $|p\rangle$ is a single-hadron state) determine the cross sections. Thus the commutator, with its causal structure, is directly accessible experimentally.

The situation is different for hadronically induced inclusive processes, such as $pp \to \mu^+\mu^-$ + anything or $pp \to \pi$ + anything. Cross sections for these processes depend on matrix elements of the type

$$\langle p_1 p_2 \,\mathrm{in}|J(x)J(0)|p_1 p_2\,\mathrm{in}\rangle\,, \tag{2}$$

which are of a different character from (1) for several reasons. First, the experiments measure the ordinary (Wightman) product, not the commutator, so that conventional causality arguments are not available for constructing a spectral representation for (2). Second, the two-particle states $|p_1 p_2\,\mathrm{in}\rangle$ behave very much like an unstable single-particle state would, and this leads to some nontrivial technical complications with the spectrum conditions. Third, one must distinguish between in and out states in (2), while this is irrelevant for (1). This is of no consequence for the general nature of the spectral representations we shall give for matrix elements of the type (2), except for certain reality properties.

In this paper, we give a spectral representation for matrix elements such as (2), based on a straightforward generalization of the conventional DGS representation for (1). The needed generalization is perfectly transparent, but the proof that it is valid is not, for the reasons given above: Causality arguments are lacking, and there are problems with spectrum conditions. In fact, we do not give a completely general proof, although one could probably be constructed along the lines of Nakanishi's[2] recent discussion of the usual DGS representation in perturbation theory.

In Sec. II, we point out that it is very simple to go from the commutation matrix element (1) to the Wightman matrix element, in the case that $|p\rangle$ is a stable single-hadron state. This is because the two orderings of Wightman products in (1) have Fourier transforms with disjoint support in momentum space. A similar line of argument fails for the commutator matrix element corresponding to (2), because the supports in momentum space overlap, as is well known. [Precisely the same problem arises in (1) if $|p\rangle$ simulates an *unstable* state.] Two complementary approaches are available for circumventing this difficulty; the first is to develop, directly from Feynman graphs for the matrix element,

$$\langle p_1 p_2 \,\mathrm{out}|\, (J(x)J(0))_+ \,| p_1 p_2 \,\mathrm{in}\rangle \,, \tag{3}$$

a representation for the time-ordered product. It is then easy to go from this to the Wightman product. Two apparent problems arise here: A variable of integration which should be positive at first sight appears to take on negative values, all the way down to negative infinity. In fact, the variable of integration can be chosen to be always positive, by deforming the contour of integration with due regard to the analytic properties of the integrand. The other apparent problem, that the Wightman product corresponding to (3) is sandwiched between an out and an in state [unlike (2)], is easily resolved in principle by taking discontinuities of spectral functions.[4]

The second general approach to a spectral representation for (2) invokes minimal analyticity properties in the external momenta p_1 and p_2. A completely satisfactory treatment can only be based on a further reduction of (2) in which, for example, the state of momentum p_1 is taken off the mass shell. Nevertheless, it is plausible that no general features of the spectral representation for (2) are changed by allowing p_1 or p_2 to have a small imaginary part. This permits the resolution of the overlap between the two orderings in the commutator matrix element, and the argument used for stable single-particle states can be taken over.[5]

In Sec. III, we give the only application of the representation that we know of, which can be made plausible without specification of detailed dynamics. It holds for the process $pp \to \mu^+\mu^-$ + anything, in which the $\mu^+\mu^-$ pair is formed from the decay of a massive timelike photon of momentum q. Let us consider the case $q^2 \gg M^2$, where M is a typical hadron mass (~1 GeV); then we argue that the cross section for the process is rather insensitive to the *perpendicular* momentum transfer squared $q_\perp{}^2$ if $q_\perp{}^2 \ll q^2$ [i.e., $q_\perp{}^2 \leq O(M^2)$]. This is, of course, in sharp contrast to hadronic processes, where the perpendicular-momentum-transfer spectrum of any single particle is rapidly decreasing.[6] We hasten to point out that this result has nothing to do with the assumption of scale invariance, light-cone dominance, or the like, nor is it a kinematic triviality; it holds only because the Fourier transform of (2) is constrained by the spectral representation we give. The available experimental evidence[7] supports the relative insensitivity of the cross section to variations in $q_\perp$, which is claimed to be independent of $q_\perp{}^2$ at least out to 1 GeV^2 (typical hadronic cross sections decrease like $e^{-6q_\perp}$).

This result is perhaps not surprising, and has been discussed by other authors[8,9] in the context of specific physical mechanisms (e.g., the parton model). Nevertheless, it is interesting that this result can be made plausible on the general grounds of the space-time structure of a product of two currents.

Unlike electroproduction, where the asymptotic cross sections are governed by the operator product of two currents near the light cone, hadronically induced inclusive processes may or may not be related to light-cone expansions.[10] One obvious question is whether the Feynman-Yang scaling law[11] for such processes can be derived from plausible light-cone expansions for the product of, e.g., two pion currents; Segrè[12] has argued that this is not possible. We argue that this scaling law does follow naturally from the free-quark model for the current product, provided that one takes into account the observed rapid decrease of the cross section in $q_\perp$. In this connection, the spectral representation is extremely useful in disentangling the s [$s=(p_1+p_2)^2$] dependence of the Fourier transform of (3) from the q dependence, a step which is necessary in establishing the relevance of light-cone expansions.

Many readers will be interested primarily in the applications of the representation, and not in its derivation, since it is such an obvious generalization of the usual DGS representation. These readers should skip (or skim) Sec. II. We go into some detail in this section primarily to allay the fears of those who discover that one variable of integration which should be positive appears to take on negative values.

II. DERIVATION OF THE SPECTRAL REPRESENTATION

We begin with a review of well-known material. The DGS representation for the single-particle matrix element of the scalar currents is (physical states covariantly normalized)

$$\langle p|[J(x),K(0)]|p\rangle = \int_0^\infty d\lambda^2 \int_{-1}^{1} d\beta\, h(\lambda^2,\beta)e^{i\beta p\cdot x} i\Delta(x;\lambda^2). \tag{4}$$

(The limits of integration are purely formal; we only require that h vanish outside the given limits.) Δ is the usual free-field commutator.[13] For the Fourier transform we have

$$\begin{aligned}\int d^4x\, e^{iqx}\langle p|[J(x),K(0)]|p\rangle &= \sum_n (2\pi)^4\delta(p+q-p_n)\langle p|J(0)|n\rangle\langle n|K(0)|p\rangle \\ &\quad -\sum_n (2\pi)^4\delta(p-q-p_n)\langle p|K(0)|n\rangle\langle n|J(0)|p\rangle \\ &= 2\pi\int d\lambda^2 d\beta\, h(\lambda^2,\beta)\epsilon(q_0+\beta p_0)\delta((q+\beta p)^2-\lambda^2).\end{aligned} \tag{5}$$

Let M_1 and M_2 be the lowest masses occurring, respectively, in the first and second sums over states in (5), and M be the mass of state $|p\rangle$, taken to be at rest ($\vec{p}=0$). If $M \leq \frac{1}{2}(M_1+M_2)$, the supports in q space of the two sums in (5) are disjoint, as is well known. Correspondingly, there must be two distinct regions of support based on the integral in (5), that is, the regions

$$q_0 = -\beta M \pm (\vec{q}^2+\lambda^2)^{1/2} \tag{6}$$

must be disjoint, the plus sign going with the first sum in (5). Throughout each of the two regions (6), the ϵ function in (5) has a unique sign, and we conclude immediately[14] that

$$\langle p|J(x)K(0)|p\rangle = \int d\lambda^2 d\beta\, h(\lambda^2,\beta)e^{i\beta p\cdot x}\Delta_+(x;\lambda^2), \tag{7a}$$

$$\langle p|K(x)J(0)|p\rangle = \int d\lambda^2 d\beta\, h(\lambda^2,-\beta)e^{i\beta p\cdot x}\Delta_+(x;\lambda^2), \tag{7b}$$

where

$$\Delta_+(x;\lambda^2) = \frac{1}{(2\pi)^3}\int d^4k\, e^{-ikx}\theta(k_0)\delta(k^2-\lambda^2) \tag{8}$$

is the free-field function for the ordinary product.

Clearly, the line of attack used here breaks down if $M \geq \frac{1}{2}(M_1+M_2)$, so that the supports of the two sums in (5) overlap. The above inequality implies $M \geq \min\{M_1, M_2\}$, so that in applications (where the currents J, K are replaced by weak or electromagnetic currents) the state $|p\rangle$ would be unstable to electromagnetic or weak decay, and perhaps unstable to strong decay as well. Of course, we are not interested in describing, say, electroproduction from an unstable target like an N^*, but this overlap of support regions inevitably occurs in hadronically induced inclusive processes, dependent on matrix elements of the type (2). Here p is replaced by p_1+p_2, with $(p_1+p_2)^2 = s \equiv W^2$, and $M_1 = M_2$ is the mass of the lowest-mass unobserved state in the process $p_1+p_2 \to$ (single observed particle) + anything. Clearly, for this process to go at all, $W > M_1$, and the supports of the two product orderings in the commutator overlap.

There is no problem in writing down the matrix elements of the commutator between two-particle states:

$$\begin{aligned}\langle p_1p_2|[J(x),J(0)]|p_1p_2\rangle &\equiv C(x^2, p_1\cdot x, p_2\cdot x, p_1\cdot p_2) \\ &= \int_0^\infty d\lambda^2\int_{-1}^{1} d\beta_1\int_{-1}^{1} d\beta_2\, h(\lambda^2,\beta_1,\beta_2;s)e^{i(\beta_1p_1\cdot x+\beta_2p_2\cdot x)} i\Delta(x;\lambda^2).\end{aligned} \tag{9}$$

This obvious generalization of the DGS representation relies on the fact that C, which depends on the listed invariants, vanishes for $x^2<0$ (as Δ does) and has a Fourier transform with respect to $p_1\cdot x$ and $p_2\cdot x$, whose support is limited by the mass spectrum to lie between -1 and $+1$. Again, it is obvious what the changes in (9) must be to represent the matrix element of the ordinary product $J(x)J(0)$; $i\Delta(x;\lambda^2)$ should be replaced by $\Delta_+(x;\lambda^2)$. It is not, however, quite obvious to prove this; the point is that there could conceivably be a term in the ordinary product which is an integral over the symmetric function $\Delta_1(x) = \Delta_+(x) + \Delta_+(-x)$ and which does not interfere with causality when the commutator is formed.

Although strictly speaking the representation (9) does not carry any information about off-shell behavior of the momenta p_1 and p_2, let us suppose that we can consider p_1 or p_2 (or both) to be complex 4-vectors, with a forward timelike imaginary part. When (9) is written out as a sum over states [cf. (5)], the first sum will have terms in $\delta(p_1+p_2+q-p_n)$ and the second terms in $\delta(p_1+p_2-q-p_n)$. Since the momenta p_n of intermediate states are real, it follows that the first sum can be nonvanishing only when q has a backward timelike imaginary part, and the second term only when q has a forward timelike imaginary part. In this way, the two overlapping supports can be resolved, and a proof similar to that leading from (4) to (7) might be constructed.[5]

While such an argument is satisfying in its generality, it lacks the details which lead to conviction. We therefore turn to perturbation theory, where a naive treatment suggests that the integral over λ^2 in (9) runs from $-\infty$ to ∞, apparently violating causality. A closer look shows that this is not the case, and λ^2 is restricted to positive values.

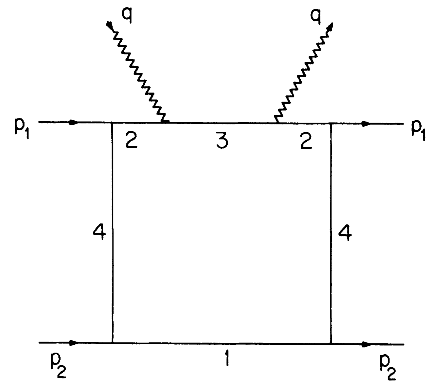

FIG. 1. A typical Feynman graph for the process $pp \to \mu^+\mu^- + \text{anything}$.

Feynman graphs are naturally suited to the discussion of matrix elements between in and out states of time-ordered products. Eventually, we want a matrix element between two in states, but changing from an out to an in state does not change the space-time character of the spectral representation, only the character of the spectral function, so we will develop from a Feynman graph the representation of the ordinary product between in and out states.

Define

$$t(q^2, p_1\cdot q, p_2\cdot q, p_1\cdot p_2) = i\int d^4x\, e^{iq\cdot x}\, T(x^2, p_1\cdot x, p_2\cdot x, p_1\cdot p_2)\,, \tag{10}$$

where

$$T(x^2, p_1\cdot x, p_2\cdot x, p_1\cdot p_2) = \langle p_1 p_2\,\mathrm{out}\,|\,(J(x)J(0))_+\,|\,p_1 p_2\,\mathrm{in}\rangle\ . \tag{11}$$

A typical Feynman graph contributing to t is shown in Fig. 1. Up to an over-all constant, t has the expression

$$t = \int \prod \frac{d\alpha_i\, \delta(1-\sum \alpha_i)\alpha_2\alpha_4}{[\alpha_3(1-\alpha_3)q^2 + 2\alpha_3(\alpha_1+\alpha_4)p_1\cdot q + 2\alpha_1\alpha_3 p_2\cdot q + \alpha_1(\alpha_2+\alpha_3)s - M^2(1-\alpha_4)^2 - M^2\alpha_4]^4}\ . \tag{12}$$

Here $s = (p_1+p_2)^2$, and all particles have mass M. With the change of variables

$$\beta_1 = \frac{\alpha_1+\alpha_4}{1-\alpha_3}\,, \quad \beta_2 = \frac{\alpha_1}{1-\alpha_3}\,, \quad \lambda^2 = \frac{M^2}{1-\alpha_3} + \frac{1}{\alpha_3}\{M^2[(\beta_1-\beta_2)^2 - (\beta_1-\beta_2)+1] - s\beta_2(1-\beta_1)\}, \tag{13}$$

Eq. (12) becomes

$$t = \int_0^1 d\beta_1 \int_0^{\beta_1} d\beta_2 \int \frac{d\lambda^2}{\alpha_3{}^4}(1-\beta_1)(\beta_1-\beta_2)\left|\frac{\partial\alpha_3}{\partial\lambda^2}\right| [(q+\beta_1 p_1+\beta_2 p_2)^2 - \lambda^2 + i\epsilon]^{-4}\,. \tag{14}$$

In (14), we are to substitute the appropriate branch of the roots of the third equation in (13):

$$\alpha_3^{\pm} = \frac{1}{2} + \frac{X^2-M^2}{2\lambda^2} \pm \frac{y}{2\lambda^2}\,, \qquad \left|\frac{\partial\alpha_3}{\partial\lambda^2}\right| = \frac{\alpha_3(1-\alpha_3)}{|y|}\,, \tag{15}$$

where

$$X^2 = M^2[(\beta_1-\beta_2)^2 - (\beta_1-\beta_2)+1] - s\beta_2(1-\beta_1)\,, \qquad y^2 = (\lambda^2 - M^2 - X^2)^2 - 4M^2X^2\,. \tag{16}$$

The fact that the q-dependent term in (14) appears to the minus fourth power is immaterial; this power can be reduced to minus one, as in the usual DGS representation, by integration by parts.

First, consider the unphysical case $0<s<4M^2$. It is not difficult to show that in such a case $y^2 \geq 0$, $X^2 \geq 0$ for allowed values of β_1 and β_2 in (14), and that the minimum value attained by λ^2 as α_3 goes from 0 to 1 is

$$\lambda_{\min}{}^2 = (M+X)^2 \quad (X>0), \tag{17}$$

corresponding to $y^2=0$. As α_3 goes from 0 to $M(M+X)^{-1}$, λ^2 goes from ∞ to $\lambda_{\min}{}^2$ corresponding to the minus sign in (15), and in the interval $M(M+X)^{-1} < \alpha_3 \leq 1$, λ^2 goes from $\lambda_{\min}{}^2$ to ∞, corresponding to the plus sign in (15). This unphysical case is entirely analogous to the ordinary one-particle DGS representation for a stable particle of momentum p_1+p_2, which can be gotten from Fig. 1 by contracting the lines labeled 4; this has the effect of setting $\beta_1=\beta_2$ in Eqs. (13), (15), and (16).

However, in the physical case $s>4M^2$, there are always values of β_1 and β_2 for which $X^2<0$, so that from (13) and (16) λ^2 apparently covers the range $-\infty<\lambda^2<\infty$ as α_3 goes from 0 to 1. These values spread out from the point $\beta_1=\beta_2=\frac{1}{2}$ at $s=4M^2$, and encroach more and more into the allowed regions of β_1 and β_2 as s increases. Call this allowed region R, and divide into two parts, R_+ for which $X^2 \geq 0$, and R_- for which $X^2 \leq 0$. Correspondingly, the integral (14) is divided into two parts; that over R_+ causes no trouble and can be handled as described in the previous paragraph. In the integral over R_+, $\lambda^2 \geq M^2$ from (17), since $X=0$ on the boundary of R_+.

In the integral over R_-, λ^2 goes *once* from $-\infty$ to ∞, instead of *twice* from M^2 to ∞ (as in R_+). It is easily verified that the appropriate branch of Eqs. (15) is $\alpha_3=\alpha_3^+$ which gives $\alpha_3=0$ at $\lambda^2=-\infty$ and $\alpha_3=1$ at $\lambda^2=+\infty$ [note that y^2 continues to be nonnegative, since the roots of $y^2=0$ are $X^2=(\lambda \pm M)^2 \geq 0$, but $X^2 \leq 0$ in R_-].

The integral over R_- in (14) is, schematically,

$$\int_{R_-} d\beta_1 d\beta_2 \int_{-\infty}^{\infty} d\lambda^2 F(\lambda^2, \beta_1, \beta_2, s) \times[(q+\beta_1 p_1+\beta_2 p_2)^2-\lambda^2+i\epsilon]^{-4}, \tag{18}$$

where F behaves no worse than λ^4 as $|\lambda^2| \to \infty$ in any direction in the complex plane, and has its only singularities in the finite plane at the roots of $y=0$. These properties of F follow from the substitution of (15) and (16) into (14), the tedious details of which we omit. Let us take $\operatorname{Im} q^2$, $\operatorname{Im} p_1 \cdot q$, $\operatorname{Im} p_2 \cdot q>0$, so that the square bracket in (18) is free of singularities for λ^2 in the lower half plane, and exchange the order of integration, leaving the λ^2 integration to the last. The integral over β_1 and β_2 in (18) will give rise to end-point singularities in λ^2, coming from a root of $y=0$ when β_1, β_2 are on the boundary of R_-. Since this boundary is just $X=0$, (16) shows that there will be an end-point singularity at $\lambda^2=M^2$. Thus (18) is an integral of the type

$$\int_{-\infty}^{\infty} d\lambda^2\, G(\lambda^2, \ldots), \tag{19}$$

where the only singularity of G in the lower half plane is a branch point at $\lambda^2=M^2$, and where G decreases like $|\lambda|^{-4}$ at infinity. It is then possible to bend the contour of integration back on itself, pivoting on the branch point at $\lambda^2=M^2$ through the lower half plane, until (19) is an integral from M^2 to ∞ of a certain discontinuity of G.[15]

At this point, we have proven the following: The contribution of Fig. 1 to T, as given by Eq. (11), has the representation

$$T=\int d\lambda^2 d\beta_1 d\beta_2\, h(\lambda^2, \beta_1, \beta_2; s) \times e^{i(\beta_1 p_1\cdot x+\beta_2 p_2 \cdot x)} i\Delta_F(x; \lambda^2), \tag{20}$$

where the minimum value of λ^2 is M^2 (more generally, $m_3{}^2$ where m_3 is the mass of the line associated with the 3 in Fig. 1). To this we add the contribution of the crossed graph, for which q in (14) is replaced by $-q$, or, alternatively, β_1 and β_2 are replaced by $-\beta_1$ and $-\beta_2$. Thus, in general, the weight function h is symmetric under this interchange. We now easily find the Wightman matrix element. For $x^0>0$,

$$T(x)=\langle p_1 p_2\, \mathrm{out}|J(x)J(0)|p_1 p_2\, \mathrm{in}\rangle \equiv W(x), \tag{21a}$$

while for $x^0<0$,

$$T(x)=\langle p_1 p_2\, \mathrm{out}|J(0)J(x)|p_1 p_2\, \mathrm{in}\rangle = W(-x), \tag{21b}$$

this latter following from translational invariance. On the other hand,

$$i\Delta_F(x; \lambda^2)=\theta(x^0)\Delta_+(x; \lambda^2)+\theta(-x^0)\Delta_+(-x; \lambda^2). \tag{22}$$

Compare (20) with (21a) for $x^0>0$, and (20) with (21b) for $x^0<0$, in the latter case replacing x by $-x$ and using the symmetry of h under $(\beta_1, \beta_2) \longleftrightarrow (-\beta_1, -\beta_2)$, to conclude

$$W(x)=\int d\lambda^2 d\beta_1 d\beta_2\, h(\lambda^2, \beta_1, \beta_2; s) \times e^{i(\beta_1 p_1\cdot x+\beta_2 p_2\cdot x)} \Delta_+(x; \lambda^2), \tag{23}$$

which is the desired representation. It is fairly clear that the proof could be generalized to any

Feynman graph, using the ideas of Nakanishi.[2]

It only remains to discuss the Wightman product between two in states. It is clear that (23) still holds, with certain restrictions on h; if $J(x)$ is a Hermitian current, h must be real. One way to show that (23) holds is to note that the requisite connected in-in matrix element in Fig. 1 can be gotten from the time-ordered product by replacing the propagators labeled 1 and 3 in Fig. 1 by $2\pi\theta(p_{0i})\delta(p_i^2-m_i^2)$ $(i=1,3)$. This can be done by taking the double discontinuity of the original Feynman graph – as given by an expression like (12), but for general masses – in m_1^2 and m_3^2, and inserting the θ functions by hand. These θ functions can be shown to be equivalent to the θ function contained in Δ_+ in (23), while taking the discontinuity produces the appropriate δ function (or derivative thereof). We omit the uninstructive details.

We end this section by noting that the parton-model results of Bjorken *et al.*[8] for inclusive processes are equivalent to the representation (23). See, e.g., their Eq. (A38), which has a δ function in it corresponding to that contained in the Fourier transform of (23).

III. APPLICATIONS TO MASSIVE LEPTON PAIR PRODUCTION

Applications of the representation (23) are of two types: Those which do and do not require assumptions about specific dynamics which go beyond those entering into the derivation of the representation itself. In this section, we mention the only application we know of which requires no dynamical hypotheses, except that the weight function h in (23) does not exhibit such pathology as infinite oscillations.

The cross section for the process $pp\to\mu^+\mu^-$ + anything can be written

$$2q_0\frac{d\sigma}{d^3q\,dq^2}=\frac{\alpha^2}{12\pi^3q^2[s(s-4M^2)]^{1/2}}(-W_\mu{}^\mu)\,,\quad(24)$$

where $\alpha=\frac{1}{137}$, $s=(p_1+p_2)^2$, the initial photon momentum is q, and we have neglected the muon mass. Here

$$W_{\mu\nu}=\int d^4x\,e^{-iqx}\langle p_1p_2\,\mathrm{in}|J_\mu(x)J_\nu(0)|p_1p_2\,\mathrm{in}\rangle_c\,,\quad(25)$$

where the subscript c indicates that only fully connected graphs are to be saved. According to the work of Mueller, Stapp, and Tan,[4] $W_{\mu\nu}$ is a certain discontinuity of the time-ordered product between in and out states, as discussed in Sec. II, and $W_{\mu\nu}$ has a representation

$$W_{\mu\nu}=\int d\lambda^2d\beta_1d\beta_2\,h_{\mu\nu}(\lambda^2,\beta_1,\beta_2;s)\theta(-q_0+\beta_1p_{10}+\beta_2p_{20})\times\delta((-q+\beta_1p_1+\beta_2p_2)^2-\lambda^2),\quad(26)$$

where $h_{\mu\nu}$ is a real tensor. The tensor indices on $h_{\mu\nu}$ are formed from the components of p_1, p_2, and q, but this tensor structure is irrelevant to our discussion, except to note that we need only save terms symmetric in exchange of μ and ν. Then $h_{\mu\nu}$ is symmetric under the operation $(\beta_1,\beta_2)\to(-\beta_1,-\beta_2)$ (crossing symmetry), and because the two initial particles are identical, $h_{\mu\nu}$ is also symmetric under $(\beta_1,\beta_2)\longleftrightarrow(\beta_2,\beta_1)$.

We are interested in the cross section (24) at large q^2, which necessarily means large s, since $q^2\leqslant(s^{1/2}-2M)^2$. Suppose for the moment that the produced initial photon were replaced by a pion. Then the differential cross section would fall rapidly in $q_\perp^2$ (perhaps like $e^{-6q_\perp}$, with $q_\perp$ in GeV), where $q_\perp$ is the component of q perpendicular to the incident beam. What we shall make plausible is that, in the case $q^2\gg q_\perp^2$ (which can be attained for virtual photons but not for pions), the cross section is either relatively *insensitive* to $q_\perp^2$, compared to typical pure hadronic cross sections, or hypersensitive to variations in q^2. Our remarks are complementary to those of other authors,[8,9] who argue that, at *large* $q_\perp$ (that is, $q_\perp^2/s$ finite), cross sections such as (24) fall off like a power, much less rapidly than the exponential behavior in $q_\perp$ that one extrapolates for inclusive production by purely hadronic processes.

As we have mentioned before, this is not a surprising result, but it is interesting to trace its origins in the space-time structure of the product of the currents, as given in (23).

The cross section (24) depends on the variables s, $p_1\cdot q$, $p_2\cdot q$, and q^2 from which we may define $q_\perp^2$ in a Lorentz-invariant way as

$$q_\perp^2=\frac{1}{s}[q\cdot(p_1+p_2)]^2-\frac{1}{s-4M^2}[q\cdot(p_1-p_2)]^2-q^2\,.\quad(27)$$

What we will show is that, in general, $W_{\mu\nu}$ is more or less equally sensitive to variations in $q_\perp^2$ or q^2. In the regime $q^2\gg q_\perp^2$, a large percentage change in $q_\perp$ corresponds to but a small percentage change in q^2, for equal variations $\Delta q^2\simeq\Delta q_\perp^2$. Barring an unforeseen hypersensitive dependence of the cross section on q^2, it cannot be as sensitive as hadronic cross sections to variations in $q_\perp^2$. In the limit $s\gg M^2$, we may express $q\cdot p_1$ and $q\cdot p_2$ in terms of $q_\perp$, $q_\parallel$, and q^2 as

$$q\cdot p_1=\tfrac{1}{2}s^{1/2}[-q_{\parallel}+(q_{\parallel}^2+q_{\perp}^2+q^2)^{1/2}]\,,$$
$$q\cdot p_2=\tfrac{1}{2}s^{1/2}[+q_{\parallel}+(q_{\parallel}^2+q_{\perp}^2+q^2)^{1/2}]\,,\qquad(28)$$

where $q_{\parallel}$ is the longitudinal momentum in the center-of-mass frame. Substitute these expressions into the δ function in (26) to find that for fixed $q_{\parallel}$ and s the sensitivity of $W_{\mu\nu}$ to variations in q^2 and $q_{\perp}^2$ is expressed through the equation ($q^2 \gg q_{\perp}^2$)

$$\Delta\delta((-q+\beta_1 p_1+\beta_2 p_2)^2-\lambda^2)=\delta'((-q+\beta_1 p_1+\beta_2 p_2)^2-\lambda^2)\times\{\Delta q^2[1-\tfrac{1}{2}s^{1/2}(\beta_1+\beta_2)(q_{\parallel}^2+q_{\perp}^2)^{-1/2}]-\Delta q_{\perp}^2\tfrac{1}{2}s^{1/2}(\beta_1+\beta_2)(q_{\parallel}^2+q_{\perp}^2)^{-1/2}\}.\qquad(29)$$

Equation (28) is to be integrated over the weight function h as in (26) to yield the variation of $W_{\mu\nu}$. After this weighted integration has been performed, it is clear that the coefficient of Δq^2 may be *large* compared to the coefficient of $\Delta q_{\perp}^2$ if the weighted value of $\beta_1+\beta_2$ is sufficiently small compared to 1, but it cannot be *small*, except for exceptional values of s, q^2, and $q_{\parallel}^2$. So $W_{\mu\nu}$ is no more sensitive to variations in $q_{\perp}^2$ than it is to those in q^2. We illustrate the significance of this fact with an example: Take

$$q_{\perp}=\Delta q_{\perp}=0.2\text{ GeV}\,,\quad q^2=4\text{ GeV}^2\,,$$
$$\Delta q_{\perp}^2=\Delta q^2\,,\quad\text{so}\quad \Delta q^2/q^2\simeq 2\%\,.$$

Now we do not expect – or observe in experiments – that the μ-pair-production cross section changes by an amount comparable to itself if $\Delta q^2/q^2\simeq(1-2)\%$. Yet, in a typical inclusive reaction in which a single hadron is observed, with cross section proportional to $e^{-6q_{\perp}}$, the cross section changes by a factor of 3 for $q_{\perp}=\Delta q_{\perp}=0.2$ GeV. It follows that, for large q^2, the μ-pair inclusive reaction is far less sensitive to $q_{\perp}$ than a typical pure hadronic reaction. This insensitivity has been recorded in the Columbia experiment,[7] where the cross section is observed to be nearly flat out to $q_{\perp}^2=1$ GeV^2.

From the above discussion we conclude that the behavior in $q_{\perp}$ of inclusive cross sections is tied to the mass of the produced particle. It is worth noting that, for the inclusive production of hadrons such as π, K, and p, the exponential decrease in $q_{\perp}$ of the cross sections is the less severe the more massive the produced hadron, i.e., the larger q^2 is.[6] This qualitative trend is to be expected on the basis of the representation (23).

IV. LIGHT CONE AND INCLUSIVE PROCESSES

There is only one class of inclusive processes for which the cross sections are necessarily dominated at high energy by the region near the light cone: the well-known electroproduction or neutrino-production process.[16,17] The situation is much less clear for inclusive processes such as $pp\to a$ + anything, where a is a μ pair or a pion of momentum q. The contribution from the light cone may[18] or may not[19] be the most significant. The problem here is that the light cone is reached in the limit $q\to\infty$,[17] with all other momenta held fixed. This is an unphysical limit for the above processes, since $q\to\infty$ requires $s\to\infty$ also. The relevance of the light cone can be established only with some specification of the s dependence of the relevant matrix element.

In this section, we study qualitatively the process $pp\to\pi^+$ + anything, using the representation (23), and find that Feynman scaling emerges in a natural way. The cross section is given by

$$2q_0\frac{d\sigma}{d^3q}=\tfrac{1}{2}(2\pi)^{-3}[s(s-4M^2)]^{-1/2}\times\int dx\,e^{-iqx}\langle p_1p_2\,\text{in}|J_+(x)J_-(0)|p_1p_2\,\text{in}\rangle_c\,,\qquad(30)$$

where $J_\pm$ are the charged pion currents. The matrix element in (29) will be constrained to reflect the following physics: (1) the fact that this cross section does indeed decrease rapidly in $q_{\perp}$; (2) the free-quark–model prediction for the small-x behavior of the current product.

The second point will be considered first. In the quark model, the pion current is

$$J_a(x)=M_QF_\pi^{-1}\bar\psi(x)i\gamma_5\lambda_a\psi(x)\,,\qquad(31)$$

where M_Q is the quark mass, and $F_\pi\simeq 94$ MeV is the pion decay constant. This combination of factors gives an approximate Goldberger-Treiman relation [with $(-G_A/G_V)\simeq 1$] for the quark axial-vector current. It is easy to extract, for free quarks, the light-cone expansion

$$J_a(x)J_b(0)\underset{x^2\to 0}{\sim}-i\left(\frac{M_Q}{2\pi F_\pi}\right)^2[\bar\psi(x)\lambda_a\lambda_b\gamma^\alpha\psi(0)-\text{H.c.}]\times\partial_\alpha\left(\frac{1}{x^2-i\epsilon x^0}\right)+\cdots\,,\qquad(32)$$

where the omitted terms are less singular in x^2. Essentially this model was used by Cornwall and Levy[20] in large-angle πN elastic scattering, where it yields a scaling law in excellent agreement with experiment. It is interesting to note that the bilocal operators displayed in (32) are exactly the same as in the light-cone expansion of the vector (or two axial-vector) currents,[17,21] as used to deal with the electroproduction and neutrinoproduction experiments.

Because of the appearance of ∂_α in (32), the representation (23) must be modified so that h contains gradient terms. We write

$$\langle p_1 p_2 \,\mathrm{in}| J_+(x) J_-(0) | p_1 p_2 \,\mathrm{in}\rangle_c = i(p_1+p_2)\cdot\partial \int h(\lambda^2, \beta_1, \beta_2; s) \times e^{ix(\beta_1 p_1 + \beta_2 p_2)} \Delta_+(x; \lambda^2)\,. \tag{33}$$

Of course, other terms may be present, involving no gradients or some other combination such as $(p_1-p_2)\cdot\partial$, but they will not change the essential features we wish to discuss. With the help of the formula

$$\Delta_+(x;\lambda^2) \underset{x^2\to 0}{\sim} \left(-\frac{1}{4\pi^2}\right)\frac{1}{x^2 - i\epsilon x^0}\,, \tag{34}$$

we use (32) in (33) to find sum rules for moments of the weight function h. Setting $x=0$ in the bilocal operator of (32) and in Eq. (33) yields a sum rule for the antisymmetric (in a, b) spectral functions h, while saving the first power of x in (32) and (33) gives sum rules for the symmetric functions. The antisymmetric sum rule reads

$$2(2M_Q/F_\pi)^2 \langle p_1 p_2 \,\mathrm{in}| J_3^\alpha(0) | p_1 p_2 \,\mathrm{in}\rangle_c = (p_1+p_2)^\alpha \int h_A(\lambda^2, \beta_1, \beta_2; s)\,, \tag{35}$$

where J_3^α is the isospin current, and $h_A = h_+ - h_-$ is the difference of spectral functions for inclusive π^+ or π^- production. In (35), if one of the states were an out state, we could use the Low[22] theorem for conserved currents to estimate the left-hand side (saving only connected states removes external-bremsstrahlung graphs). Let us estimate the in-in matrix element by the absolute value of the out-in matrix element (see Sanda and Suzuki[23] for further discussion), and ignore off-mass-shell derivatives which would enter because saving only connected states does not exactly correspond to the matrix element discussed by Low. Then we shall use the isospin analog of Low's result[22]:

$$\text{nonsingular part of } \langle p_1 p_2 \,\mathrm{out}| J_3^\alpha(0) | p_1 p_2 \,\mathrm{in}\rangle = (p_1+p_2)^\alpha \frac{\partial T}{\partial s}\,, \tag{36}$$

where T is the forward scattering amplitude for $pp\to pp$. At infinite s, $\partial T/\partial s \to i\sigma_{\mathrm{tot}}$, and our estimate for the left-hand side of (35) leads to

$$\int h_A(\lambda^2, \beta_1, \beta_2; s) \simeq (2M_Q/F_\pi)^2 \sigma_{\mathrm{tot}}\,. \tag{37}$$

A similar sum rule holds for the integral of $(\beta_1+\beta_2)(h_+ + h_-)$, yielding a value of the order of the right-hand side of (37). This sum rule is based on the Low theorem as applied to the stress-energy tensor, which emerges[19] (in the free-quark model) as one of the operators in the first-order expansion of the bilocal operator in (32). We shall see that the essential feature for Feynman scaling is that these integrals be independent of s.

Nevertheless, h must depend on s, in order that the cross section (30) show the usual sharp decrease in $q_\perp$. Introduce the Feynman scaling variable $y = q_\parallel/p \to 2s^{-1/2} q_\parallel$ (the second form holding when $s \gg M^2$), where $q_\parallel$ is the center-of-mass longitudinal momentum. We assume in what follows that $q_\parallel > 0$; if $q_\parallel < 0$, the role of p_1 and p_2 should be exchanged in the discussion below. From (28), the following limits are found for $s \gg M^2$, $q^2 = m_\pi^2$:

$$p_1\cdot q \to \frac{q_\perp^2 + m_\pi^2}{2y}\,,\quad p_2 \cdot q \to \tfrac{1}{2} s y\,. \tag{38}$$

The cross section (30) involves an integral of h over a δ function, as in (26); the argument of this δ function is

$$m_\pi^2 - \frac{\beta_1}{y}(q_\perp^2 + m_\pi^2) - \beta_2 s y + s\beta_1\beta_2 + M^2(\beta_1-\beta_2)^2 - \lambda^2\,. \tag{39}$$

In view of the arguments of Sec. III, if this quantity depends sensitively on $q_\perp^2$, it will depend hypersensitively on s. The s dependence in (39) must be suppressed, and one simple way to do this is to assign the following s dependence to h:

$$h = sF(\lambda^2, \beta_1, s\beta_2) + (\beta_1 \longleftrightarrow \beta_2)\,, \tag{40}$$

which is consistent with the s independence of (37). With the substitution $z = s\beta_2$, (33), (38), and (40) lead to a cross section

$$2q_0 \frac{d\sigma}{d^3 q} = \frac{y}{16\pi^2}\int d\lambda^2 d\beta_1 dz\, F(\lambda^2, \beta_1, z)\,\theta(-q_0 + \beta_1 p_{10} + (z/s) p_{20}) \times \delta(m_\pi^2 - (\beta_1/y)(q_\perp^2 + m_\pi^2) - zy + z\beta_1 + M^2(\beta_1-\beta_2)^2 - \lambda^2) + O(s^{-1})\,, \tag{41}$$

where part of the $O(s^{-1})$ term comes from that part of (40) in which β_1 is exchanged with β_2. As Feynman scaling demands,[11] the leading term of the right-hand side of (41) has the form $\gamma(y, q_\perp^2)$, independent of s at fixed y. This s independence is a consequence of the ansatz (40), which was motivated by the requirements: that (37) be independent of s, and that the s dependence not be coupled strongly to the $q_\perp$ dependence, as would be the case in general [Eq. (39) and Sec. III].

V. DISCUSSION

Inclusive reactions with two hadrons in the initial state are considerably more complex, both kinematically and dynamically, than those (such as electroproduction) with only one hadron in the initial state. The most obvious extra complication is the dependence on s of two-particle matrix elements, and little of use can be extracted from the representation (23) without some knowledge of the dependence of the spectral function h on s. Section IV is a very preliminary step in that direction, but it must be considerably augmented before one can make any decision on the relevance of the light cone[16,17,21] and associated operator expansions.[24,25] To repeat, the reason is that light-cone dynamics can only freely be applied when s is fixed and $q\to\infty$, but we can only do the experiment discussed in Secs. III and IV with $s > q^2$. For this reason, it is presently of little value to catalog the kinematic sum rules relating integrals over cross sections to various singularities at small x^2, which have proven to be so useful in electroproduction.[3] Such sum rules will be important if the arguments concerning s dependence in Sec. IV can be strengthened and extended, but even then formidable difficulties ensue, such as separating out the fully connected part of the sum rules and dealing with matrix elements having in states only. These latter problems are equivalent to the problem of taking a certain discontinuity of Feynman amplitudes of the type (11).[4] With such a great deal of dynamics remaining to be supplied, it is no surprise that present applications, such as in Sec. III, are so limited.

We have not dealt here with an analysis like that of Sec. IV for the process $pp\to\mu^+\mu^-$ + anything,[7,18,19,23] which could be based on light cone dynamics[17,21] and the Low theorem[22] extended to the other conserved operators. Certain difficulties arise which remain to be fully analyzed; we hope to return to this subject in a future publication.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor R. E. Norton for invaluable discussions concerning the material presented in Sec. II, and for carefully reading the manuscript.

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*Work supported in part by the National Science Foundation.

†Alfred P. Sloan Foundation Fellow.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 11 1 JUNE 1972

Two-Variable Expansions and the $K \to 3\pi$ Decays

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(Received 2 February 1972)

Previously suggested two-variable expansions of three-body decay amplitudes in terms of harmonic functions of an O(4) group are discussed and applied to analyze the Dalitz-plot distribution of over 3.2 million $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$ decay events. Among the general features of the O(4) expansions we wish to stress that they are written in the c.m. system of two of the final particles, the angular momentum of which is displayed explicitly, and that each term in the expansion has a good behavior at the threshold, pseudothreshold, and at the boundary of the physical region. We analyze the recent data of Ford *et al.* on charged $K \to 3\pi$ decays, using both O(4) expansion and the standard power-series expansion in terms of the Dalitz-Fabri variables. In both cases it is perfectly adequate to keep four terms in the corresponding expansion. The χ^2 fit is marginally better for the O(4) expansion. We conclude that the $K \to 3\pi$ Dalitz plot has too little structure in it to provide a real test of the advantages or disadvantages of different treatments. It is thus most desirable to apply the O(4) expansions to Dalitz plots of other processes, like $\eta \to 3\pi$ or $\bar{p}n \to \pi\pi\pi$. No conclusive evidence is found for *CP* violation. However, the "linear" term in the O(4) expansion of the difference between the squared matrix elements for K^+ and K^- decays does differ from zero by more than two standard deviations. The effect is stable with regard to the number of terms kept in the expansions. An important distinctive feature of the O(4) expansions is their intimate relation to two-variable O(3, 1) expansions of physical scattering amplitudes.

I. INTRODUCTION

In a previous publication[1] (to be referred to as I), we presented a general formalism for performing harmonic analysis on Dalitz plots, i.e., for analyzing Dalitz-plot distributions for three-body decays, involving particles of spin zero. The main purpose of this paper is to apply the formalism to $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$ decays, using the new data of Ford *et al.*,[2,3] and also to discuss some further features of our approach.

The formalism presented in I consists of two-variable expansions of decay amplitudes in terms of basis functions of irreducible representations of the group O(4). It is actually an extension and modification of an approach developed in a series of previous articles,[4-11] devoted mainly to two-body scattering. The purpose of the whole approach is to develop a reaction theory based on two-variable expansions of relativistic amplitudes and thus to display the entire dependence on the kinematic parameters $s=(p_1+p_2)^2$ and $t=(p_1-p_3)^2$ explicitly in certain special functions, whereas the entire dynamics of the process under consideration is summarized in the expansion coefficients, which we call the Lorentz amplitudes. The motivation is thus twofold, theoretical (the incorporation of general principles, the formulation of dynamical hypotheses) and phenomenological – the fitting of larger bodies of data than can be fitted by single-variable expansions. In this article the phenomenological aspect is stressed.

For scattering, the two-variable expansions are obtained, making use of little else than Lorentz invariance. Indeed, consider the reaction

$$1+2 \to 3+4 \tag{1}$$

and let the particles have arbitrary masses but zero spins. The scattering amplitude $f(s,t)$ can be considered to be a function $M(p_1, \ldots, p_4)$ of the momenta $p_1, \ldots, p_4$, each on its own mass shell. Lorentz invariance and conservation laws natural-