[1]C. R. Johnson, Phys. Rev. D 4, 295 (1971); 4, 318 (1971); 4, 3555 (1971). These papers will be referred to as papers I, II, and III, respectively. It is assumed that the reader is familiar with these earlier papers. Unless otherwise indicated, the notation used in this paper will be the same as that used in the earlier papers.

[2]See Secs. IV B and V A in paper I.

[3]A discussion of these solutions can be found in Sec. V B and the references in paper I and also in paper III.

[4]See Sec. V B and Appendix A in paper I.

[5]See Sec. V B and the references in paper I.

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Scalar-Tensor Theory of Gravitation

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A scalar-tensor theory of gravitation is constructed using the Weyl formulation of Riemannian geometry. The scalar field is given an important geometrical role to play and is related to the integrable change in length of a vector as it is transported from point to point in space-time. The geometry uses modified covariant derivatives and a metric tensor which is not covariantly constant. The field equations can be written down very simply in terms of a modified curvature tensor. The theory agrees with the usual Lagrangian formalism in its experimental predictions and offers a reformulation or reinterpretation of the transformation of units considered by Dicke.

I. INTRODUCTION

Brans and Dicke[1] have formulated a scalar-tensor theory of gravitation using a Lagrangian formalism in which the coupling of the scalar field is $1/\omega$. The theory reduces to Einstein's tensor theory if the parameter ω is infinite. Observations on the perihelion rotation of Mercury agree with the value calculated from the Einstein theory but allow for a contribution from the scalar field if the sun is oblate. Dicke and Goldenberg[2] have measured this oblateness and find that 8% of the residual precession may be due to a solar quadrupole moment. Ingersoll and Spiegel[3] have suggested that a surface temperature differential between the pole and the equator in the sun could account for these oblateness data. The lithium-beryllium abundances observed at the sun's surface lends support to a quadrupole moment however.[4] The recent planetary radar reflection experiment of Shapiro *et al.*[5] found $\lambda = 1.02 \pm 0.05$ where λ is a parameter which is 1.0 for general relativity and less than 1.0 for the scalar-tensor theory. This result suggests that the fractional contribution of the scalar field is at most 3% or 4%.

The present work considers a scalar-tensor theory of gravitation which incorporates a scalar field into general relativity in a very direct unambiguous way, using a modification of Riemannian geometry originally considered by Weyl[6-8] in connection with his work on a unified field theory. We treat the scalar field from the beginning as an object with well-defined geometrical meaning in the spirit of general relativity and arrive at a formalism quite different from the usual Lagrangian theory of Brans and Dicke. The two formalisms agree in their predictions of experimental results, however. In Sec. II we will define the geometry we use. The field equations for nonempty space and the geodesic equations of motion are discussed in Secs. III and IV. The spherically symmetric, time-independent solution is given in Sec. V. This is used to compute experimental results for red shift, deflection of starlight, and perihelion advance in Sec. VI.

II. DEFINITION OF THE GEOMETRY

The geometry that we will use is based upon

$$dl = l T_\alpha dx_\alpha \quad \text{(linear form)} \tag{1}$$

and

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{(bilinear form with symmetric } g_{\mu\nu}\text{)} . \tag{2}$$

T_α is a vector which is related to the scalar field in (33). The l appearing in (1) is the length of an arbitrary vector where, for a general vector ξ_μ, we have

$$l^2 = \xi_\mu \xi^\mu \equiv g_{\mu\nu} \xi^\mu \xi^\nu . \tag{3}$$

Equation (1) thus specifies how the length of a vector changes under displacement through space-time. This T_α is related to gauge transformations in the following way. Let us transform gauges such that in the new gauge

$$\hat{T}_\alpha = T_\alpha + \tfrac{1}{2}(\ln f)_{|\alpha}\,, \tag{4}$$

where $f(x^\beta)$ is an arbitrary scalar field and the single short vertical bar denotes ordinary partial differentiation with respect to x^α. From Weyl's work we then have in the new gauge

$$\hat{g}_{\alpha\gamma} = f(x^\beta) g_{\alpha\gamma} \tag{5}$$

and

$$\hat{\Gamma}^\alpha{}_{\beta\rho} = \Gamma^\alpha{}_{\beta\rho}\,. \tag{6}$$

The metric tensor undergoes a stretching transformation. By allowing gauge transformations of the type shown in (5), Weyl used a more general geometry than the geometry used in ordinary general relativity in which a vector does not change length under displacement.

Weyl[8] also introduced the concept of the weight of a tensor under gauge transformations. If we consider a given contravariant vector b^α to be given independently of the metric, then the covariant form

$$b_\alpha = g_{\alpha\pi} b^\pi \tag{7}$$

depends upon the metric and transforms under a gauge transformation (5) as

$$\hat{b}_\alpha = f(x^\beta) b_\alpha\,. \tag{8}$$

The vector b_α is said to be of weight unity. A tensor is of weight n if it is multiplied by a factor $f(x^\beta)^n$ under a gauge transformation (5). For our field equations involving $g_{\mu\nu}$ we will be interested in tensors of weight zero (which are said to be gauge-independent).

In this geometry we also have the law of parallel displacement which specifies how the components of a vector change under displacement,

$$d\xi^\alpha = \Gamma^\alpha{}_{\beta\gamma} \xi^\beta dx^\gamma\,. \tag{9}$$

In (9), the $\Gamma^\alpha{}_{\beta\gamma}$ are the coefficients of the affine connection. In ordinary Riemannian geometry, we have

$$\Gamma^\alpha{}_{\beta\gamma} = -\left\{ {\alpha \atop \beta\gamma} \right\}, \tag{10}$$

where $\left\{ {\alpha \atop \beta\gamma} \right\}$ is the Christoffel symbol of the second kind. In the present geometry, the connections are given by

$$\Gamma^\alpha{}_{\beta\gamma} = -\left\{ {\alpha \atop \beta\gamma} \right\} + g^{\pi\alpha}(g_{\pi\beta}T_\gamma + g_{\pi\gamma}T_\beta - g_{\beta\gamma}T_\pi)\,. \tag{11}$$

We can easily derive (11) from

$$d(l^2) = d(g_{\mu\nu}\xi^\mu \xi^\nu) = 2l(l T_\alpha dx^\alpha)\,. \tag{12}$$

From (11) we note that

$$\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}\,. \tag{13}$$

Since we have an explicit form for $\Gamma^\alpha{}_{\beta\gamma}$, covariant derivatives can be defined as

$$\xi^\alpha{}_{\|\beta} = \xi^\alpha{}_{|\beta} - \Gamma^\alpha{}_{\beta\gamma}\xi^\gamma \tag{14}$$

and

$$\xi_{\alpha\|\beta} = \xi_{\alpha|\beta} + \Gamma^\gamma{}_{\alpha\beta}\xi_\gamma\,, \tag{15}$$

where a double vertical bar denotes covariant differentiation. These covariant derivatives transform as tensors.

An important consequence of the above geometry is that the metric tensor is no longer covariantly constant. In fact we have

$$g^{\alpha\beta}{}_{\|\gamma} = g^{\alpha\beta}{}_{|\gamma} - \Gamma^\beta{}_{\pi\gamma} g^{\alpha\pi} - \Gamma^\alpha{}_{\pi\gamma} g^{\pi\beta} \tag{16}$$

and substituting (11) gives

$$g^{\alpha\beta}{}_{\|\gamma} = -2g^{\alpha\beta} T_\gamma\,. \tag{17}$$

Similarly,

$$g_{\alpha\beta\|\gamma} = 2g_{\alpha\beta}T_\gamma \tag{18}$$

and

$$g^\alpha{}_{\beta\|\gamma} = 0\,. \tag{19}$$

In general g_{ir} does not commute with covariant differentiation and one must be careful in doing tensor manipulations. From (17) we also find the interesting relationship

$$T^\alpha = -\tfrac{1}{2} g^{\pi\alpha}{}_{\|\pi}\,. \tag{20}$$

Thus T^r is just the divergence of the metric. This expression does not imply that a knowledge of g^{lr} leads to a knowledge of T^r, however, because the T^r arises from the operational definition of the covariant derivative.

Now the condition for integrability of (1) is just

$$T_{\alpha|\beta} - T_{\beta|\alpha} = 0\,. \tag{21}$$

If (21) holds, we can always carry out a gauge transformation (4) in such a way that in the new gauge $\hat{T}_\alpha = 0$.[9] In other words, (21) is the necessary and sufficient condition that a Weyl geometry can be reduced to a Riemannian geometry by a proper gauge transformation. The Weyl formalism can be useful even in a pure Riemannian geometry. It allows a wider choice of metric tensor as in (5) at the expense of having vectors change length as they are moved by parallel displacement. In the original metric $g_{\alpha\gamma}$ in (5), we have $T_\alpha = 0$ and the lengths of vectors are preserved under parallel displacement from (12). In the new metric $\hat{g}_{\alpha\gamma}$ in (5), the square of the length of a vector is

$$\hat{l}^2 = \hat{g}_{\alpha\beta}\xi^\alpha\xi^\beta = f(x^\lambda) l^2 \tag{22}$$

and is a function of position. The same displacement law (9) using $\hat{g}_{\alpha\gamma}$ leads to

$$\frac{d\hat{l}}{\hat{l}} = \tfrac{1}{2}(\ln f)_{|\lambda}\, dx^{\lambda}\,, \tag{23}$$

which is the same as (22). Thus $\hat{T}_\alpha = \frac{1}{2}(\ln f)_{|\alpha}$ from (1) which is the same as (4).

In the work that follows, (21) will always hold as we shall see in (31). Thus we are in a *pure Riemannian geometry*. We shall take advantage of the versatility afforded by the Weyl formalism to choose a modified metric (5), however. Relative to this modified metric, vectors change length as they are displaced. The transformation (5) is very similar to the "units transformations" discussed by Dicke.[10]

We can now introduce the extended curvature tensor by way of

$$\xi^{\gamma}{}_{\|\alpha\|\beta} - \xi^{\gamma}{}_{\|\beta\|\alpha} = S^{\gamma}{}_{\pi\alpha\beta}\xi^{\pi}\,, \tag{24}$$

where

$$S^{\gamma}{}_{\pi\alpha\beta} = -\Gamma^{\gamma}{}_{\pi\alpha|\beta} + \Gamma^{\gamma}{}_{\pi\beta|\alpha} + \Gamma^{\gamma}{}_{\rho\beta}\Gamma^{\rho}{}_{\pi\alpha} - \Gamma^{\gamma}{}_{\rho\alpha}\Gamma^{\rho}{}_{\pi\beta}\,. \tag{25}$$

We can use (11) to write this out in terms of $g_{\mu\nu}$ and T_α as

$$S^{\gamma}{}_{\pi\alpha\beta} = R^{\gamma}{}_{\pi\alpha\beta} + (T_\alpha \text{ terms})\,, \tag{26}$$

where $R^{\gamma}{}_{\pi\alpha\beta}$ is the usual Riemann curvature tensor.

Now let us consider integrability conditions in this geometry as a springboard to the field equations. The condition for integrability of the law of parallel displacement, (9), is just

$$S^{\alpha}{}_{\beta\gamma\delta} = 0\,. \tag{27}$$

The condition for integrability of (1) is just (21). In analogy to ordinary general relativity, it is then very reasonable to take (21) and (27) to be our field equations in completely empty space.

Now we would like to write down the field equations for nonempty space. Let us first, however, consider the symmetry and gauge-invariance properties of $S^{\alpha}{}_{\beta\delta\gamma}$. From (13) we have

$$S^{\alpha}{}_{\beta\delta\gamma} = -S^{\alpha}{}_{\beta\gamma\delta}\,. \tag{28}$$

This is the only symmetry $S^{\alpha}{}_{\beta\gamma\delta}$ possesses. $R_{\alpha\beta\gamma\delta}$ on the other hand is also antisymmetric under interchange of α and β and symmetric under interchange of the pairs $\alpha\beta$ and $\gamma\delta$.

We note from (25) that $S^{\gamma}{}_{\pi\alpha\beta}$ is made up entirely of $\Gamma^{\gamma}{}_{\pi\beta}$ and is thus gauge-invariant. Quantities such as $S_{\gamma\pi\alpha\beta}$, $S^{\gamma\pi}{}_{\alpha\beta}$, $S^{\gamma\pi\alpha}{}_{\beta}$, $S^{\gamma\pi\alpha\beta}$, etc. are not gauge-invariant since $g_{\mu\nu}$ also occurs when we write them out.

III. FIELD EQUATIONS FOR NONEMPTY SPACE

In analogy with general relativity we form contractions of $S^{\alpha}{}_{\beta\pi\delta}$ to get the field equations for nonempty space. If we restrict ourselves to gauge-invariant equations, the only possible contractions are $S^{\alpha}{}_{\beta\alpha\gamma}$, $S^{\alpha}{}_{\beta\gamma\alpha}$, and $S^{\alpha}{}_{\alpha\beta\gamma}$. We thus postulate the following field equations using (28):

$$S^{\alpha}{}_{\beta\alpha\delta} = -S^{\alpha}{}_{\beta\delta\alpha} = 0 \tag{29}$$

and

$$S^{\alpha}{}_{\alpha\beta\delta} = 0\,. \tag{30}$$

These equations are unique in the sense that we have used all possible contractions of the gauge-invariant form of the extended curvature tensor. These equations are purely geometrical and hold at space-time points where matter and charge densities vanish. In ordinary general relativity (30) would vanish identically from symmetry.

Our field equation (30) can be written

$$S^{\alpha}{}_{\alpha\beta\delta} = 4(T_{\delta\|\beta} - T_{\beta\|\delta}) = 4(T_{\delta|\beta} - T_{\beta|\delta}) = 0\,, \tag{31}$$

and (29) yields finally

$$\begin{aligned} S^{\gamma}{}_{\pi\gamma\beta} &= R^{\gamma}{}_{\pi\gamma\beta} - 3T_{\pi\|\beta} + T_{\beta\|\pi} - (g_{\pi\beta}T^{\gamma})_{\|\gamma} \\ &\quad + 2T_\beta T_\pi - 2g_{\pi\beta}T^{\alpha}T_\alpha \\ &= 0\,. \end{aligned} \tag{32}$$

Our field equations are thus (31) and (32). Since (31) always holds, we can write

$$T_\alpha = \varphi_{\|\alpha} = \varphi_{|\alpha}\,, \tag{33}$$

where φ is the fundamental scalar field in our theory. The condition (31) that T_α can be written as $\varphi_{|\alpha}$ is also the necessary and sufficient condition that we have a Riemannian geometry. If a scalar field is to be put into the formalism as in (33) we are forced to have a Riemannian geometry. We note that (31) is the same as (21), our integrability condition for dl. Thus (21) follows quite naturally from the extended curvature tensor $S^{\alpha}{}_{\beta\delta\gamma}$. It need not be postulated separately. In the present work dl is always integrable and we are in a classical Riemannian geometry in contrast to Weyl's unified field theory where dl is not integrable if an electromagnetic field is present. Since (1) is always integrable in our theory, we escape a major objection to the use of Weyl geometry. In 1918 Einstein[11] pointed out that two atoms located at a given point in a spherically symmetric electric field could have different periods in the Weyl unified field theory if they had different previous histories. In other words, since (1) is not integrable in Weyl's unified field theory, the period of an atom depends upon the path that a given atom follows to a given space-time point. In the present theory, on the other hand, this path dependence does not exist since (1) is always integrable.

Also we note that if we postulate that (31) always holds, then in addition to the symmetry (28) we also have

$$S^{\gamma\pi}{}_{\alpha\beta} = -S^{\pi\gamma}{}_{\alpha\beta} \tag{34}$$

and

$$S_{\alpha\beta}{}^{\gamma\pi} = S^{\gamma\pi}{}_{\alpha\beta} \tag{35}$$

so that $S^{\alpha}{}_{\beta\gamma\delta}$ has the same symmetry properties as $R^{\alpha}{}_{\beta\gamma\delta}$.

It is important to emphasize at this point that (31) and (32) are completely gauge-invariant since $S^{\gamma}{}_{\pi\alpha\beta}$ is made up entirely of $\Gamma^{\gamma}{}_{\rho\alpha}$ from (25) and the $\Gamma^{\gamma}{}_{\beta\alpha}$ are unchanged under gauge transformations from (6). Thus (31) will hold in all gauges. Also (32) will hold in all gauges even though its form will change. Under a gauge transformation both the $R^{\gamma}{}_{\pi\gamma\beta}$ and the T_α parts of (32) will change. In the gauge in which $T_\alpha = 0$, for example, the $R^{\gamma}{}_{\pi\alpha\beta}$ part will carry the information previously carried by the T_α. This information is carried in the modified $f(x^\beta)$ factor which now multiplies the $g_{\alpha\beta}$ in the new gauge as in (5).

In what follows we will work in a specific gauge. Since (32) is gauge-invariant, it is immaterial what gauge we choose in this equation. It is clearly necessary to choose a specific gauge since we identify the scalar field φ with $T_\alpha = \varphi_{|\alpha}$ in (33). Gauge transformations (4) will add an arbitrary scalar function to φ. This is clearly inadmissible if φ is to represent a physical field. The gauge we choose to work in is the one in which the entire physical scalar field φ appears in

$$dl = l\,\varphi_{|\alpha} dx^\alpha \tag{36}$$

for the change in length of a vector. This gauge is defined uniquely up to an additive constant. The metric tensor is not the usual one of general relativity. In fact, in this gauge we can use (33) to write the field equation (32) in terms of φ as

$$\begin{aligned} S_{\pi\beta} &= R_{\pi\beta} - 2\varphi_{|\pi\|\beta} - (g_{\pi\beta} g^{\gamma\alpha}\varphi_{|\alpha})_{\|\gamma} \\ &\quad + 2\varphi_{|\beta}\varphi_{|\pi} - 2 g_{\pi\beta} g^{\alpha\gamma}\varphi_{|\gamma}\varphi_{|\alpha} \\ &= 0\,, \end{aligned} \tag{37}$$

where $R_{\pi\beta}$ is the usual contracted Riemann curvature tensor written in terms of our modified metric and $S_{\pi\beta} \equiv S^{\alpha}{}_{\pi\alpha\beta}$. Equation (37) is our field equation for the modified metric in this particular gauge, given φ. We still need an equation for φ in this gauge. This equation will clearly depend on the gauge because φ by definition is gauge-dependent. We will obtain the required equation for φ using (37) and a variational principle.

The field equations of general relativity can be obtained from the variational principle

$$\delta \int R\sqrt{-g}\, d^4x = 0\,, \tag{38}$$

where R is the fully contracted scalar curvature and the variation is carried out with respect to the metric tensor. In our modified metric, this variational principle becomes

$$\delta \int S\sqrt{-g}\, d^4x = 0\,, \tag{39}$$

where $S \equiv g^{\pi\beta} S_{\pi\beta}$. When (39) is varied with respect to the modified metric tensor, a gauge-invariant Euler-Lagrange equation results since terms such as $\partial S/\partial g^{\alpha\beta}$ are of weight zero (both S and $g^{\alpha\beta}$ are of weight -1). From gauge invariance and the definition of $S_{\pi\beta}$ in (25) this Euler-Lagrange equation is just our gauge-invariant field equation (32). This is obvious since in the gauge where $S \equiv R$, we get the $R_{\mu\nu} = 0$ version of (32). When (39) is varied with respect to φ, the resulting Euler-Lagrange equation is no longer gauge-invariant because terms such as $\partial S/\partial\varphi$ are not of weight zero and because the specific φ dependence of S depends on the gauge we choose. In the above gauge, (39) becomes

$$\delta \int (R - 6 g^{\alpha\gamma}\varphi_{|\alpha\|\gamma} - 6\varphi^{|\alpha}\varphi_{|\alpha})\sqrt{-g}\, d^4x = 0\,, \tag{40}$$

where we used (37). The Euler-Lagrange equations for φ are

$$-\left(\frac{\partial S}{\partial \varphi_{|\alpha\|\beta}}\right)_{\|\alpha\|\beta} + \left(\frac{\partial S}{\partial \varphi_{|\alpha}}\right)_{\|\alpha} - \frac{\partial S}{\partial \varphi} = 0\ . \tag{41}$$

This becomes

$$+6 g^{\alpha\beta}{}_{\|\alpha\|\beta} - 12\varphi^{|\alpha}{}_{\|\alpha} = 0 \tag{42}$$

and using (17) and (33) gives finally

$$\varphi^{|\alpha}{}_{\|\alpha} = 0\,, \tag{43}$$

where $\varphi^{|\alpha} \equiv g^{\alpha\beta}\varphi_{|\beta}$. Equation (43) is clearly gauge-dependent. It follows from (37) and the variational principle (39) and can be thought of as the gauge condition, specifying the above gauge. It might be mentioned at this point that our choice of gauge is equivalent to the units chosen in the original Brans-Dicke theory[1] in which $\hbar$, c, and particle masses are constant. Geodesic equations are obtained in that theory but Einstein's field equations are not valid [see (37)] and the gravitational constant G depends on the scalar field [see (69)].

Equations (37) and (43) constitute our field equations in nonempty space in terms of φ. If we write these out in terms of the usual Christoffel symbols (defined in terms of the modified metric), we get

$$S_{\pi\beta}=R_{\pi\beta}-2\varphi_{|\pi|\beta}-2\varphi_{|\pi}\varphi_{|\beta}+2g_{\pi\beta}\varphi^{|\alpha}\varphi_{|\alpha}-g_{\pi\beta}g^{\gamma\alpha}\varphi_{|\alpha|\gamma}+2\varphi_{|\alpha}\left\{\begin{matrix}\alpha\\ \pi\beta\end{matrix}\right\}+g_{\pi\beta}g^{\gamma\alpha}\varphi_{|\delta}\left\{\begin{matrix}\delta\\ \alpha\gamma\end{matrix}\right\}=0 \tag{44}$$

and

$$\varphi^{|\alpha}{}_{||\alpha}=g^{\alpha\beta}\varphi_{|\alpha|\beta}-g^{\alpha\beta}\varphi_{|\pi}\left\{\begin{matrix}\pi\\ \alpha\beta\end{matrix}\right\}-4\varphi_{|\alpha}\varphi^{|\alpha}=0, \tag{45}$$

respectively. We used (14), (15), and (16) with (11) substituted in to write out the derivatives.

Our curvature tensor satisfies the Bianchi identity

$$S^{\alpha}{}_{\beta\gamma\delta||\pi}+S^{\alpha}{}_{\beta\delta\pi||\gamma}+S^{\alpha}{}_{\beta\pi\gamma||\delta}=0 \tag{46}$$

which we can easily see by explicit calculation using the properties of the $\Gamma^{\alpha}{}_{\beta\gamma}$. We can use (46) to show

$$g^{\alpha\pi}(S_{\pi\beta}-\tfrac{1}{2}g_{\pi\beta}S)_{||\alpha}=0, \tag{47}$$

where $S\equiv g^{\alpha\pi}S_{\alpha\pi}$. Thus if we take (47) to be what we mean by the divergence of a quantity, we can write our field equation (37) in divergenceless form as

$$S_{\pi\beta}-\tfrac{1}{2}g_{\pi\beta}S=0. \tag{48}$$

This form is also completely gauge-invariant since we can write the last term as $-\frac{1}{2}g_{\pi\beta}g^{\alpha\gamma}S_{\alpha\gamma}$ and this combination of metric tensors in invariant (S itself is not gauge-invariant, for example). The present theory holds only in regions of space which have no charge and mass densities. As in general relativity, we can generalize (48) to include a source term by writing

$$S_{\pi\beta}-\tfrac{1}{2}g_{\pi\beta}S=-8\pi KT_{\pi\beta}/c^2, \tag{49}$$

where K is the gravitational constant and $T_{\pi\beta}$ the energy-momentum tensor. Equation (43) should also have a source term on the right-hand side. The precise form of this equation for φ goes beyond the geometrical considerations above and requires a Lagrangian formalism with energy-momentum terms included as in the work of Brans and Dicke.[1] We shall limit ourselves to regions where charge and mass densities vanish.

IV. GEODESIC EQUATIONS OF MOTION

In the present theory test particles will follow geodesics of the geometry given by the gauge-independent equation

$$\frac{d^2x^{\mu}}{ds^2}-\Gamma^{\mu}{}_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}=0. \tag{50}$$

These differ from the usual geodesic equations because of the presence of φ terms in $\Gamma^{\mu}{}_{\alpha\beta}$ but still represent geodesics of our modified geometry. The geodesic equations (50) follow from our field equations exactly as in general relativity. This can be shown by substituting our modified $\Gamma^{\mu}{}_{\alpha\beta}$ for $-\left\{\begin{smallmatrix}\mu\\ \alpha\beta\end{smallmatrix}\right\}$ in the derivations.[12]

Let us now look at the weak-field limit of (50). Consider a time-independent metric

$$g_{\mu\nu}=g^{(L)}_{\mu\nu}+\epsilon\gamma_{\mu\nu}, \tag{51}$$

where $g^{(L)}_{\mu\nu}$ is the Lorentzian flat-space metric with a signature (1, −1, −1, −1). If we assume that $\varphi_{|0}=0$ and $\varphi_{|\alpha}$ is of order ϵ for all $\alpha\neq 0$ and work to first order in ϵ and v/c only, then (50) is satisfied identically for $\mu=0$ and for $\mu=i\neq 0$ can be written

$$\frac{d^2x^{i}}{dt^2}+c^2\left\{\begin{matrix}i\\ 00\end{matrix}\right\}-c^2\varphi^{|i}=0. \tag{52}$$

In vector notation this becomes

$$\frac{d^2\vec{x}}{dt^2}=-\tfrac{1}{2}c^2\epsilon\vec{\nabla}\gamma_{00}-c^2\vec{\nabla}\varphi. \tag{53}$$

For a single central point mass M, correspondence with Newton's theory then requires

$$1-\frac{2KM}{rc^2}=g_{00}+2\varphi, \tag{54}$$

where K is the gravitational constant. We will use this result in Sec. V.

V. SCHWARZSCHILD SOLUTION

Let us now consider the solution to our field equations for the case of a point particle of mass M at the origin assuming time independence and spherical symmetry. From symmetry the metric can be written as

$$g_{\mu\nu}=\begin{pmatrix} e^{\nu(r)} & & & \\ & -e^{\lambda(r)} & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix}. \tag{55}$$

Let us also denote $d\varphi(r)/dr\equiv f(r)$. From (44) we have for the S_{00}, S_{11}, and S_{22} equations, respectively,

$$\nu''-\tfrac{1}{2}\nu'\lambda'+\tfrac{1}{2}\nu'^2+2\nu'/r-2f'+\lambda'f-3f\nu'+4f^2-4f/r=0, \tag{56}$$

$$\nu''+\tfrac{1}{2}\nu'^2-\tfrac{1}{2}\lambda'\nu'-2\lambda'/r-6f'-\nu'f+3\lambda'f-4f/r=0, \tag{57}$$

$$\tfrac{1}{2}\nu'/r-\tfrac{1}{2}\lambda'/r+1/r^2-e^{\lambda}/r^2-4f/r+2f^2-f'+\tfrac{1}{2}f\lambda'-\tfrac{1}{2}f\nu'=0. \tag{58}$$

The S_{33} equation is identical with (58) and all $S_{\pi\beta}$ with $\pi \neq \beta$ are identically zero. To first order the solution of (45) is

$$\varphi(r) = A + B/r, \tag{59}$$

where A and B are constants. It is interesting that this leads to a change in the square of the length of a vector from (22) and (4) of $\hat{l}^2 = e^{-2B/r} l^2$. The metric tensor must of course also be modified from the usual one of general relativity so that this variable length can be incorporated into Riemannian geometry in accord with our arguments following (21). We see this explicitly in (72) and (73). Equations (56), (57), and (58) are all self-consistent. In fact, an *exact* solution to these equations for the approximate $\varphi(r)$ given in (59) is just

$$e^{\lambda} = (1 + B/r)^2/(1 + C_1 e^{B/r} B/r) \tag{60}$$

and

$$e^{\nu} = e^{2B/r} + (C_1 B/r) e^{3B/r}, \tag{61}$$

where C_1 is a constant. It is interesting mathematically that this exact solution exists, but physically it is accurate to first order only since we used (59). To first order these can be written

$$e^{\lambda} = 1 + 2B/r - C_1 B/r \tag{62}$$

and

$$e^{\nu} = 1 + 2B/r + C_1 B/r. \tag{63}$$

From the correspondence principle and in order for (63) to be consistent with (54) we must have $A = 0$ in (59) and $C_1 \equiv -4 - 2m/B$, where $m = KM/c^2$. Then,

$$e^{\nu} = 1 - (2m + 2B)/r \tag{64}$$

and

$$e^{\lambda} = 1 + (2m + 6B)/r \tag{65}$$

to first order in m/r and B/r. B is the parameter which gives the strength of the scalar field. We are assuming that B and m are approximately the same order. Using (64) and (65) in (45) lets us calculate $\varphi(r)$ to second order giving

$$\varphi(r) = B/r + mB/r^2 + 4B^2/r^2. \tag{66}$$

Using (66) in (56), (57), and (58) then gives to second order

$$e^{\nu} = 1 - 2m/r - 2B/r - 4mB/r^2 - 2B^2/r^2 \tag{67}$$

and

$$e^{\lambda} = 1 + 2m/r + 6B/r + 4m^2/r^2 + 45B^2/r^2 + 26mB/r^2. \tag{68}$$

Let us now use these solutions to calculate experimental results.

VI. TESTS OF THE THEORY

The gravitational weight of a body is determined by g_{00} to lowest order and so is the gravitational red shift. Thus we expect the same result for the gravitational red shift as in Einstein's theory. From (64) we must redefine the gravitational constant K so that

$$K'M/c^2 = B + KM/c^2, \tag{69}$$

where K' is the gravitational constant measured experimentally. In terms of $m' \equiv K'M/c^2$, we have

$$m' = B + m. \tag{70}$$

Putting everything in terms of m' gives to second order

$$\varphi(r) = B/r + m'B/r^2 + 3B^2/r^2, \tag{71}$$

$$e^{\lambda} = 1 + 2m'/r + 4B/r + 4m'^2/r^2 + 18m'B/r^2 + 23B^2/r^2, \tag{72}$$

$$e^{\nu} = 1 - 2m'/r - 4m'B/r^2 + 2B^2/r^2. \tag{73}$$

The deflection of starlight also depends on g_{11} so the results of general relativity are modified. It is easy to show using the first-order terms in (71), (72), and (73) that to first order the deflection is

$$\delta = (\text{general-relativity result})(1 + B/m'). \tag{74}$$

The advance in the perihelion of a planetary orbit requires g_{00} to second order and g_{11} to first order. Taking these from (72) and (73) above gives the result,

$$\text{shift/revolution} = (\text{general-relativity result}) \times (1 + \tfrac{4}{3}B/m' - \tfrac{1}{3}M^2/m'^2). \tag{75}$$

Brans and Dicke[1] calculate a correction factor $(3 + 2\omega)/(4 + 2\omega)$ in (74) and $(4 + 3\omega)/(6 + 3\omega)$ in (75). Our results agree with this to first order if

$$B/m' = -\tfrac{1}{2}(1/\omega). \tag{76}$$

Thus the two forms of the scalar-tensor theory predict the same experimental results. We see that if $\omega \geq 6$, then $|B/m'| \leq \frac{1}{12}$ and B^2/m'^2 corrections are small. The results of Shapiro *et al.*[5] suggest that B is probably smaller than this by at least a factor of 3.

VII. CONCLUSION

The present scalar-tensor theory of gravity is an alternative to the usual Brans-Dicke formalism in regions of space free of mass and charge densities. The scalar field enters the two theories very differently. In the present work it is related to the integrable change in length of a vector as it is transported from place to place. This results in a Riemannian geometry which appears quite different from the one used in general relativity with modified derivatives and a metric tensor which is not covariantly constant. Having paid this price, we end with an unambiguous theory with very simple field equations directly related to the curvature tensor as in general relativity, in which the scalar field plays a rather elegant geometrical role.

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Gravitational Lagrangian

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The method of recasting the Newman-Penrose formalism for the gravitational field equations into a Yang-Mills–type theory is reviewed. The free-field gravitational Lagrangian density structured along the lines of the free-field Yang-Mills Lagrangian density by Kibble is generalized to give the complete set of gravitational field equations one obtains in the Newman-Penrose formalism.

I. INTRODUCTION

In Yang-Mills[1] theory one assumes that at each space-time point there exists a 2-dimensional internal space. Under an isotopic gauge transformation $S(x)$, the 2×2 matrix potential and matrix field then transform according to

$$B'_\mu = S^{-1}B_\mu S - S^{-1}\partial_\mu S\,,\qquad F'_{\mu\nu} = S^{-1}F_{\mu\nu}S\,, \tag{1.1}$$

where

$$F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + [B_\mu, B_\nu]\,, \tag{1.2}$$

and $[B_\mu, B_\nu] = B_\mu B_\nu - B_\nu B_\mu$. The action principle applied to the Lagrangian density

$$\mathfrak{L} = -\tfrac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\,, \tag{1.3}$$

where Tr denotes trace, then gives rise to the free gauge field equation

$$\partial_\beta F^{\alpha\beta} - [B_\beta, F^{\alpha\beta}] = 0\,. \tag{1.4}$$

In introducing the gravitational field from a generalized Poincaré invariance, Kibble[2] has extended the above Lagrangian density into

$$-\tfrac{1}{4}(-g)^{1/2}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\,, \tag{1.5}$$

describing the free gravitational field. Here $g = \det g_{\mu\nu}$ and $g_{\mu\nu}$ is the geometrical metric.

More recently Carmeli,[3] who has shown[4,5] that the Newman-Penrose (NP) formalism[6] for the gravitational field equations can be cast into a Yang-Mills–type theory by use of the group SL(2, C), used a first-order form of the Lagrangian density (1.5) to obtain the vacuum NP equations.

The question arises as to whether one can generalize the Lagrangian density (1.5) into one which