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Unitarization of the Pion Electromagnetic Form Factor in the Smoothed Veneziano Model*

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We present a simple method to unitarize the pion electromagnetic form factor by smoothing the Veneziano-type model. The weight function is chosen so as to satisfy the unitarity in the elastic region. Two typical types of the Veneziano form factors are unitarized and compared with the experimental data.

I. INTRODUCTION

Within the framework of dual models several representations for the pion electromagnetic form factor $F(t)$ have been proposed.[1-4] Among the various forms are

$$F_1(t)=c_1'\frac{\Gamma(1-\alpha(t))}{\Gamma(\lambda-\alpha(t))} \tag{1}$$

and

$$F_2(t)=c_2'\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t))}{\Gamma(\frac{5}{4}-\frac{1}{2}\alpha(t))}, \tag{2}$$

where $\alpha(t)$ denotes the ρ trajectory and c_1' and c_2' are normalization constants. Equation (1) with $\lambda=2.5$ was obtained by Oyanagi[1] by means of current commutation relations with the assumption that the axial-vector current was coupled to π and A_1 only. Although there is ambiguity in this method coming from the so-called subtraction constants, (1) may be regarded as the simplest form in the framework of dual models. The present authors[3] obtained $\lambda=2.46$ in Eq. (1) by using unitarity at threshold. Equation (2) was obtained by Suura[2] by making use of the similarity between a pair of chirally conjugate currents. In this method it is essential that $F(t)$ has poles alternately on the trajectory.

Neither Eq. (1) nor Eq. (2) is unitary. Steps toward unitarization have been taken recently. One approach is to take a complex trajectory for $\alpha(t)$ and impose the elastic unitarity condition on $F(t)$.[5] Another method is to solve a Muskhelishvili-Omnès type equation with (1) and (2) as starting points.[6] The former method has, however, complex poles on the first sheet of the t plane instead of on the second sheet as required by analyticity. In the latter case there is in principle the difficulty of treating the integration of phase shift up to infinity and the well-known ambiguity of an arbitrary entire function multiplying the solution.

In this paper we propose a simple method for unitarizing a form factor. First of all, a continuous spectrum is achieved for $F(t)$ by smoothing out the original forms (1) and (2). This method is similar to the work of Bali *et al.*[7] and Huang[8] for smoothing out the scattering amplitude in the Veneziano model. We use a simple and reasonable weight function with one parameter. This is, of course, not unique. Secondly, we determine the parameter so as to satisfy the unitarity requirement that in the elastic region the phase of $F(t)$ is equal to the phase of the p-wave amplitude of $\pi\pi$ scattering. For the latter phase shift we use the experimental data[9-11] obtained from the $\pi N \to \pi\pi N$ reaction. This reaction has also been studied by Roberts and Wagner[12] in terms of the Veneziano model with Lovelace's prescription for interpreting the amplitude.

The resulting form factor has the correct analyticity with poles on the second sheet of the complex t plane and satisfies the unitarity in the elastic region. It reduces to the original form when the width of the resonance goes to zero. For the higher-t region we cannot say anything about unitarity although our form factor has an infinite number of thresholds. One might, however, hopefully expect that this form factor would work well for higher t, in view of the fact that the dual scattering model seems to satisfy unitarity in the average sense.[13]

The unitarizations for (1) and (2) are shown in Sec. II. In Sec. III comparison with experimental data[14,15] and discussion are given. Recently, Antoniou *et al.*[16] have also presented a smoothed Veneziano-type form factor by using a different smoothing method. They do not, however, take unitarity into account. In Sec. III we have also tested the unitarity requirement for their model.

II. FORMULATION

The smoothed-out form for (1) is given by

$$F_1(t)=c_1\int_{x_1}^{\infty}\frac{\Gamma(1-\alpha(t)+x)}{\Gamma(\lambda-\alpha(t)+x)}\rho_1(x)dx\,. \tag{3}$$

x_1 is so determined that (3) has the threshold at $t=4\mu^2\equiv t_0$, where μ is the pion mass,

$$x_1=\alpha(t_0)-1. \tag{4}$$

c_1 is given by the normalization $F(0)=1$. We retain the value $\lambda=2.46$ as before.[3] The weight function $\rho_1(x)$ should be chosen in such a way that (i) $F(t)$ has poles on the second sheet, (ii) it has the correct threshold behavior, and (iii) Eq. (3) reduces to (1) in the limit of zero width. We use a simple $\rho_1(x)$ which satisfies the above requirements:

$$\rho_1(x)=\frac{\epsilon/\pi}{x^2+\epsilon^2}\left(\frac{x-x_1}{-x_1}\right)^{3/2}e^{-qx}\,. \tag{5}$$

Here $\epsilon=\alpha' m\Gamma$, m and Γ are the mass and the width of the ρ meson, respectively, and q is determined by the unitarity requirement,

$$\frac{\mathrm{Im}F(t)}{\mathrm{Re}F(t)}=\tan\delta(t) \tag{6}$$

in the elastic region. $\delta(t)$ is the p-wave phase shift of $\pi\pi$ scattering. In this region the imaginary part of $F_1(t)$ is given by

$$\mathrm{Im}F_1(t)=\frac{c_1}{\Gamma(\lambda-1)}\frac{\epsilon}{[\alpha(t)-1]^2+\epsilon^2}\times\left(\frac{\alpha(t)-\alpha(t_0)}{1-\alpha(t_0)}\right)^{3/2}e^{-q[\alpha(t)-1]}\,, \tag{7}$$

and the real part is

$$\mathrm{Re}F_1(t)=c_1\,\mathrm{P}\int_{x_1}^{\infty}\frac{\Gamma(1-\alpha(t)+x)}{\Gamma(\lambda-\alpha(t)+x)}\times\frac{\epsilon/\pi}{x^2+\epsilon^2}\left(\frac{x-x_1}{-x_1}\right)^{3/2}e^{-qx}dx\,. \tag{8}$$

With (7) and (8), we calculate the left-hand side of (6) and adjust q to give the best fit to the experimental data of $\delta(t)$. The results are not very sensitive to q. This means that the form (5) for $\rho_1(x)$ is quite reasonable.

A similar procedure is used to unitarize (2),

$$F_2(t)=c_2\int_{x_2}^{\infty}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t)+x)}{\Gamma(\frac{5}{4}-\frac{1}{2}\alpha(t)+x)}\rho_2(x)dx\,. \tag{9}$$

In this case x_2 and $\rho_2(x)$ are given by

$$x_2=\tfrac{1}{2}[1-\alpha(t_0)]\,, \tag{10}$$

$$\rho_2(x)=\frac{\epsilon/2\pi}{x^2+\frac{1}{4}\epsilon^2}\left(\frac{x-x_2}{-x_2}\right)^{3/2}e^{-qx}\,. \tag{11}$$

In the elastic region

$$\mathrm{Im}F_2(t)=\frac{c_2}{\Gamma(\frac{3}{4})}\frac{2\epsilon}{[\alpha(t)-1]^2+\epsilon^2}\times\left(\frac{\alpha(t)-\alpha(t_0)}{1-\alpha(t_0)}\right)^{3/2}e^{-q[\alpha(t)-1]/2} \tag{12}$$

and

$$\mathrm{Re}F_2(t)=c_2\,\mathrm{P}\int_{x_2}^{\infty}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(t)+x)}{\Gamma(\frac{5}{4}-\frac{1}{2}\alpha(t)+x)}\times\frac{\epsilon/2\pi}{x^2+\frac{1}{4}\epsilon^2}\left(\frac{x-x_2}{-x_2}\right)^{3/2}e^{-qx}dx\,. \tag{13}$$

It is noted that the unitarized form factors have the same asymptotic behaviors as the original

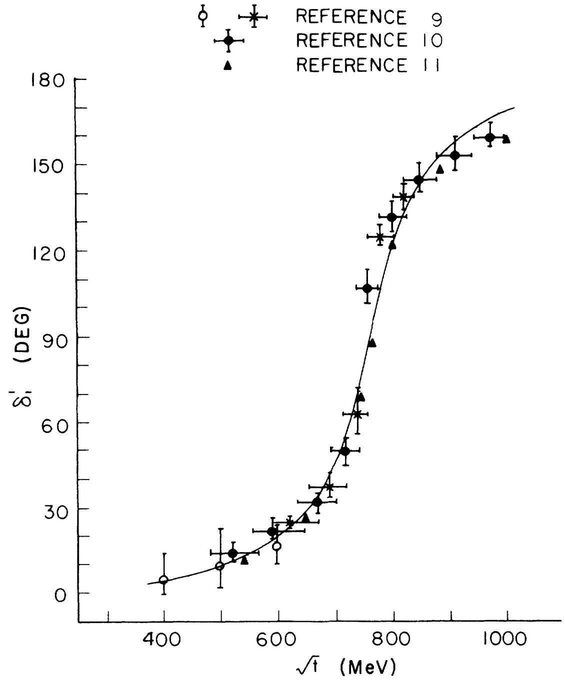

FIG. 1. The p-wave $\pi\pi$ scattering phase shifts calculated from $F_1(t)$. The experimental data are from Refs. 9–11.

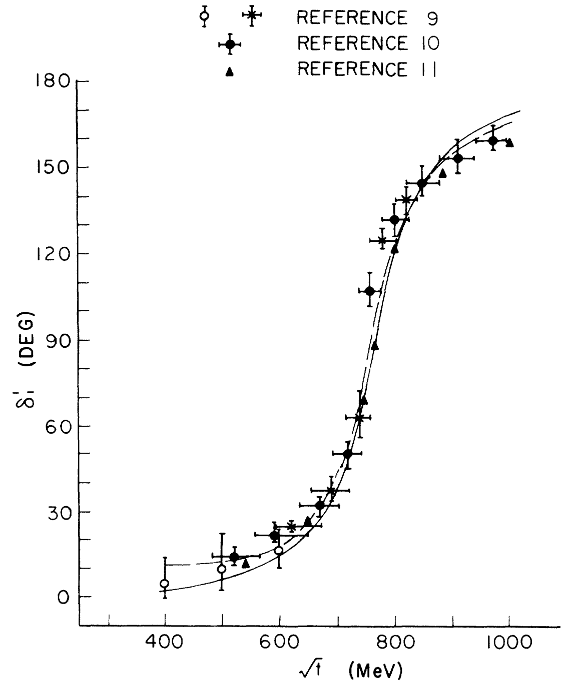

FIG. 2. The p-wave $\pi\pi$ scattering phase shifts calculated from $F_2(t)$ (solid curve), and from $F_3(t)$ (dashed curve).

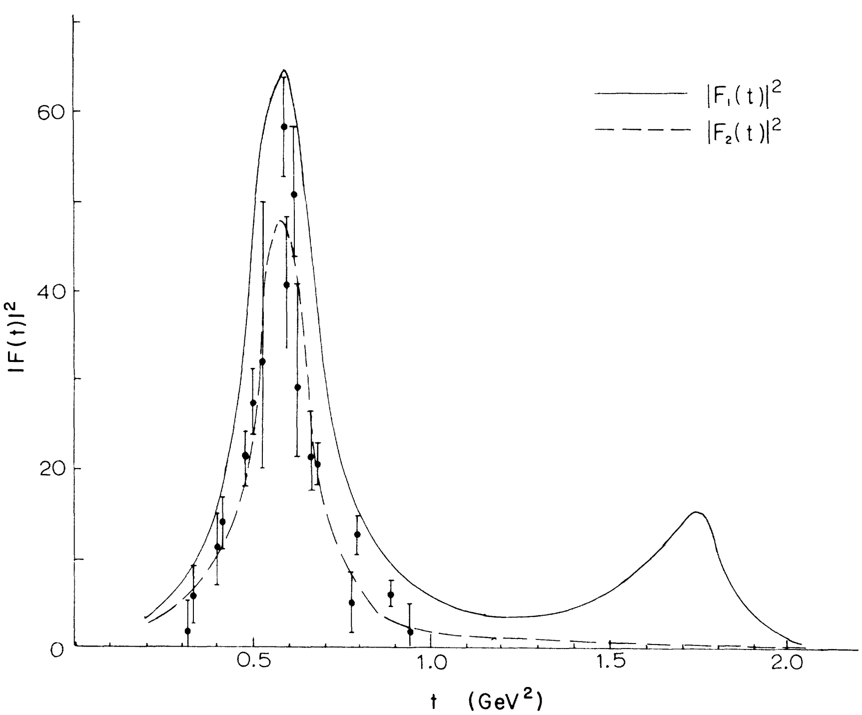

FIG. 3. $|F_1(t)|^2$ and $|F_2(t)|^2$ in the timelike region with data from Ref. 14.

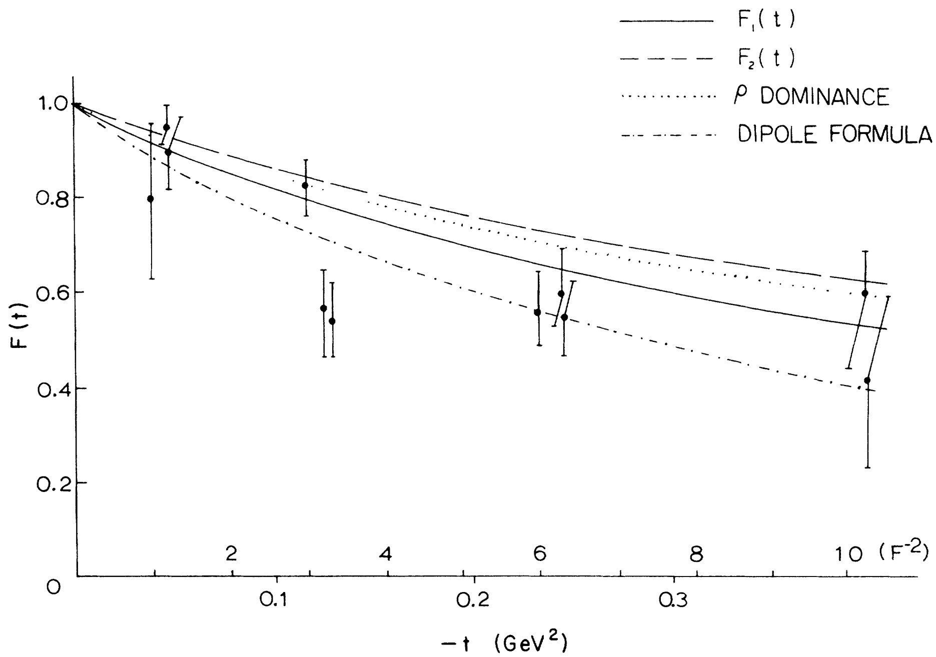

FIG. 4. $F_1(t)$ and $F_2(t)$ in the spacelike region with data from Ref. 15.

ones, namely,

$$F_1(t) \underset{t\to\infty}{\sim} [-\alpha(t)]^{1-\lambda} c_1 \int_{x_1}^{\infty} \rho_1(x)dx$$

for (3) and

$$F_2(t) \underset{t\to\infty}{\sim} [-\tfrac{1}{2}\alpha(t)]^{3/4} c_2 \int_{x_2}^{\infty} \rho_2(x)dx$$

for (9).

III. NUMERICAL RESULTS AND DISCUSSIONS

First we calculate the phase shift $\delta(t)$ by using (6), (7), (8), (9), (11), (12), and (13) and find that $q = 2.5$ for $F_1(t)$ and $q = 5.5$ for $F_2(t)$ give the best fit to the experimental data. This is shown in Fig. 1 and Fig. 2. We have not used a χ^2 test, since the calculated $\delta(t)$ is not so sensitive to q and $|F(t)|^2$ even less so. $|F_1(t)|^2$ and $|F_2(t)|^2$ are plotted in Fig. 3 for the timelike region and Fig. 4 shows $F_1(t)$ and $F_2(t)$ for the spacelike region. The value of the width used for $F_1(t)$ is $\Gamma = 140$ MeV and for $F_2(t)$ it is $\Gamma = 110$ MeV. The mass of the ρ meson is taken to be 762 MeV and the ρ trajectory used is $\alpha(t) = 0.86t + 0.5$.

The above unitarization scheme leaves the form factor in the spacelike region almost unchanged, but the peak near $t = m_\rho^2$ becomes higher if the same width is used as in the original model. Thus we have used a larger value for Γ than in Ref. 3. The charge radii of the pion are

$$\langle r^2 \rangle^{1/2} = 0.72 \times 10^{-13} \text{ cm for } F_1(t)$$

and

$$\langle r^2 \rangle^{1/2} = 0.64 \times 10^{-13} \text{ cm for } F_2(t),$$

which are also only slightly changed.

It has been pointed out by Acharya *et al.*[5] and also by Drago and Grillo[6] that $F_2(t)$ gives a better fit to the experimental data than $F_1(t)$ near $t \simeq m_\rho^2$. In our opinion, we think both $F_1(t)$ and $F_2(t)$ can give a good fit to the data near $t \simeq m_\rho^2$ provided that a different width Γ is used. In fact, analysis of πN-$\pi\pi N$ by Lovelace's unitarized form of the Veneziano model[12] favors the larger width $\Gamma \simeq 140$ MeV. Thus $F_1(t)$ may seem to be preferred. A true test will be in the neighborhood of $t \simeq 1.7$ $(\text{GeV}/c)^2$ where $F_1(t)$ has a second peak whereas $F_2(t)$ does not. In addition, $F_1(t)$ and $F_2(t)$ of course have quite different asymptotic behavior in the spacelike region.

Finally we have also calculated the phase shifts from the form factor proposed by Antoniou *et al.*[16] Their smoothed-out form factor is

$$F_3(t) = \int_{\lambda=0}^{\lambda_m} \Phi(\lambda) \frac{\Gamma(1 - \alpha_0 - b\lambda t)}{\Gamma(p - \alpha_0 - b\lambda t)} d\lambda, \tag{14}$$

with

$$\Phi(\lambda) = \frac{g(\lambda_m - \lambda)^{3/2}}{(\lambda - 1)^2 + (\Gamma_\rho / m_\rho)^2} + g'(\lambda_m - \lambda)^{3/2}, \tag{15}$$

$$\lambda_m = m_\rho{}^2/4m_\pi{}^2, \quad m_\rho = 760\text{ MeV},$$

$$\Gamma_\rho = 112\text{ MeV}, \quad p = \tfrac{5}{2}\ ,$$

and

$$g = 1.42\times 10^{-3}, \quad g' = 1.20\times 10^{-3}\ .$$

Thus basically their smoothing is achieved by integrating with respect to the slope of the trajectory, whereas ours is done with respect to the intercept. The calculated phase from (14) is shown in Fig. 2. It is seen that although the result lies within the rather large experimental error bars, in the low energy region (up to $\sqrt{t}\simeq 400$ MeV) it is about three times as large as the Lovelace-Roberts-Wagner[12] (L-R-W) phase shift with $\Gamma\simeq 110$ MeV. The phase shifts of $F_1(t)$ and $F_2(t)$ agree with the L-R-W[12] phase shift in the elastic region. Improvements can be achieved for Antoniou's form factor if a larger width ($\Gamma\simeq 135$ MeV) and a smaller g' ($\simeq 0.2\times 10^{-3}$) are used.

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σ-Commutator Term and Pion-Pion Scattering

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The pion-pion scattering amplitude on the mass shell is constructed for low energy in terms of the σ term. A couple of models are discussed in connection with this approach and the pion-pion S-wave scattering lengths are estimated to be compared with Weinberg's prediction in the σ model. It is found that the ratio of the scattering lengths generally depends only on the ratio of the isospin components in the σ commutator.

I. INTRODUCTION

The subject of pion scattering lengths has been carefully studied in the soft-pion limit by Weinberg.[1] With the hypothesis of partially conserved axial-vector current (PCAC) the scattering amplitude can be approximated by the amplitude with two pions off their mass shells as derived from current algebra. Two of the terms in the amplitude ensue from the use of standard commutation relations. The third term which contains the time-ordered product of two axial-vector currents may include possible pole contributions. As in most current-algebra calculations our knowledge in this term (except possibly for the pole contribution) is very limited. It was pointed out in Ref.