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Tests of Complex Scaling and Regge Behavior of Compton Amplitudes*

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A numerical evaluation of the complex-scaling sum rules is given. The results indicate that the concept of precocious complex scaling is in agreement with the present data on $W_2(\nu, q^2)$. It is also shown that these sum rules can discriminate between different Regge fits to the electroproduction data and place restrictions that go beyond those imposed by the finite-energy sum rules.

I. INTRODUCTION

In a recent paper[1] one of us derived a set of new sum rules for off-shell Compton scattering. These sum rules follow essentially from analyticity in two variables and "precocious" complex scaling inside the analyticity domain. They also contain contributions from the nonscaling Regge region.

The sum rules in I are generally of two classes. The first are relatively insensitive to the contributions from the Regge region. These, as stressed in I, provide a test of the concept of complex scaling and its "precocious" nature. The second class of sum rules is sensitive to the Regge contribution and, if complex scaling is valid, they are expected to differentiate between different Regge fits to the SLAC-MIT electroproduction data.

This brief paper is mainly devoted to the evaluation of the above-mentioned sum rules using the data on $W_2(\nu, q^2)$ from the SLAC-MIT collaboration.[2]

Our main results are the following: First, complex scaling is consistent with the data on $W_2(\nu, q^2)$, and the sum rules testing complex scaling are satisfied to within 10%. This is the same accuracy to which real scaling has been verified. The precocious nature of complex scaling also turns out to be consistent with the data. Second, after verifying complex scaling, the second type of sum rule was used to differentiate between different Regge fits to νW_2. The main result here is to show that the complex-scaling sum rules provide restrictions on the Regge fits which go beyond the restrictions imposed by the finite-energy sum rules (FESR). Some Regge fits which satisfy the FESR turn out not to satisfy the sum rules of I. It is interesting to note that these fits are mainly the ones that do not have a fixed $J=0$ pole in $T_2(\nu, q^2)$.

II. REVIEW OF SUM RULES

The main result of I was to find special analytic closed contours in the four-dimensional ν, q^2 complex space such that

$$\int_C \nu(\tau) T_2(\nu(\tau), q^2(\tau)) d\tau = 0.$$

The contours C remained inside the analyticity domain, in this case the forward tube. Here T_2 is the invariant off-shell Compton amplitude and is related to W_2, the inelastic structure function, by $\mathrm{Im}\,T_2 = \pi W_2$. The contours C consisted of two parts. Along the first part both ν and q^2 are real and q^2 is spacelike. Thus, along this segment T_2 is determined by the data on W_2 through the standard fixed-q^2 forward dispersion relation. Along the second part of the contour both ν and q^2 are complex. In this region $T_2(\nu, q^2)$ is determined by complex scaling when both $|\nu|$ and $|q^2|$ are large, and by the Regge fits when $|\nu|$ is large but $|q^2|$ is not.

The integration above leads to essentially two classes of sum rules. The first is given by

$$2\int_{-1}^{+1} dx\, \nu(x) \int_{\nu_t(x)}^{\infty} d\nu' \, \frac{\nu' W_2(\nu', q^2(x))}{\nu'^2 - \nu^2(x)} \cong -2i\int_{\phi_1}^{\pi} d\phi\, e^{i\phi}\omega(e^{i\phi}) \int_1^{\infty} d\omega' \, \frac{F_2(\omega')}{\omega'^2 - [\omega(e^{i\phi})]^2} - i\int_0^{\phi_1} d\phi\, e^{i\phi} R(q^2(\phi), \nu(\phi)). \tag{2.1}$$

Here the functions $q^2(z)$, $\nu(z)$, and $\omega(z)$ are given by

$$q^2(z) = -m_0{}^2 \frac{z-1}{z-b}, \tag{2.2}$$

$$\nu(z) = m_0 \left[(\tfrac{1}{2}\sqrt{b}) \frac{1}{b-z} + \frac{z-1}{2\sqrt{b}} \right], \tag{2.3}$$

$$\omega(z) = \frac{2m_p \nu(z)}{-q^2(z)}. \tag{2.4}$$

The parameters b and m_0 are chosen such that both are real and positive and vary in the ranges

$$2 \leq b \leq 3 \tag{2.5}$$

and

$$m_0 > 3m_p, \tag{2.6}$$

where m_p is the proton mass.

The left-hand side of (2.1) is completely determined by the SLAC-MIT data on W_2, with $q^2(x)$ varying in the range

$$-\frac{2}{b+1} m_0^2 \leq q^2(x) \leq 0, \tag{2.7}$$

as x varies in $-1 \leq x \leq +1$.

On the right-hand side $F_2(\omega')$ is the usual Bjorken scaling function defined as $\lim_{\nu\to\infty} \nu W_2(\nu, q^2) = F_2(\omega)$, where $\omega = -2m_p \nu/q^2$ is kept fixed as the limit is taken. The right-hand side is obtained by integrating $\nu(z) T_2(\nu(z), q^2(z))$ over the unit semicircle in the z plane, $z = e^{i\phi}$, and $0 < \phi < \pi$. The parameter ϕ_1 defines the transition point between the Regge nonscaling region, $0 < \phi < \phi_1$, and the scaling region, $\phi_1 < \phi < \pi$. We shall let ϕ_1 vary in our evaluations of (2.1); however, we always keep it in the neighborhood of $O(m^2/m_0^2)$.

In the Regge region $|q^2(\phi)|$ varies in the range

$$0 < |q^2(\phi)| < \frac{2}{b-1} m_p^2, \quad \phi_1 \approx 2\frac{m^2}{m_0^2}, \tag{2.8}$$

and in the complex scaling region $\phi_1 < \phi < \pi$ we have the bounds

$$\frac{2}{b-1} m^2 < |q^2(\phi)| < \frac{2}{b+1} m_0^2, \quad \phi_1 \approx 2\frac{m^2}{m_0^2}. \tag{2.9}$$

The Regge fit $R(q^2(\phi), \nu(\phi))$ has the standard form

$$R(q^2, \nu) = i\beta_1(q^2) + (i-1)\beta_2(q^2)(\nu)^{-1/2} + (i+1)\beta_3(q^2)(\nu)^{-3/2} - \beta_{\text{FP}}(q^2)\nu^{-1}, \tag{2.10}$$

where FP means fixed pole.

These fits are usually determined for real spacelike q^2 and large values of ν/q^2. In R we shall use the same functional form for complex q^2 as long as $|q^2| \lesssim m_p^2$.

The amplitude $T_2(\nu, q^2)$ vanishes as $q^2 \to 0$. This of course is also true in the Regge fits one uses in (2.1) for $R(q^2(\phi), \nu(\phi))$. One then expects that the low-q^2 region makes a small contribution to the Regge term in (2.1). This fact turns out to be true for the whole Regge region, $0 < \phi < \phi_1$, when one considers the imaginary parts of both sides in (2.1). In this region the Pomeranchukon contributes mainly to ImR and in taking the imaginary part of the sum rule ImR is multiplied with a factor of $\sin\phi$, which is small in the range $0 < \phi < \phi_1$. This will be borne out by the results in the next section.

The sum rule obtained by taking the real part of (2.1) turns out to be sensitive to different Regge fits and will help to exclude some of them as seen below.

The second class of sum rules given in I was obtained by dividing T_2 by $q^2/1\,\text{BeV}^2$ and thus removing the zero at $q^2 = 0$. This sum rule should be even more sensitive to the low-q^2 data. It is given by

$$\frac{1}{m_0^2} 2\int_{-1}^{+1} dx\, \nu(x)\left(\frac{x-b}{x-1}\right)\int_{\nu_t(x)}^{\infty} d\nu' \frac{\nu' W_2(\nu', q^2(x))}{\nu'^2 - \nu^2(x)}$$
$$\cong -2i\frac{1}{m_0^2}\int_{\phi_1}^{\pi} d\phi\, e^{i\phi}\left(\frac{e^{i\phi}-b}{e^{i\phi}-1}\right)\omega(e^{i\phi})\int_1^{\infty} d\omega' \frac{F_2(\omega')}{\omega'^2 - \omega^2(e^{i\phi})} - i\frac{1}{m_0^2}\int_0^{\phi_1} d\phi\, e^{i\phi}\left(\frac{e^{i\phi}-b}{e^{i\phi}-1}\right) R(q^2(\phi), \nu(\phi)). \tag{2.11}$$

Note that in Eq. (2.11) m_0^2 should actually read $m_0^2/1\,\text{BeV}^2$, so that both sides of the equation will be dimensionless.

Finally, we should point out that there is a small difference in the normalization of amplitudes given in this note and in I; the relevant factor is $2/m_p^2$. Here a m_p^{-1} factor is absorbed into the definition of W_2 such that νW_2 is dimensionless. Note also that in this paper we have defined $\nu = q\cdot p/m_p$.

III. EVALUATION OF THE SUM RULES

We will now discuss the numerical evaluation of the sum rules (2.1) and (2.11). We denote the sum rule (2.1) by "A" and (2.11) by "B". For convenience we define the following three integrals:

$$L = 2\int_{-1}^{+1} dx\, \rho(x)\nu(x)\int_{\nu_t(x)}^{\infty} d\nu' \frac{\nu' W_2(\nu', q^2(x))}{\nu'^2 - \nu^2(x)}, \tag{3.1}$$

$$C_S = -2i\int_{\phi_1}^{\pi} d\phi\, e^{i\phi}\rho(e^{i\phi})\omega(e^{i\phi})\int_1^{\infty} d\omega' \frac{F_2(\omega')}{\omega'^2 - \omega^2(e^{i\phi})}, \tag{3.2}$$

$$C_R = -i\int_0^{\phi_1} d\phi\, e^{i\phi}\rho(e^{i\phi}) R(q^2(e^{i\phi}), \nu(e^{i\phi})), \tag{3.3}$$

with

$$\rho(z) = \begin{cases} 1 & \text{in sum rule A} \\ \dfrac{1}{m_0^2}\left(\dfrac{z-b}{z-1}\right) & \text{in sum rule B,} \end{cases} \tag{3.4}$$

and $q^2(z)$, $\nu(z)$, and $\omega(z)$ given by (2.2), (2.3), and (2.4), respectively.

The sum rules can be written as

$$L = C_S + C_R \equiv C. \tag{3.5}$$

In evaluating L we effectively have to integrate $W_2(\nu, q^2)$ over a path in the ν-q^2 real plane shown schematically in Fig. 1. We note that this path is by no means a fixed-q^2 path as in the usual FESR. The path of integration passes from the lower edge of the Regge region at $x=1$, through the resonance region, and up into the Johnson-Low-Bjorken region at $x=-1$.

To evaluate L, we divided the data into four regions: (a) In the resonance region ($W=\sqrt{s} \leq 2$ BeV) we used a linear interpolation of the 6° and 10° data[2] along lines of fixed W. (b) In the scaling region ($W > 2$ BeV and $|q^2| > 1$ BeV^2) we used the following fit to the data for $\omega < 12$ (see Ref. 2):

$$\nu W_2 = 2.33\left(1-\frac{1}{\omega}\right)^3 - 2.67\left(1-\frac{1}{\omega}\right)^4 + 0.91\left(1-\frac{1}{\omega}\right)^5, \quad 1 \lesssim \omega \lesssim 4,$$

$$\nu W_2 = 0.35, \quad 4 \lesssim \omega \lesssim 6 \tag{3.6}$$

$$\nu W_2 = 0.369 - 0.0033\omega, \quad 6 \lesssim \omega \lesssim 12.$$

(c) In the region $W > 2$ BeV, $|q^2| < 1$ BeV^2, $\omega < 12$, we used the fit[3]

$$\nu W_2 = 0.557\left(1-\frac{1}{\omega'}\right)^3 + 2.1978\left(1-\frac{1}{\omega'}\right)^4 - 2.5954\left(1-\frac{1}{\omega'}\right)^5, \tag{3.7}$$

where

$$\omega' = \omega - \frac{m_p^2}{q^2}. \tag{3.8}$$

(d) For $\omega > 12$ and $W > 2$ BeV we used the various Regge fits which are being tested below.

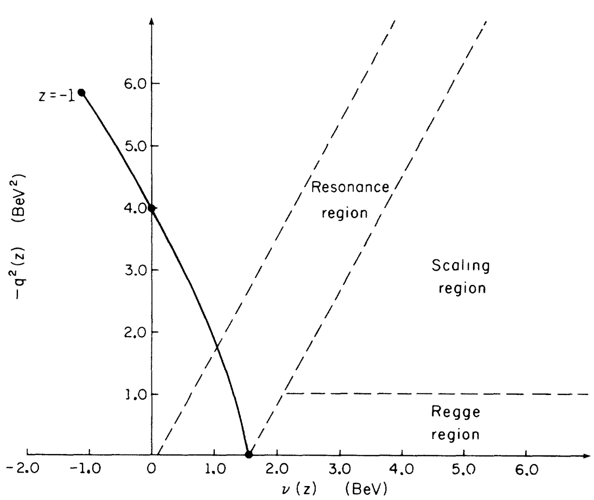

FIG. 1. Path of integration of W_2 in the ν-q^2 real plane.

In taking the imaginary part of L the integration over ν' becomes trivial and we end up with a single integral over νW_2 along the path shown in Fig. 1. The real part of L, however, involves a principal-value integration over ν'. This integral we cut off at $\nu'_{\max} - \nu_t(x) = 20$ BeV. Increasing this cutoff to 40 BeV produced no change in ReL at the 1% level.

Both the real and imaginary parts of C_S were determined from (3.2) with $F_2(\omega')$ given by (b) for $\omega' < 12$ and by

$$\lim_{q^2\to\infty} \frac{1}{\pi} \operatorname{Im} R(q^2, \omega') \quad \text{for } \omega' > 12.$$

The integral over ω' was cut off at $\omega'_{\max} = 20$ and did not vary more than 1% as $\omega'_{\max}$ was varied from 20 to 40.

The Regge fits we tested were all of the form given in (2.10) with Regge residues of the form

$$\begin{aligned}
\beta_1(q^2) &= \frac{-\pi q^2}{-q^2+\mu^2}\,\alpha,\\
\beta_2(q^2) &= \frac{-\pi q^2}{-q^2+\mu^2}\left(\frac{-q^2+m^2}{2m_p}\right)^{1/2}\beta,\\
\beta_3(q^2) &= \frac{-\pi q^2}{-q^2+\mu^2}\left(\frac{-q^2+m^2}{2m_p}\right)^{3/2}\gamma,\\
\beta_{\rm FP}(q^2) &= -\pi q^2\delta.
\end{aligned} \tag{3.9}$$

Note that the residue of the fixed pole, $\beta_{\rm FP}$, is assumed to be linear in q^2. The constants α, β, γ, δ, m^2, and μ^2 for various fits are listed in Table I.

Fits 1–4 (see Table I) are the "preferred" fits of Close and Gunion.[4] They are not exactly fits but solutions to a certain set of restrictions. These restrictions are the following:

(i) $F_2^p(\omega)$ is dominated by three Regge trajectories for $\omega > 12$. These are the Pomeranchukon ($\alpha=1$), the $f_0 - A_2$ ($\alpha = \frac{1}{2}$), and an effective background trajectory with $\alpha \cong -\frac{1}{2}$.

(ii) $F_2^p(\omega)$ satisfies a finite-energy sum rule with a $J=0$ fixed pole having a residue linear in q^2. The coefficient of q^2 in the residue of the fixed

TABLE I. Parameters of Regge fits tested.

Fit	α	β	γ	δ	μ	m
1	0.12	0.462	4.02	1	0.37	0.22
2	0.06	0.618	4.64	1	0.37	0.22
3	0.05	0.645	4.75	1	0.37	0.22
4	0.07	0.663	3.67	0	0.37	0.22
5	0.17	0.113	3.42	1	0.44	0.44
6a	0.28	0.18	0.0	−0.2	0.5	0.5
6b	0.28	0.18	0.0	−0.2	0.447	0.447
7	0.11	0.68	0.0	−0.42	0.316	0.316

TABLE II. Results of evaluation of sum rule A with $m_0^2=10$ BeV^2, $b=2.5$, $q_c^2=1.0$ BeV^2.

Fit	1	2	3	4	5	6a	6b	7
ImL	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
ImC	0.18	0.17	0.17	0.16	0.14	0.15	0.14	0.17
ImC_R	−0.01	−0.01	−0.01	−0.02	−0.05	−0.04	−0.04	−0.02
ReL	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17
ReC	0.15	0.14	0.14	0.07	0.16	0.15	0.14	0.18
ReC_R	0.10	0.09	0.09	0.02	0.10	0.10	0.09	0.13

pole is

$$R_p \equiv \delta = \begin{cases} 1 & \text{(fits 1–3)} \\ 0 & \text{(fit 4).} \end{cases} \tag{3.10}$$

(iii) $F_2^p(\omega) - F_2^n(\omega)$ obeys the quark-charge sum rule.[5]

(iv) The neutron fixed pole is also linear in q^2 with coefficient

$$R_n = \begin{cases} 0 & \text{(fits 1, 4)} \\ \frac{2}{3} & \text{(fit 2)} \\ 1 & \text{(fit 3).} \end{cases}$$

Fit 5 is the one Close and Gunion find by using a modified quark charge sum rule[4] with $R_p=1$ and $R_n=0$ and the other constraints unchanged.

Fits 6 and 7 are those obtained by Pagels[6] with fixed-pole residues as calculated by Elitzur[7] using the FESR. Pagels and Elitzur had fits only for large q^2. To get low-q^2 fits from theirs we used the forms (3.9) and chose $\mu^2=m^2$ to give a good fit to the low-q^2 Regge region. In both these fits $\beta_3(q^2)\equiv 0$. Fit 6 corresponds to $\sigma_s/\sigma_t=0.18$ and fit 7 to $\sigma_s/\sigma_t=0$.

Table II contains the results for sum rule A evaluated with $m_0^2=10$ BeV^2, $b=\frac{5}{2}$, and $q_c^2 \equiv |q^2(e^{i\phi_1})| = 1$ BeV^2. Table III contains the results of sum rule B with the same m_0^2, b, and q_c^2. In Table IV we show how the evaluation of sum rule A varies when q_c^2 varies but $m_0^2=10$ BeV^2, $b=2.5$ are kept fixed. Only Regge fit 1 is used in Table IV. In Table V we give the results of sum rule A with $q_c^2=1$ BeV^2 and m_0^2 and b variable, again using Regge fit 1.

IV. DISCUSSION AND CONCLUSIONS

The first point to note from the numbers in the tables is that all the results for the integrals ReL, ImL, ReC, and ImC are roughly of the same order of magnitude as the average value of νW_2 itself. We therefore cannot expect them to be correct to less than 10%, the combined statistical, systematic, and interpolation uncertainty of the scaling data themselves.

From Table II we see that along the real axis the integrations over νW_2 lead to values of ReL and ImL that are practically independent of the Regge fit used.

We noted in Sec. II that we expected in sum rule A that the Regge contribution to the imaginary part of the integral along the circle will be small. This is clearly supported by the numbers in Table II. Note that ImC_R, the Regge contribution, is about 10% of the total value of ImC for fits 1–4 and 7. Further, we see from Table II that for all the fits Im$L\cong$ImC to within 10–20%. The errors are given as a percentage of the magnitude of L. This gives us a test of complex scaling alone, and it is evident that the agreement is as good as can be expected. The best results are for the cases where the Regge region makes a small contribution. Thus, as far as can be learned from sum rule A, complex scaling, as defined in I for values of $|q^2|$ between 1 and 10 BeV^2, seems to be as good as real scaling. Further tests of this concept with sum rules of the same general type, but which are independent of the Regge input, are being carried

TABLE III. Results of evaluation of sum rule B with $m_0^2=10$ BeV^2, $b=2.5$, $q_c^2=1.0$ BeV^2.

Fit	1	2	3	4	5	6a	6b	7
ImL	0.43	0.42	0.42	0.42	0.43	0.43	0.43	0.44
ImC	0.43	0.41	0.41	0.25	0.41	0.32	0.35	0.48
ImC_R	0.31	0.29	0.29	0.13	0.29	0.20	0.23	0.36
ReL	0.09	0.07	0.07	0.09	0.10	0.10	0.11	0.11
ReC	0.08	0.06	0.06	0.08	0.17	0.15	0.16	0.12
ReC_R	0.12	0.10	0.10	0.12	0.22	0.19	0.20	0.16

TABLE IV. Results of evaluation of sum rule A with $m_0^2 = 10$ BeV^2, $b = 2.5$, q_c^2 variable, and using fit 1 only.

q_c^2 (BeV^2)	0.25	0.50	0.75	1.00	1.25
ImL	0.20	0.20	0.20	0.20	0.20
ImC	0.18	0.17	0.17	0.18	0.17
ImC_R	−0.01	−0.01	−0.01	−0.01	−0.01
ReL	0.17	0.17	0.17	0.17	0.17
ReC	0.14	0.14	0.15	0.15	0.11
ReC_R	0.03	0.05	0.09	0.10	0.10

TABLE V. Results of evaluation of sum rule A with $q_c^2 = 1.0$ BeV^2, m_0^2 and b variable, and using fit 1 only.

m_0^2 (BeV^2)	10	10	10	20	30
b	2.0	2.5	3.0	2.5	2.5
ImL	0.19	0.20	0.19	0.15	0.13
ImC	0.16	0.18	0.18	0.13	0.11
ImC_R	−0.01	−0.01	−0.01	−0.01	−0.00
ReL	0.17	0.17	0.18	0.16	0.14
ReC	0.15	0.15	0.13	0.14	0.13
ReC_R	0.09	0.10	0.11	0.06	0.04

out.[8] The preliminary results are positive.

The real part of the integral along the semicircle, ReC, receives approximately equal contributions from the Regge part and from the scaling region (with the exception of fit 4). Again we see from Table II that for sum rule A, Re$L \cong$ ReC to within about 10% for all fits except fit 4. For fit 4, ReL is more than twice ReC.

Sum rule B is more sensitive to the precise Regge fit used. As shown in Table III, both Re$L \cong$ ReC and Im$L \cong$ ImC are satisfied to within 10% for Regge fits 1, 2, 3, and 7. For Regge fits 5 and 6b the agreement is only slightly less good. Regge fit 6a produces a discrepancy of 25%, so the choice of masses used in fit 6b is preferable. Fit 4 again fails badly.

On the basis of these two sum rules and with the accuracy of the present data only fit 4 can be conclusively ruled out. It is interesting to note here that this is the only one of the fits above that does not have a fixed pole. But the important point to remember is that fit 4, like all the other fits used here, is a solution of the usual finite-energy sum rules. The fact that it fails to satisfy the sum rules A and B of I shows that analyticity in the two variables ν and q^2 can lead to restrictions on Regge fits that go beyond those obtained from the FESR.

Since complex scaling appears to be a reasonable hypothesis for $1 < |q^2| < 10$ BeV^2, we varied q_c^2 to see if complex scaling can be extended (in some average sense) to even lower values of $|q^2|$. From Table IV we see that as q_c^2 decreased from 1 BeV^2 to 0.25 BeV^2, the agreement between the two integrals gradually becomes worse. However, even for $q_c^2 = 0.25$ BeV^2, both real and imaginary parts agree to better that 20% using fit 1. Increasing q_c^2 to 1.25 BeV^2 shows that the Regge fit cannot be expected to work for $|q^2| > 1$ and small values of $|\nu|$. (For q_c^2 in the range studied, $|\nu| \cong 2$ BeV when $|q^2| = q_c^2$ and it is surprising that the Regge fits work as well as they do for $0.5 < |q^2| < 1.0$ BeV^2.)

Table V displays the results of varying m_0^2 and b, with $q_c^2 = 1$ BeV^2 and using only Regge fit 1. With one exception these sum rules all work equally well. The exception is the case $m_0^2 = 10$ BeV^2, $b = 3$, and there is a clear reason why this case cannot be expected to work, namely, for these values of m_0^2 and b the point $z = 1$ falls in the middle of the resonance region. Consequently, for z on the unit circle near the point $z = 1$ we cannot expect the Regge fit for T_2 to work, and the sum rule cannot be expected to hold.

In conclusion we would like to emphasize the following two points. First, it is clear that several different sum rules of the general type discussed here should be derived and compared with the data, in order to make certain that the apparent validity of complex scaling is not due to the specific forms of $\nu(z)$ and $q^2(z)$ chosen in I. This analysis is being carried out.[8] Second, a class of sum rules more sensitive to the Regge region could also be developed. Such sum rules would not only limit further the class of allowable Regge fits but also, hopefully, would discriminate between fits in which the sign of the fixed-pole residues are different. This last point might have to await the availability of more accurate data.

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Finite-Energy Sum Rules, Regge Cuts, and Forward πN Charge-Exchange Scattering

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Finite-energy sum rules (FESR) with logarithmic terms are formulated in a way amenable to simple and accurate approximation, thus allowing their convenient use in data fitting with Regge cuts. It is observed that a typical cut FESR contribution is similar to that of a pole lying perhaps half a unit lower in the J plane. Therefore $\rho+\rho'$ fits to $\pi^- p \to \pi^0 n$ seem to imply decoupling of the $\rho\otimes P$ Regge cut from the nonflip amplitude at $t=0$. This is investigated in more detail, and its implications are pointed out.

I. INTRODUCTION

Finite-energy sum rules (FESR) are usually written for amplitudes with assumed power (Regge pole) asymptotic behavior.[1] However, now it is believed that at high energy there are also significant Regge-cut (i.e., logarithmic) contributions,[2] and so to help in the construction of realistic phenomenological models it is desirable to generalize the sum rules to include such terms.

Unfortunately this is impossible in closed form[3-5] (except in special models[6]), and the repeated numerical evaluations necessary in typical parameter-search calculations are often prohibitively time consuming.

In this paper we first examine the structure of FESR's with logarithmic terms and put them into a form where a simple approximation can be made to overcome the time problem, so giving a practical way to include FESR constraints in data fitting with general types of Regge cuts.

We then observe that in a FESR the simplest kind of Regge cut behaves like a pole with intercept lower in the J plane by about $\frac{1}{4}$ to $\frac{1}{2}$ a unit. Consequently, for example, some $\rho+\rho'$ models of $\pi^- p \to \pi^0 n$ may be good approximations to the physics of a ρ pole plus a $\rho\otimes P$ cut and imply that the cut essentially decouples at $t=0$. (We use the notation $A\otimes B$ to denote the cut resulting from the simultaneous exchange of Regge poles A and B in the t channel.)

The situation is explored in more detail, and quantitative results are given. We point out the major implications of a small cut amplitude at $t=0$, regarding especially the crossover mechanism and the difference of total cross sections at high energy, and note the similarity of the situation in charged-pion photoproduction.

II. SUM RULES

A standard FESR derivation[3] deals with an amplitude $F(\nu)$ $[\nu=(s-u)/4m]$ at fixed t or u (suppressed) with the usual analytic properties and an assumed asymptotic ($|\nu|\geq N$) model form. Cauchy's theorem $[\int_C F(\nu)d\nu=0]$ is used, where the contour C lies along the real axis above the physical cuts from $-N$ to N and closes with a semicircle $|\nu|=N$ in the upper half-plane. Thus the low-energy amplitude is integrated from $-N$ to N, and the asymptotic model round the semicircle (not to threshold, nor to ∞). For an amplitude of definite crossing symmetry $[F(-\nu)=\pm F^*(\nu)]$ the left- and right-hand parts of the low-energy integral can be combined.

A crossing-odd amplitude with Regge-pole (power) behavior,

$$F(\nu)=i\gamma(-i\nu)^{\alpha}=i\gamma e^{-i\pi\alpha/2}\nu^{\alpha},\quad |\nu|\geq N \tag{1}$$

is then found to obey the FESR

$$\frac{1}{N}\int^{N}\mathrm{Im}F(\nu)d\nu=\gamma\frac{N^{\alpha}}{\alpha+1}\cos\tfrac{1}{2}\pi\alpha\,. \tag{2}$$

A Regge-cut amplitude contains powers of $\ln\nu$, and a simple crossing-odd term of the type suggested by current models[3-8] is