Ambiguities in Scattering Amplitudes Resulting from Insufficient Experimental Data*

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We present a systematic study of the problem of reconstructing the scattering matrix from experimental data. Insufficient data lead to ambiguities; a general technique for finding ambiguities is given. The cases of isospinless spin-0–spin-$\frac{1}{2}$ scattering (two amplitudes) and pion-nucleon scattering (four amplitudes) are treated in detail. The ambiguities present in both cases when the R and A parameters are unknown have a profound effect on partial-wave analysis, leading to confusion and misidentification of resonances. It is shown that only eight measurements are needed to reconstruct the pion-nucleon amplitudes, rather than nine as averred by Bilenkii and Ryndin.

I. INTRODUCTION

Because experimental data have been insufficient to allow an actual measurement of scattering amplitudes, physicists have relied on fitting theoretical models to the available data. It is well known that ambiguities may arise when data are lacking, but the effects of such ambiguities on model fitting have been largely ignored. In this paper we attempt a systematic study of how ambiguities arise and how they affect the reconstruction of the amplitude.

To our knowledge, the first description of an ambiguity was that of Minami.[1] Subsequent papers by Wolfenstein[2] and by Puzikov, Ryndin, and Smorodinsky[3] furthered the analysis of the scattering matrix for processes involving spin, and somewhat later Bilenkii and Ryndin[4] showed that the pion-nucleon scattering amplitudes could be determined unambiguously from nine experimental quantities. More recently, Daum, Michael, and Schmid[5] studied the ambiguity in Regge analysis of high-energy scattering.

In Sec. II we begin by giving a general treatment of the question of ambiguities and suggest a technique for reconstructing the scattering amplitudes. The specific case of spin-0–spin-$\frac{1}{2}$ scattering is treated in Sec. III. It is shown that knowing only differential cross sections and polarizations leaves an infinite set of transformations on the amplitudes which do not alter these data. The general effect of such transformations is to change the partial-wave amplitudes completely, as we show in some simple cases. The constraint of unitarity is most easily applied in partial-wave form; for purely elastic amplitudes, unitarity rules out all but the Minami ambiguity. When there is inelasticity, however, the constraint is much weaker, and the ambiguities may survive.

The four-amplitude case of pion-nucleon scattering is treated in Sec. IV. If only differential cross sections and polarizations for three processes $\pi^{\pm}p \to \pi^{\pm}p$ and $\pi^- p \to \pi^0 n$ are known, the same ambiguities described in Sec. III are present in each isospin channel. We show that a unique solution can be obtained by measuring only two additional quantities, rather than three as suggested by Bilenkii and Ryndin.

II. GENERAL THEORY OF AMBIGUITIES

Let us consider a two-body scattering process described by N invariant amplitudes F_α. At an arbitrary scattering angle away from the forward direction, the available experimental data will consist of a set of r real values d^i for corresponding real quadratic forms of the F_α,

$$d^i = M^i_{\alpha\beta} F^*_\alpha F_\beta \,. \tag{1}$$

The reality of the d^i implies that the $M^i_{\alpha\beta}$ are Hermitian.

An ambiguity in reconstructing the amplitudes corresponds to the existence of a transformation on the F_α under which all of the equations (1) are invariant. The simplest of these is an over-all phase transformation $F_\alpha \to e^{i\lambda} F_\alpha$, showing that we can measure only the relative phases of the amplitudes. We shall investigate other less trivial linear and antilinear transformations in this paper. Clearly, if the data are sufficient to rule out any such transformation, then a solution, if one exists, must be unique. Unfortunately, there are also nonlinear transformations, and because of them our method cannot provide a general technique for solving Eq. (1). In many cases, however, we feel that it may be useful in exposing ambiguities corresponding to linear and antilinear transformations.

Suppose that one of the data always corresponds to the differential cross section, and that we choose the amplitudes in such a way that it corresponds to $M^1_{\alpha\beta}=\delta_{\alpha\beta}$, i.e.,

$$d^1=\delta_{\alpha\beta}F^*_\alpha F_\beta=\sum_\alpha |F_\alpha|^2 . \qquad (2)$$

Requiring that a linear transformation $F_\alpha \to T_{\alpha\beta}F_\beta$ conserves Eq. (2) does not imply that T is unitary, since F is not an arbitrary vector. But any nonunitary T maintaining (2) must be unitary in the subspace into which it can map F, so we can consider only unitary transformations. Then an ambiguity will correspond to the existence of a unitary U which commutes with all of the M_i. If H denotes a Hermitian matrix satisfying

$$[H, M^i]=0 \qquad (3)$$

for all i, then it follows that $U=e^{i\varphi H}$ also commutes with the M^i for any φ. Consequently the data cannot distinguish F from

$$F'=e^{i\varphi H}F . \qquad (4)$$

We shall call the group of transformations (4) a linear ambiguity group.

To determine whether such a group exists, one must search for an H satisfying Eq. (3). This can be done systematically using the algebra of SU(N). We know that since the M^i are Hermitian, they can be expanded as

$$M^i=m^i_\lambda G_\lambda , \qquad (5)$$

where G_λ are the generators of SU(N) and m^i_λ are real. Similarly we may write

$$H=h_\mu G_\mu . \qquad (6)$$

Then Eq. (3) becomes

$$m^i_\lambda h_\mu [G_\lambda , G_\mu]=m^i_\lambda h_\mu i g_{\lambda\mu\nu}G_\nu=0 , \qquad (7)$$

where $g_{\lambda\mu\nu}$ are the structure constants of SU(N). It then follows from the independence of the G_ν that

$$(m^i_\lambda g_{\lambda\nu\mu})h_\mu=R^i_{\nu\mu}h_\mu=0 , \qquad (8)$$

i.e., that h_μ is an eigenvector of the matrix $R^i_{\nu\mu}=m^i_\lambda g_{\lambda\nu\mu}$ with zero eigenvalue. [Since $h=m^i$ is a nontrivial solution of Eq. (8), the determinant of this matrix certainly vanishes.] If there is a common null eigenvector h^0 for all of the R^i, then it follows that $H=h^0_\alpha G_\alpha$ generates a linear ambiguity group.

Next let us investigate briefly the antilinear transformations obtained via $F'_\alpha=A_{\alpha\beta}F^*_\beta$. Once again we consider only those transformations for which A is unitary, so that $F'^*_\alpha F'_\alpha=F^*_\alpha F_\alpha$. Then the effect of such a transformation is given by $M^{i\prime}_\alpha=A^\dagger \tilde{M}^i A$, where the tilde denotes transposition. Consequently any A for which

$$A M^i A^\dagger=\tilde{M}^i \qquad (9)$$

defines an antilinear ambiguity. A systematic generation of such transformations does not appear possible, but it may be noted that if

$$M^i_{\beta\alpha}=e^{i[k(\alpha)-k(\beta)]}M^i_{\alpha\beta} , \qquad (10)$$

then d^i is invariant under changing F_α to

$$F'_\alpha=e^{ik(\alpha)}F^*_\alpha , \qquad (11)$$

i.e., $A_{\alpha\beta}=e^{ik(\alpha)}\delta_{\alpha\beta}$. It is not possible to say generally whether such a symmetry will hold for all M^i (or to some set $M^{i\prime}$ equivalent to the original set). If an appropriate set of $k(\alpha)$ does exist, however, then Eq. (11) can be combined with any available linear ambiguity group to generate an antilinear ambiguity group.

These considerations prompt the following conjectures about solving simultaneous quadratic equations. Since $2N-1$ real numbers are to be determined, at least $2N-1$ measurements are required. The manipulations involved in solving the equations are equivalent to converting the data matrices M^i into a "soluble" set. The latter should contain N linearly independent diagonal matrices, yielding solutions for the magnitudes $|F_\alpha|$, plus at least $N-1$ off-diagonal ones giving the relative phases $\eta_{\alpha\beta}=\eta_\alpha-\eta_\beta$. It is easily seen, however, that a single measurement determines only $\cos\eta_{\alpha\beta}$ or $\sin\eta_{\alpha\beta}$, leaving in either case a sign ambiguity in the phase. To resolve this ambiguity at least one more measurement is required, determining some linear combination of all $\eta_{\alpha\beta}$ and thereby fixing the signs. Consequently the minimum number of measurements required is $2N$, and we conjecture correspondingly that the smallest number of linearly independent matrices M^i which can rule out all of the above transformations is also $2N$. To solve the equations, one finds N mutually commuting linear combinations of the M^i, diagonalizes them, and solves for the magnitudes. The remaining N equations may then be solved manually for the relative phases.[6]

III. CONSEQUENCES FOR MESON-NUCLEON SCATTERING

We treat first the simplest case of the scattering of a spin-0 meson by a spin-$\frac{1}{2}$ nucleon. The transition matrix, in the usual form

$$T=F+iG\hat{n}\cdot\vec{\sigma} ,$$

is determined by measuring the differential cross section, polarization, and spin-rotation parameters,

$$d^1=\frac{d\sigma}{dr}=|F|^2+|G|^2 ,$$

$$d^2 = P\frac{d\sigma}{d\Omega} = -2\,\mathrm{Im}\,F^*G\,,$$
$$d^3 = R\frac{d\sigma}{d\Omega} = (|F|^2 - |G|^2)\cos\theta + 2\,\mathrm{Re}(F^*G)\sin\theta\,, \tag{12}$$
$$d^4 = A\frac{d\sigma}{d\Omega} = -(|F|^2 - |G|^2)\sin\theta + 2\,\mathrm{Re}(F^*G)\cos\theta\,,$$

where θ is the scattering angle. The matrices corresponding to d^i are clearly

$$M^1 = 1,\quad M^2 = -\sigma_y\,,$$
$$M^3 = (\sigma_z\cos\theta + \sigma_x\sin\theta),$$
$$M^4 = (-\sigma_z\sin\theta + \sigma_x\cos\theta),$$

where σ_i are the Pauli matrices.

In the most common situation only d^1 and d^2 are experimentally measured. It follows that a linear ambiguity group is generated by choosing $H = \sigma_y$; d^1 and d^2 are invariant under

$$\begin{pmatrix}F\\G\end{pmatrix} \to \begin{pmatrix}F'\\G'\end{pmatrix} = e^{i\varphi\sigma_y}\begin{pmatrix}F\\G\end{pmatrix} = \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}F\\G\end{pmatrix} \tag{13}$$

for any φ.[7] To see the meaning of this ambiguity it is convenient to transform to new amplitudes

$$\begin{pmatrix}f\\g\end{pmatrix} = e^{i\pi\sigma_x/4}e^{i\theta\sigma_z/2}\begin{pmatrix}F\\G\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\theta}F + ie^{-i\theta}G\\ ie^{i\theta}F + e^{-i\theta}G\end{pmatrix} \tag{14}$$

for which the data matrices become simply $M^i = 1, \sigma_z, \sigma_x, \sigma_y$. Thus

$$d^1 = |f|^2 + |g|^2\,,$$
$$d^2 = |f|^2 - |g|^2\,,$$

so we can solve for $|f|$ and $|g|$, but the phase between them is completely undetermined.

Furthermore, since Eq. (13) refers only to data at a single fixed energy and scattering angle, φ can depend *arbitrarily* on those variables; the only limitation we impose here (aside from such trivial ones as continuity, etc.) is that $\varphi(\theta=0)=0$ in order to ensure that the spin-flip amplitude G vanishes in the forward direction. Considering $\varphi(\theta)$ as an arbitrary function has far-reaching effects on the partial-wave analysis of $F(\theta)$ and $G(\theta)$. For example, we take the simplest case $\varphi = \theta$. Using the customary expansions

$$F(\theta) = \sum_l [(l+1)f_{l+} + lf_{l-}]P_l(\cos\theta)\,,$$
$$G(\theta) = \sum_l (f_{l-} - f_{l+})P_l^1(\cos\theta)\,,$$

and the recursion relations for Legendre polynomials, we find that (13) is equivalent to replacing $f_{l\pm}$ by

$$f_{l+}' = [2lf_{l-1,+} + f_{l+1,-}]/(2l+1)\,,$$
$$f_{l-}' = [2(l+1)f_{l+1,-} - f_{l-1,+}]/(2l+1)\,. \tag{15}$$

A similar procedure with $\varphi = -\theta$ leads instead to

$$f_{l+}'' = [2(l+2)f_{l+1,+} - f_{l+1,-}]/(2l+3)\,,$$
$$f_{l-}'' = [2(l-1)f_{l-1,-} + f_{l-1,+}]/(2l-1)\,. \tag{16}$$

These results can be iterated n times to yield the transformations corresponding to $\varphi = \pm n\theta$. One other possibility we have considered is $\varphi = \epsilon\sin\theta$, for some small value of ϵ. A similar procedure here leads to new partial-wave amplitudes given, to order ϵ, by

$$f_{l+}(\epsilon) = f_{l+} + \epsilon\left[\frac{2l+1}{(2l+1)(2l+3)}f_{l+1,-} - \frac{l+2}{2l+3}f_{l+1,+} + \frac{l}{2l+1}f_{l-1,+}\right],$$
$$f_{l-}(\epsilon) = f_{l-} + \epsilon\left[\frac{l+1}{2l+1}f_{l+1,-} - \frac{l-1}{2l-1}f_{l-1,-} - \frac{2l}{(2l+1)(2l-1)}f_{l-1,+}\right]. \tag{17}$$

In contrast to Eqs. (15) and (16), the transformations (17) allow the partial-wave amplitudes to be varied continuously.

Equations (15), (16), and (17) describe the effects on the partial-wave amplitudes of three simple choices for the ambiguity φ in (13). Before discussing the consequences of these transformations, let us turn to the antilinear ambiguities present when only differential cross sections and polarizations are measured. It is easily verified that 1 and σ_y satisfy (10) with $k(1) - k(2) = \pi$. Consequently the data are invariant under $F \to F^*$, $G \to -G^*$, which corresponds to the replacement of $f_{l\pm}$ by

$$f_{l+}^c = -(f_{l+}^* + 2lf_{l-}^*)/(2l+1)\,,$$
$$f_{l-}^c = -[2(l+1)f_{l+}^* - f_l^*]/(2l+1)\,. \tag{18}$$

In addition, this transformation may be combined with any linear ambiguity. For example, if Eq. (16) is applied to the results of Eq. (18), one obtains the well-known Minami transformation

$$f_{l\pm}^M = -f_{l\pm1,\mp}^*\,. \tag{19}$$

The same result follows if (18) is applied to the results of (15).

The constraint of unitarity can now be applied to these transformations by considering the partial-wave amplitudes. If the $f_{l\pm}$ describe elastic scattering below the inelastic threshold, they must satisfy $\text{Im} f_{l\pm} = |f_{l\pm}|^2$. It is then easy to show that the amplitudes obtained by any of the transformations (15), (16), (17), and (18) violate unitarity; only the Minami transformation (19) survives. In most partial-wave analyses, however, elastic unitarity does not hold. If the $f_{l\pm}$ describe elastic scattering above the inelastic threshold, the weaker condition $\text{Im} f_{l\pm} \geq |f_{l\pm}|^2$ applies, while if they describe an inelastic process one has only $|f_{l\pm}| \leq \frac{1}{2}$. In either of these cases it is possible for the transformations to change the amplitudes without violating unitarity. In fact, one can show that f_{l+}', f_{l-}'', and f^c_{l+} are certainly well behaved; while the unitarity of the others depends on the inelasticities of the initial $f_{l\pm}$.

It is tempting to speculate on the relation between these ambiguities and the many solutions usually found for the phase shifts at a given energy. One possibility is that many of these solutions are related by transformations such as those given above. At the other extreme, it is conceivable that applying these transformations to those solutions would lead to a plethora (perhaps even a continuum) of possible solutions. In the latter case it will be very hard to defend the traditional "shortest path" techniques of joining phase-shift analyses at different energies.

The identification of resonances via loops in the Argand diagram will also be subject to confusion. Let us suppose, for example, that the $f_{l\pm}$ are the true partial-wave amplitudes and that there is a resonance in f_{L-}. If the transformation (15) is made at all energies, the resulting $f_{l\pm}'$ show twin resonances in $f_{L-1,\pm}'$. If (16) is used instead, we find a resonance in $f_{L+1,-}''$, while the loop appears *inverted* in $f_{L-1,+}''$, i.e., the resonant energy would correspond to the bottom of the loop. The latter behavior in an elastic amplitude would not be called a resonance. Similar results hold for a resonance in f_{L+}. The antilinear transformation (18), on the other hand, will always change a counterclockwise loop into a clockwise one (and vice versa), thereby eliminating the resonance.

Data on the spin-rotation parameters are needed in order to avoid these ambiguities. There is no matrix which commutes with M^2 and M^3 or M^4, so all of the linear ambiguity groups (except the trivial one corresponding to the over-all phase) are removed if d^3 or d^4 is known. To consider the antilinear ambiguities, it is convenient to transform to the amplitudes (14). If d^3 is measured, the fact that σ_x is symmetric implies that (10) holds with $k(\alpha) \equiv 0$, i.e., the data do not distinguish (f, g) from (f^*, g^*). Similarly, a measurement of d^4 only leaves an ambiguity under $(f, g) \to (f^*, -g^*)$. Both of these transformations simply reflect an unknown sign in the relative phase of f and g.

IV. EXTENSION TO PION-NUCLEON SCATTERING

Finally we shall consider here the ambiguities in analyzing the elastic pion-nucleon scattering amplitudes. Here there are four scalar amplitudes, corresponding to spin flip and nonflip in each isospin channel. If we write the transition operator in the form

$$\mathcal{T} = A + iB\hat{n}\cdot\vec{\sigma} + C\vec{\pi}\cdot\vec{\tau} + iD\hat{n}\cdot\vec{\sigma}\vec{\pi}\cdot\vec{\tau},$$

where $\frac{1}{2}\vec{\tau} = \vec{I}$ is the nucleon isospin and $\vec{\pi}$ is the pion isospin, then the available data take the form

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\pi^+p) &= d^1 = |A+C|^2 + |B+D|^2,\\
P\frac{d\sigma}{d\Omega}(\pi^+p) &= d^2 = -2\,\text{Im}[(A+C)^*(B+D)],\\
\frac{d\sigma}{d\Omega}(\pi^-p) &= d^3 = |A-C|^2 + |B-D|^2,\\
P\frac{d\sigma}{d\Omega}(\pi^-p) &= d^4 = -2\,\text{Im}[(A-C)^*(B-D)],\\
\frac{d\sigma}{d\Omega}(\pi^-p\to\pi^0 n) &= d^5 = 2(|C|^2 + |D|^2),\\
P\frac{d\sigma}{d\Omega}(\pi^-p\to\pi^0 n) &= d^6 = -4\,\text{Im}(C^*D).
\end{aligned} \tag{20}$$

The corresponding data matrices are conveniently written in direct product form, taking $F_\alpha = (A, B, C, D)$, as[8]

$$\begin{aligned}
M^1 &= (1+\tau_x)\otimes 1, & M^2 &= -(1+\tau_x)\otimes\sigma_y,\\
M^3 &= (1-\tau_x)\otimes 1, & M^4 &= -(1-\tau_x)\otimes\sigma_y,\\
M^5 &= (1-\tau_z)\otimes 1, & M^6 &= -(1-\tau_z)\otimes\sigma_y.
\end{aligned} \tag{21}$$

Since six data d^i are insufficient to determine seven real numbers (four moduli and three phases), one expects to find an ambiguity group. Because only 1 and σ_y occur in the latter part of the direct products (which corresponds to having only differential cross sections and polarizations), it is easily shown that

$$H = 1\otimes\sigma_y = \begin{pmatrix} 0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix} \tag{22}$$

satisfies $[H, M^i] = 0$ for all i. Consequently $\exp(i\varphi H)$ generates a linear ambiguity group. By explicit expansion, we find

$$e^{i\varphi H} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0\\ -\sin\varphi & \cos\varphi & 0 & 0\\ 0 & 0 & \cos\varphi & \sin\varphi\\ 0 & 0 & -\sin\varphi & \cos\varphi\end{pmatrix} \tag{23}$$

representing an ambiguity which mixes only (A, B) and (C, D) separately. All of the analysis of Sec. II is therefore applicable here also, provided that the same ambiguity transformations are applied simultaneously in both isospin channels.

Let us note the effect of the transformation (14) in this case. In terms of

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = e^{i\pi(1\otimes\sigma_x)/4} e^{i\theta(1\otimes\sigma_z)/2} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta}A + ie^{-i\theta}B \\ ie^{i\theta}A + e^{-i\theta}B \\ e^{i\theta}C + ie^{-i\theta}D \\ ie^{i\theta}C + e^{-i\theta}D \end{pmatrix}, \tag{24}$$

we have M^1, M^3, and M^5 unchanged while $\sigma_y \to \sigma_z$ in M^2, M^4, and M^6. (In addition, the explicit θ dependence is removed from the spin-rotation parameters.) By linear combination one can then convert the M^i into four diagonal matrices plus two off-diagonal ones. The former can be solved for four magnitudes, yielding

$$\begin{aligned} |\alpha|^2 &= \tfrac{1}{4}(d^1 + d^3 - d^5 + d^4 + d^2 - d^6), \\ |\beta|^2 &= \tfrac{1}{4}(d^1 + d^3 - d^5 - d^4 - d^2 + d^6), \\ |\gamma|^2 &= \tfrac{1}{4}(d^5 + d^6), \\ |\delta|^2 &= \tfrac{1}{4}(d^5 - d^6), \end{aligned} \tag{25}$$

while the off-diagonal ones give the phases between (α, γ) and between (β, δ) as

$$\begin{aligned} 2|\alpha||\gamma|\cos\eta_{13} &= \tfrac{1}{4}(d^1 - d^3 - d^4 + d^2), \\ 2|\beta||\delta|\cos\eta_{24} &= \tfrac{1}{4}(d^1 - d^3 + d^4 - d^2). \end{aligned} \tag{26}$$

The ambiguity group represents the fact that the phase η_{12} between α and β is completely unknown. In addition, of course, the antilinear ambiguity described in Sec. III is present and may be combined with any of the transformations (23). In fact, none of the M^i mix the (α, γ) and (β, δ) subspaces, so the antilinear transformations in those subspaces may be applied independently. That is, the data are invariant under $(\alpha\beta\gamma\delta) \to (\alpha\beta^*\gamma\delta^*)$ and $(\alpha\beta\gamma\delta) \to (\alpha^*\beta\gamma^*\delta)$ as well as $(\alpha\beta\gamma\delta) \to (\alpha^*\beta^*\gamma^*\delta^*)$.

The ambiguities present when only these six data are known, then, may be summarized as (a) the phase η_{12} between α and β is completely free; (b) the signs of η_{13} and η_{24}, the relative phases of (α, γ) and of (β, δ), are unknown. At least two more measurements are therefore required in order to determine all four amplitudes up to a common phase – one to determine the magnitude of η_{12} (or some function of η_{12}), the other to fix an independent combination of the three phases. These data must be chosen carefully, however, in such a way as to remove the above ambiguities. For example, two different R measurements, $R\,d\sigma/d\Omega(\pi^{\pm}p)$, will yield two new matrices which are both symmetric, and therefore the ambiguity $(\alpha\beta\gamma\delta) \to (\alpha^*\beta^*\gamma^*\delta^*)$ will remain. Likewise measuring R and A for a single reaction is insufficient, since both determine a *single* phase, without removing the other sign ambiguities. It can be shown that the data are invariant in that case under the transformation $(\alpha\beta\gamma\delta) \to (\alpha^*, \beta^* e^{2i\psi}, \gamma^*, \delta^* e^{2i\psi})$ where ψ is the phase determined by R and A.

Thus one must choose a combination such as $A\,d\sigma/d\Omega(\pi^-p)$ and either $R\,d\sigma/d\Omega(\pi^+p)$ or $A\,d\sigma/d\Omega(\pi^+p)$. The solution obtained by Bilenkii and Ryndin used all three of these quantities, but it is not difficult to see that only two are necessary. Knowing $R\,d\sigma/d\Omega(\pi^+p)$, for example, determines the magnitude of the phase between $(\alpha + \gamma)$ and $(\beta + \delta)$. Since the magnitudes of η_{13} and η_{24} are also known, this measurement reduces the number of possible solutions to eight – one for each choice of the three signs. Four of these solutions would yield the same $A\,d\sigma/d\Omega(\pi^+p)$, but only *one* will yield a particular value for $A\,d\sigma/d\Omega(\pi^-p)$. Thus the latter measurement will yield a unique solution.

V. CONCLUSIONS

We have shown how insufficient data – particularly the lack of knowledge of spin-rotation parameters – produce ambiguity in the determination of the scattering amplitudes. The consequences of such ambiguities are particularly important in partial-wave analysis. It is ordinarily true that direct fitting of differential cross-section and polarization data produces a large number of possible phase-shift solutions. Ambiguity transformations such as (15), (16), (17), (18), and (19) seem likely to increase that large number still significantly further. With such an abundance of possible solutions, the need for additional *theoretical* input to choose the correct one becomes acute.

For example, the use of dispersion relations for the partial-wave amplitudes, the requirement that high partial waves approach the Born approximation, and other similar ideas have frequently been used to distinguish the true phase shifts. Energy-independent theoretical constraints are clearly preferable for this purpose, since they will actually remove some of the ambiguity, whereas energy-dependent techniques will only relate solutions at neighboring energies. In the pion-nucleon phase-shift analysis of Bareyre *et al.*,[9] for one example among many, the data at each energy were fitted with essentially no such assumptions. "Shortest-path" techniques were then used to choose the correct solution at each energy on the basis of the smoothest energy variation. But suppose that applying our ambiguity transformations would, say, double the number of possible solutions at each energy. There is no compelling reason to believe that the increased flexibility could not lead to a completely different shortest-path solution.

In summary, these results indicate that phase-shift analyses performed without spin-rotation parameter data or fixed-energy theoretical constraints to eliminate ambiguities may not be trustworthy. The use of energy-smoothness criteria in this case is questionable unless all of the ambiguity-related solutions are included at each energy. Resonances found only in such an analysis, without corroborating evidence in mass plots, total cross sections, etc., should therefore be viewed with some suspicion.

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Absorptive Separable Potentials Constructed from πN Data*

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A simple method for constructing an absorptive (complex) separable potential directly from the πN phase shifts and absorption parameters is presented. This permits the transition matrix to be calculated both on and off the energy shell as required in the calculation of the π-nucleus optical potential. The solution of the inverse scattering problem presented here is an extension of a previous solution to include absorption, when the phase shifts are complex, and relativistic kinematics. The potentials for all S- and P-wave πN eigenchannels are constructed and displayed. The construction of a complex separable potential is also applicable to other systems such as NN, ΛN, and KN.

I. INTRODUCTION

Interest in pion-nucleus scattering is being stimulated by the advent of meson factories. It is hoped that the pion provides a useful, additional probe of nuclei provided one can extract nuclear-structure information with a reliable pion-nucleus scattering theory.[1] Various descriptions of pion-nucleus scattering have been suggested and applications near the 3-3 resonance region have been qualitatively successful.[2,3] A basic ingredient of these theories is the underlying pion nucleon interaction, which is, however, not completely known.

In this paper, a simple phenomenological model of the pion-nucleon interaction is presented. A