Total Cross Sections for the Reactions $2\gamma \to 3\pi$ and $e^+e^- \to e^+e^- 3\pi$*

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Using the model-independent $2\gamma \to 3\pi$ amplitude recently given by Adler, Lee, Treiman, and Zee, we calculate the total cross section for $2\gamma \to \pi^+\pi^-\pi^0$, and the total cross section for $e^+e^- \to e^+e^-\pi^+\pi^-\pi^0$ in the Weizsäcker-Williams approximation. Some comments are also made on the cross section for $\mu^+\mu^- \to \pi^+\pi^-\pi^0$ and its relevance to the decay $K_L^0 \to \mu^+\mu^-$.

Recently there has been some discussion of low-energy theorems for the amplitude $2\gamma \to 3\pi$. A model-independent prediction has now been made by Adler, Lee, Treiman, and Zee[1] which generalizes and corrects the work of the other authors.[2] Both the matrix elements for $2\gamma \to 3\pi^0$ and $2\gamma \to \pi^+\pi^-\pi^0$ have been given and we will concentrate here on the charged-pion case, as it is more interesting from the viewpoint of experiment. The matrix element is the sum of a π^0 pole term, bremsstrahlung radiation from the charged pions, and a contact term necessary to maintain gauge invariance. As the complete matrix element is explicitly given in Ref. 1, it is not necessary to write it here; we merely comment that the overall normalization is set by the $\gamma \to 3\pi$ coupling constant $F^{3\pi}$ and the charged-pion decay constant f^π. The parameter x which is proportional to the isotensor component of the "σ term" is expected to be small, so we have taken it to be zero. $F^{3\pi}$ can be related to the π^0 decay coupling constant F^π so all the parameters in the amplitude are known. Knowledge of this amplitude allows us to calculate the total cross sections for several interesting processes, such as $2\gamma \to 3\pi$, $e^+e^- \to e^+e^-3\pi$, and $\mu^+\mu^- \to 3\pi$.

The simplest process to discuss is $2\gamma \to \pi^+\pi^-\pi^0$. We used an algebraic computer program[3] to square the matrix element and then calculated the total cross section by doing the four-dimensional integral numerically. The result is given in Fig. 1, where we have averaged over the photon polarizations. We then used this $2\gamma \to \pi^+\pi^-\pi^0$ cross section as input in the Weizsäcker-Williams approximation for the $e^+e^- \to e^+e^-\pi^+\pi^-\pi^0$ cross section.[4] Our answer for the latter cross section is given in Fig. 2, where we also plot for comparison 10^{-3} times the total cross section for $e^+e^- \to e^+e^-\pi^+\pi^-$ taken from Ref. 4. It is obvious that the cross section for the three-pion production is significantly smaller than that for the two-pion production.

Since we believe quantum electrodynamics for muons, the cross section for $\mu^+\mu^- \to 3\pi$ must be smaller than that for $2\gamma \to 3\pi$ by at least a factor of $(\alpha/\pi)^2$. Although we have not explicitly calculated this cross section, a closer examination of the amplitude for $\mu^+\mu^- \to 3\pi$ reveals that there is a logarithmically divergent piece from the π^0 pole term and a finite contribution from the charged-pion bremsstrahlung. The π^0 pole term, however, must vanish when the muon mass vanishes, so this dangerous term is further suppressed. It is clearly impossible, therefore, to have any anomalously large $\mu^+\mu^- \to 3\pi$ cross section from such a matrix element. The main interest in such a cross section is its possible influence on the $K_L^0 \to \mu^+\mu^-$ decay rate.[5] The question to ask in this context is

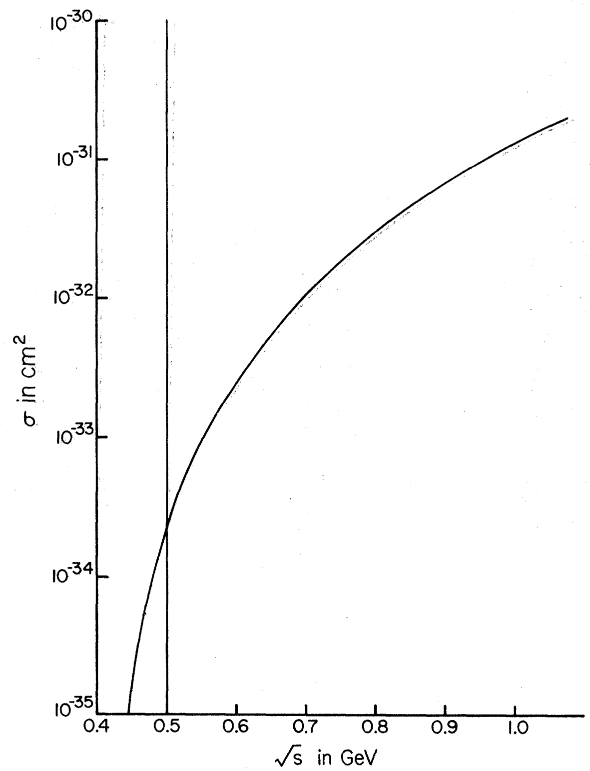

FIG. 1. $\sigma(\gamma\gamma \to \pi^+\pi^-\pi^0)$ in cm^2 versus $\sqrt{s}$, the total energy in GeV in the c.m. system. The straight line gives $\sqrt{s} = M_K$.

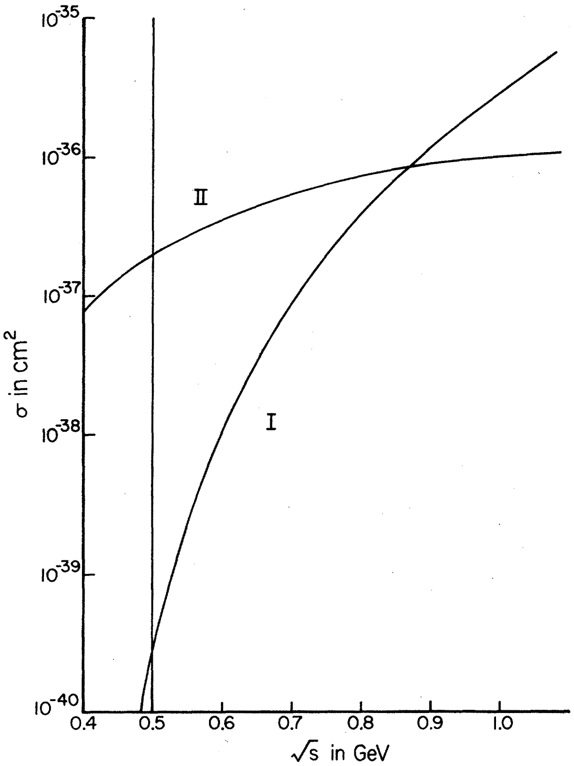

FIG. 2. $\sigma(e^+e^- \to e^+e^-\pi^+\pi^-\pi^0)$ (curve I) and $10^{-3}\times\sigma(e^+e^- \to e^+e^-\pi^+\pi^-)$ (curve II) versus $\sqrt{s}$ in GeV. The latter curve is taken from Ref. 4. (Note that our values of $\sqrt{s}$ are rather low, so we had to extrapolate the curve given in Ref. 4.) The straight line gives $\sqrt{s} = M_K$.

the following. How large a cross section for $\mu^+\mu^- \to 3\pi$ do we need in order to produce any significant interference in the $K_L^0 \to 2\mu$ problem?[6] The answer to this question is available in Eq. (3.9) of the paper of Martin, de Rafael, and Smith.[7] This equation gives the Schwarz inequality for the 3π contribution to the K_L^0 absorptive part, assuming CP invariance in the decay. This contribution, called $\Omega(3\pi)$, is bounded once we know the cross section for $(\mu^+\mu^-)_{^1S_0} \to 3\pi$ and the rate for $K_L^0 \to 3\pi$. As we know the magnitude of the 2γ contribution to the absorptive part we know the *magnitude* of $\Omega(3\pi)$ necessary to produce any significant interference between these two contributions. In the units of Ref. 7, $\Omega(3\pi)$ must be larger than 10^{-13}, so we can use Eq. (3.9) to find the necessary magnitude of the $(\mu^+\mu^-)_{^1S_0} \to \pi^+\pi^-\pi^0$ cross section. We find that this cross section at the squared c.m. energy $s = M_K^2$ must be larger than 0.6×10^{-34} cm². Hence, taking into account the factor of $(\alpha/\pi)^2$, we require the cross section for $2\gamma \to 3\pi$ to be roughly 10^{-29} cm² at this energy. Indeed, a slightly different argument[8] has already yielded an even larger estimate for the required magnitude of this cross section, namely that it must be around 10^{-28} cm². If we now look at Fig. 1 we see that the $2\gamma \to \pi^+\pi^-\pi^0$ cross section at $s = M_K^2$ is 2.0×10^{-34} cm², so we conclude that the $2\gamma \to \pi^+\pi^-\pi^0$ amplitude given in Ref. 1 is far too small to be of any importance in the $K_L^0 \to \mu^+\mu^-$ puzzle. This was to be expected because the model used here is very similar to the models we investigated previously.[8] Similar conclusions have also been made independently by Aviv and Sawyer.[9]

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