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Infinity Suppression in Gravity-Modified Electrodynamics. II

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It is argued that the use of a visibly localizable parametrization of the gravitational interaction yields a number of advantages. First, the question of ambiguities can be completely solved: According to a theorem of Lehmann and Pohlmeyer there exists in such theories a *unique* "minimally singular" solution which it is natural to adopt as the physical one. Second, it is possible to show that this solution satisfies the usual requirements of analyticity and unitarity in the sense of perturbation theory. These points are reviewed in this paper, the main object of which is to introduce a new technique for the treatment of those nonpolynomial Lagrangians in which the interaction terms are intimately associated with the free part and contain derivatives. The gravity-modified theories exemplify this type of Lagrangian: In such theories the zero-graviton approximant to any process is "cradled" in a sequence of graphs with arbitrarily large numbers of gravitons whose sum exists, is finite, and free of ambiguities. Since the problem of preserving electromagnetic (and gravitational) gauges is also the problem of derivatives occurring either in interaction Lagrangians or in the propagators, our general treatment of derivatives is expected to resolve such gauge difficulties. In particular, we show that both the gravity-modified photon renormalization constant and the gravity-modified electromagnetic self-mass of the electron up to order $\alpha \log G_N m^2$ (where G_N is the Newtonian constant) are gauge-invariant.

I. INTRODUCTION

Field-theoretic infinities – first encountered in Lorentz's computation of electron self-mass – have persisted in classical electrodynamics for seventy years and in quantum electrodynamics for some thirty-five years. These long years of frustration have left in the subject a curious affection for the infinities; perhaps even a belief that they will never be circumvented.[1]

As is well known, the infinities result from a lack of proper definition of singular distributions which occur in field theory. One of the major obstacles to progress in the subject has been the uncertainty of whether these singularities have their origin in the circumstance that a perturbation expansion is being made or whether it is the form of the Lagrangian – assumed to be polynomial in field variables – which is at fault. An important suggestive advance in resolving this uncertainty has been the work of Glimm and Jaffe[2] who, working with exact and mathematically well-defined solutions of *polynomial* Lagrangian field theories (in two and three space-time dimensions) have shown that infinities persist even in exact solutions. If their conclusions may be extrapolated to physical four-dimensional space-time, it would seem that the origin of the infinities is not so much in the inadequate mathematics of the perturbation solution. Rather, the fault lies with the inadequate physics of the assumed polynomial character of the electromagnetic interaction.

Now nonpolynomial Lagrangian theories have been studied since 1954 (in fact they date back to the Born-Infeld nonlinear electrodynamics of the 1930's) and it is well known that a variety of these do indeed possess perturbation solutions free of infinities. However, in modifying electrodynamics to a nonpolynomial version one has been presented with two dilemmas:

(1) There are many nonpolynomial ways of "completing" the conventional polynomial version. Which represents physics?

(2) Since the methods developed for solving nonpolynomial theories are radically different from those for polynomial theories – for example they involve analytic continuation procedures in an essential manner – one would wish to be sure that the field-theory solutions thus defined do satisfy the conventional canons of *good* field theories,

like appropriate analyticity, unitarity, positive definiteness, and Froissart boundedness.

In respect of the first problem, i.e., that of discovering the missing (nonpolynomial) physics, which should complete conventional electrodynamics, we revived in a series of earlier papers the conjecture of Landau, Klein, Pauli, Deser, De Witt, and others[3] which suggested that it may be the neglect (of the intrinsic nonpolynomial character) of tensor gravity – and the associated curvature of space-time produced by an electron or a photon in the space surrounding it – which may be the direct cause of the electron's and photon's self-mass and self-charge infinities.

In respect of the second problem, an advance has just recently been made by Lehmann and Pohlmeyer[4] and Taylor[5] who have shown rigorously that the analytic procedures developed in earlier papers by Volkov, Filippov, Salam, Strathdee, and others[6-8] do indeed define *good* field theories, *good* in the perturbational sense, provided the associated nonpolynomial theory falls into the *localizable* class, satisfying the principle of microcausality.

The advance of Lehmann and Pohlmeyer[4] and Taylor[5] is an important one. Of peculiar relevance to our work is their insistence on *localizability, microcausality,* and their consequences. In our earlier papers,[9] following Efimov and Fradkin,[10] we had worked with *nonlocalizable* nonpolynomial theories. This had led to a number of serious shortcomings which were noted in Ref. 9. Although we were able to show by actual computation that, when tensor gravity effects were properly taken into account, the conventional logarithmically infinite expressions $|\alpha \log 0|$ for self-charge and self-mass do become realistically regularized to $|\alpha \log(\kappa^2 m^2)|$ where $16\pi\kappa^2$ is the Newtonian constant G_N, there were still a number of problems the computation left unresolved.

Mathematically:

(1) The results were not (electromagnetic) gauge-invariant.

(2) In obtaining the results, use was made of a Borel summation of a divergent series – a procedure open to ambiguities.

(3) The results were obtained, using a particular choice of the gravitational field variables – viz., the one which treated the contravariant field $g^{\mu\nu}$ as the fundamental field with the covariant field $g_{\mu\nu}$ expressed in terms of it. Since field-theoretic equivalence theorems would seemingly permit either field being treated as basic, the role of such transformations was not clear.

Physically:

It was not clear whether it was true *tensor* gravity which was responsible for the finite computation of the renormalization constants or whether it was some *scalar* version of it.

It is the purpose of this paper to show that these shortcomings of the earlier papers are circumvented, provided we work with a *localizable,* visibly *microcausal* version of Einstein's gravity theory. Notwithstanding this change, it turns out that our numerical results to the order we computed are unaltered.

The plan of the paper is as follows. In Sec. II we discuss localizable theories in general and the localizable parametrization of gravity theory in particular. (Since we shall be dealing in a later section with spin-$\frac{1}{2}$ particles, it is necessary, as is well known, to work with the *vierbein* formalism of the spin-2 gravity field. We wish to emphasize that for a quantum-field theorist, it is a mistake to get too involved in the geometry associated with *vierbein* quantities or indeed even the geometry of the metric tensor. All one needs to know is that the *vierbein* field is related to the "square root" of the metric tensor field. Reference 8 may be consulted for further details.) In Sec. III a number of technical points relating to the mathematics of singular distributions and their space-time derivatives are discussed and we formulate a "law" of conservation of derivatives to give a precise meaning to products of derivatives of singular distributions and to eliminate tadpoles of the second kind from the theory. In the last subsection of Sec. III, a model Lagrangian is considered in order to show the heart of the ideas involved in the detailed calculations of gravity-modified photon self-energy and electron self-mass presented in Secs. IV and V. Appendix A describes a very simple calculation which illustrates the basic ideas behind the quantization of a nonpolynomial Lagrangian field theory. Appendix B contains some remarks on the method of "kinking" and "cradling."

II. LOCALIZABLE GRAVITATIONAL FIELD THEORY

A. Localizable Theories in General

Consider a nonderivative Lagrangian:

$$\mathcal{L}_{\text{int}}(\phi) = \sum_{n=0}^{\infty} \frac{v(n)}{n!} :\phi^n:, \tag{2.1}$$

where the double dots denote normal ordering [i.e., we agree to "renormalize" $D^n(x-x) = \lim_{x\to 0}(-1/x^2)^n$ to the value zero for $n>0$]. According to Jaffe's classification, $\mathcal{L}(\phi)$ defines a *localizable* theory, with operators $\mathcal{L}(\phi)$ satisfying the microcausality relation

$$[\mathcal{L}_{\rm int}(\phi(x)), \mathcal{L}_{\rm int}(\phi(0))] = 0\,, \quad x^2 < 0 \tag{2.2}$$

provided the spectral function $\rho(p^2)$ associated with the two-point function increases for large $\|p^2\|$ no faster than $\exp\|p^2\|^\alpha$ with $\alpha < \frac{1}{2}$. For a growth like $\exp\|p^2\|^{1/2}$ we shall say that the theory is *just localizable*. When $\alpha > \frac{1}{2}$ the theory is *not localizable*. To compute $\rho(p^2)$ for the Heisenberg operator in a Lagrangian theory, one conventionally uses second-order perturbation theory in the major coupling constant. There is no reason to believe that perturbation theory gives the correct high-energy behavior of $\rho(p^2)$. These perturbation estimates, however, typically give for a zero-mass field:

$$\mathcal{L}(\phi) = g\!: e^{\kappa\phi} - 1\!: \quad \textit{localizable}\,, \qquad \rho \simeq \exp\|p^2\|^{1/3} \tag{2.3}$$

$$= g\!: e^{-\kappa^2\phi^2} - 1\!: \quad \textit{just localizable}\,, \qquad \rho \simeq \exp\|p^2\|^{1/2} \tag{2.4}$$

$$= g\!:\frac{1}{1+\kappa\phi} - 1\!: \quad \textit{nonlocalizable}\,, \qquad \rho \simeq \exp\|p^2\|\,. \tag{2.5}$$

In general, with $\mathcal{L}(\phi)$ given by (2.1), the theory is localizable if $|v(n)| < A^n n^{\sigma n}$ with $0 < \sigma < \frac{1}{2}$.

Let us list the reasons for preferring, at this stage of the development of the theory, the class of localizable Lagrangians.

(1) *Elimination of Borel ambiguities*. The superpropagators

$$\mathcal{G}(x) = \langle\, \mathcal{L}(\phi(x))\mathcal{L}(\phi(0))\rangle_+$$

for the *localizable* and *nonlocalizable* theories (2.3) and (2.5) are, respectively,

$$\mathcal{G}_{\rm L}(x) = g^2 \sum_{n=1}^{\infty} \frac{(\kappa^2)^n}{n!}[D(x)]^n\,, \tag{2.6}$$

$$\mathcal{G}_{\rm NL}(x) = g^2 \sum_{n=1}^{\infty} n!\,(\kappa^2)^n [D(x)]^n\,. \tag{2.7}$$

Notice that $\mathcal{G}_{\rm L}(x)$ is an entire function in the $(\kappa^2 D)$ complex plane, while $\mathcal{G}_{\rm NL}$ is a divergent series. Efimov, Fradkin,[10] and we ourselves in our earlier papers[9] worked with rational normally ordered Lagrangians of the nonlocalizable variety and were faced with the problem of defining the sums of divergent series like (2.7). We adopted the Borel summation procedure; this, however, necessarily introduces a source of ambiguity. *By working always with localizable theories we avoid this ambiguity completely*. An n-point superpropagator in a theory with the localizable Lagrangian (2.3) is expressed in the form

$$(g)^n \exp\Big(\kappa^2 \sum_{i<j} D(x_i - x_j)\Big)\,, \tag{2.8}$$

again an entire function in the $D(x_i - x_j)$ plane.

(2) *Distribution-theoretic ambiguities*. Both *localizable* and *nonlocalizable* theories suffer from one further set of ambiguities. These are the *distribution-theoretic* ambiguities met with in the definition of the time-ordered product of field operators. Specifically

$$\langle\, T(\,:\phi^n(x):\,:\phi^n(0):\,)\rangle \tag{2.9}$$

equals $n!\,[D(x)]^n$ with ambiguities up to terms of the type

$$\sum_{r\geq 2}^{n} b_r(\partial^2)^{r-2}\delta(x)\,. \tag{2.10}$$

[There is no ambiguity in the Wightman product $(\,:\phi^n(x):\,:\phi^n(0):\,)$; it is the lack of precise definition of the time-ordered product at $x_\mu = 0$ which introduces this ambiguity in all field theories.]

Now Lehmann and Pohlmeyer[4] show that these particular ambiguities can be turned into a positive virtue so far as certain localizable nonpolynomial field theories are concerned, marking them out as superior not only to nonlocalizable theories but also to the conventional polynomial ones. This is because one can sharply distinguish between terms like (2.6) and (2.7) and the ambiguous terms (2.10) in a distribution-theoretic sense. Their first remark is that localizability implies a restriction on b_n's in (2.10) such that the function $b_n z^n$ is entire of order $\alpha < \frac{1}{2}$. Secondly, in Fourier space one can verify that $\tilde{\mathcal{G}}_L(p)$, for example, in (2.6) falls to zero along some direction in the complex p^2 plane for large $\|p^2\|$. There is, however, no direction along which the ambiguity terms (2.10) can fall. This is guaranteed by the fact that (2.10) must be of order less than $\frac{1}{2}$ for the localizable case. However, no such distinction can be made between $\mathcal{G}_{\rm NL}$ in (2.7) and the corresponding ambiguous terms. Lehmann and Pohlmeyer thus define a class of minimally singular superpropagators which are ambiguity free for localizable theories. This class coincides with the class previously considered by Volkov, Filippov and other authors. Using this, Lehmann and Pohlmeyer show that the theory thus obtained possesses conventional analyticity and unitarity properties to *all* orders in the major coupling constant, g. The same result has been independently established by Taylor.[5] Their proofs can be extended to establish positive definiteness also.

(3) *Froissart boundedness*. Glaser, Martin, and Epstein,[11] in a fundamental paper, have shown rigorously that mass-shell S-matrix elements for two-particle scattering in localizable theories must possess Froissart boundedness at high energies. There is no such result known for non-

localizable theories. (This aspect of the superiority of localizable theories may, however, be illusory. This is because the Volkov-Lehmann minimally singular perturbation expansion in the major constant does not exhibit this behavior if any single term in this expansion is considered. Presumably one must sum chains of supergraphs – as one does for polynomial Lagrangian theories if one wishes to exhibit Regge or eikonal high-energy behavior – a behavior not characteristic of individual graphs. It is conceivable that the same treatment may yield Froissart-bounded high-energy behavior for both localizable and nonlocalizable theories.)

(4) *Equivalence transformations of field variables.* For the purposes of this paper, the most important basis of the superiority of localizable over nonlocalizable theories lies (together with the elimination of Borel and other distribution-theoretic ambiguities) in the circumstance that for these theories we can make field transformations at will. Since localizable theories are microcausal, and microcausality is the basis of Borchers's theory of equivalence classes, we shall take over Borchers's results and assert that those field transformations which transform one localizable theory into another do respect the equivalence theorems regarding the equality of mass-shell S-matrix elements.

In the rest of this paper we shall freely make such field transformations and, as we shall see, this will assist us greatly in the discussion of electromagnetic gauge invariance.

To summarize, localizable theories are superior to nonlocalizable theories for five reasons:

(a) There are no problems of Borel ambiguities for the former.

(b) The remaining distribution-theoretic ambiguities can be eliminated using the Lehmann-Pohlmeyer minimality ansatz which holds only for nonpolynomial localizable theories.

(c) The Glaser-Epstein-Martin theorem assures Froissart boundedness of localizable theories.

(d) We can make field transformations at will and expect that on-shell S-matrix elements will remain unaltered.

(e) The Lehmann-Pohlmeyer and Taylor proof of appropriate unitarity and analyticity is available for localizable theories.

We close this section with two remarks:

(1) Localizability implies only microcausality of the theory. Whether it corresponds to the macrocausal behavior of field theories is an unresolved problem.

(2) As was emphasized in Ref. 12, Sec. D, a rational Lagrangian-like $1/(1+\kappa\phi)$ is nonlocalizable only when normally ordered, i.e., when $D(0) = \lim_{x\to 0} 1/x^2$ is renormalized to the finite value zero. If $D(0)$ is renormalized to a finite value, all rational Lagrangians can be shown to fall into the just-localizable class. In this paper we shall always normal order. This may well be the real source of the paradoxes which arise when one is considering problems of equivalence of Lagrangians under field transformations.

B. Localizable Parametrization of Gravity

In our earlier paper[9] we assumed that the fundamental gravitational field was the contravariant field $g^{\mu\nu}(x)$. In the limit of an asymptotically flat space-time this field splits up in general into the sum of its Minkowskian expectation value

$$\eta^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.11}$$

and a functional of the physical interpolating field. At this stage it is possible to use a wide variety of parametrizations. One such is the "rational" parametrization

$$g^{\mu\nu}(x) = \eta^{\mu\nu} + \kappa\phi^{\mu\nu}(x), \tag{2.12}$$

where $\phi^{\mu\nu}(x)$ is the physical graviton field which possesses in and out states. The covariant field $g_{\mu\nu}(x)$ is then given as the ratio of two polynomials in $\phi^{\mu\nu}$ of degree 3 and 4, respectively:

$$g_{\mu\nu}(x) = \frac{4\epsilon_{\mu\alpha\beta\gamma}\epsilon_{\nu\alpha'\beta'\gamma'}g^{\alpha\alpha'}g^{\beta\beta'}g^{\gamma\gamma'}}{\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\alpha'\beta'\gamma'\delta'}g^{\alpha\alpha'}g^{\beta\beta'}g^{\gamma\gamma'}g^{\delta\delta'}}\,. \tag{2.13}$$

An alternative (and by the mathematicians the more favored) is an exponential parametrization[13,14]

$$g^{\mu\nu} = [\exp\kappa(\phi)]^{\mu\nu}, \tag{2.14}$$

where $\phi^{\alpha\beta} = \phi^{\beta\alpha}$ are the basic interpolating fields. The covariant tensor $g_{\mu\nu}(x)$ is simple and is given by

$$g_{\mu\nu}(x) = \{\exp[-\kappa(\phi)]\}_{\mu\nu}\,. \tag{2.15}$$

Similarly, the *vierbein* gravity field $L^{\mu a}$ can be parametrized as

$$L^{\mu a} = [\exp(\tfrac{1}{2}\kappa\phi)]^{\mu a}\,. \tag{2.16}$$

More generally, instead of the exponential parametrization, one may consider any other entire function parametrization in (2.14). Since in gravity theory one always assumes that $\det g \neq 0$, it is clear from (2.12) that if $g^{\mu\nu}(x)$ is entire, so is $g_{\mu\nu}(x)$.

Throughout this paper we shall, for the sake of simplicity, use a Euclidean rather than Minkowskian metric, transforming back to the correct metric at the appropriate stage. This should cause no confusion. Notice that the Minkowskian form of Eq. (2.14) would be

$$(\eta^{1/2}g\eta^{1/2})^{\mu\nu}=[\exp\kappa(\eta^{1/2}\phi\eta^{1/2})]^{\mu\nu}, \tag{2.17}$$

where

$$\eta^{1/2}=\begin{pmatrix}1&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&i\end{pmatrix}.$$

The exponential can be expanded to give the matrix formula

$$\eta^{1/2}g\eta^{1/2}=1+\kappa\eta^{1/2}\phi\eta^{1/2}+\tfrac{1}{2}\kappa^2\eta^{1/2}\phi\eta\phi\eta^{1/2}+\cdots,$$

i.e.,

$$g=\eta+\kappa\phi+\tfrac{1}{2}\kappa^2\phi\eta\phi+\cdots.$$

In particular one finds

$$\begin{aligned}(-\det g)^{1/2}&=\exp\kappa\,\mathrm{Tr}(\eta\phi)\\&=\exp\kappa(\phi^{\alpha}{}_{\alpha})\end{aligned} \tag{2.18}$$

for the determinant of the Minkowskian tensor.

We shall call the rational parametrizations (2.12) and (2.13) nonlocal, while (2.14) will be referred to as the local parametrization of gravity. To justify this nomenclature, consider the nonderivative parts of gravity-matter interaction and assume that the gravity-gravity interaction can be neglected. One can easily show that the superpropagators of cotensors $\langle g_{\mu\nu}(x_1),\ g_{\alpha\beta}(x_2),\ldots\rangle$ in the rational parametrization (2.13) give rise to nonlocalizable high-energy behavior, while the exponential parametrization (2.15) leads to a behavior characteristic of localizable theories.

When derivative couplings of $g^{\mu\nu}$ are additionally considered (including the nonpolynomial graviton-graviton couplings characteristic of Einstein's theory) this conclusion may alter, though the presumption is that (2.14) is still a localizable theory. This is because an interaction term like $(\partial_\mu\phi)(\partial_\mu\phi)\times\exp(\kappa\phi)$ can be majorized – so far as the high-energy behavior of the superpropagators in momentum space is concerned – by a (localizable) term like $\phi^4\exp(\kappa\phi)$, where each derivative ∂_μ is replaced by a field function ϕ. Such a majorization procedure is, however, likely to be misleading when applied to the nonlocalizable version of the theory (2.12). The reason is that with derivative couplings there is the possibility of enormous numbers of cancellations which may reduce the seemingly nonlocalizable behavior of (2.12) and (2.13) to a less singular localizable one. The majorization which replaces $(\partial_\mu\phi)$ by ϕ^2 is likely to conceal this.

To summarize, in a full derivative-containing gravity theory, we believe that the parametrization (2.14) does give us a localizable theory. What we cannot assert is that the seemingly nonlocalizable theory, represented by the parametrizations (2.12) and (2.13), may not after all also be localizable. In this paper we shall take no chances and will work with the parametrization (2.14), leaving open the question as to whether or not the rational and exponential parametrizations of gravity after quantization represent the same theories in the sense of field-theoretic equivalence theorems. It is important to stress that the parametrization (2.14) is only one of a class of parametrizations which may be classified as localizable. The common characteristic of the elements of this class is that they are represented by entire functions of the variables $\phi^{\mu\nu}$. Borchers's theorem should permit us to make field transformations between members of this class.

III. GRAVITY-MODIFIED ELECTRODYNAMICS

A. The Lagrangian

The gravity-modified Lagrangian for quantum electrodynamics may be written in the form shown below in Eq. (3.3*). [We use the notation of Ref. 9. Equations "parallel" to those from the latter reference will carry an asterisk to distinguish them from the equation numberings of this paper. Equation (3.3*) below is parallel to Eq. (3.3) of Ref. 9 in the sense that when w_e, the weight of the electron field, is set equal to zero in (3.3*) we recover Eq. (3.3) of Ref. 9.] As stated in the introduction, the spinor character of the electron field necessitates the introduction of a *vierbein* version of gravity, the *vierbein* spin-2 field $L^{\mu a}(x)$ being simply related to the metric field $g^{\mu\nu}(x)$ by the relation

$$g^{\mu\nu}(x)=L^{\mu a}(x)L^{\nu b}(x)\delta_{ab}.$$

In terms of the field $L^{\mu a}$, the electrodynamic Lagrangian reads[9]

$$\begin{aligned}L_{\text{total}}=L_{\text{gravity}}&+\frac{1}{(\det L)^{2w_e+1}}[\tfrac{1}{2}iL^{\mu a}(\bar\psi\gamma_a\psi_{;\,\mu}-\bar\psi_{;\,\mu}\gamma_a\psi)-m_0\bar\psi\psi+e_0\bar\psi\gamma_a A_\mu\psi L^{\mu a}]\\&-\frac{1}{4(\det L)}(g^{\mu\nu}g^{\kappa\lambda}F_{\mu\kappa}F_{\nu\lambda})+3i\delta^4(0)\ln|\det L|^{-2w_e+1},\end{aligned} \tag{3.3*}$$

where

$$\psi_{;\,\mu} = \partial_\mu\psi - \tfrac{1}{4}iB_\mu^{ab}\sigma_{ab}\psi + w_e(\det L)_{,\mu}\psi\,,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu;\,\mu} - A_{\mu;\,\nu}\,, \qquad (3.4^*)$$
$$\det L \equiv \det L^{\mu a}\,.$$

The parameter w_e denotes the weight of the electron field. It can be changed at will by making field transformations and, if Borchers's theorem holds, it should not appear in the physical S matrix. Writing

$$L^{\mu a} = [\exp(\tfrac{1}{2}\kappa\phi)]^{\mu a} \qquad (3.1)$$

we recover

$$g^{\mu\nu} = (\exp\kappa\phi)^{\mu\nu}\,. \qquad (2.14)$$

The *vierbein* connection, $B_{\mu ab}$, is the product of L's and their derivatives, its full expression being given in Eq. (2.9) of Ref. 9. It is worth remarking that the factor $\det L$ in (3.3*) (so crucial in Ref. 9 for infinity suppressing) acquires a very simple form in exponential parametrization. In fact

$$\det L = \exp(\tfrac{1}{2}\kappa\,\mathrm{Tr}\,\phi)\,. \qquad (3.2)$$

B. Scalar Gravity

In this section we wish to show that a scalar gravity theory is unlikely to suppress infinities in electrodynamics.

The scalar gravity Lagrangian can be recovered from (3.3*) by substituting

$$w_e = 0\,,$$
$$L^{\mu a} = \exp(\tfrac{1}{2}\kappa\phi)\delta^{\mu a}\,,$$
$$g^{\mu\nu} = \exp(\kappa\phi)\delta^{\mu\nu}\,, \qquad (3.3)$$
$$\det L = \exp(2\kappa\phi)\,.$$

L_{total} reduces to the form

$$L_{\text{total}} = L_{\text{gravity}} + i[\bar\psi\gamma_\mu(\partial_\mu - ie_0A_\mu)\psi]\exp(-\tfrac{3}{2}\kappa\phi) - m_0\bar\psi\psi\exp(-2\kappa\phi) - \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}\,. \qquad (3.4)$$

Note the crucial circumstance that the photon field and the scalar graviton do not couple, a result well known in general relativity theory from the conformal invariance of scalar gravitons and photons.

Let us now make a further field transformation:

$$\psi' = \exp(-\tfrac{3}{4}\kappa\phi)\psi\,. \qquad (3.5)$$

This has the effect of decoupling the electron and the graviton also, except from the mass term for the electron. In the limit $m_0 = 0$, even the electrons do not interact with scalar gravitons.

Now if δm and δe were strict physical mass-shell quantities, *one would unhesitatingly have said that scalar gravity plays no regularizing role for electrodynamics of zero (bare) mass electrons.* One cannot make this negative assertion with confidence for two reasons: First, in the exact theory δm and δe are both expressed as integrals of (off-mass-shell) spectral functions. Although both δm and Z_3 share with the strict mass-shell quantities the property of electromagnetic gauge invariance (unlike Z_2), there are no results known at present which should imperatively guarantee that the mass-shell S-matrix equivalence theorems apply also for the case of these off-mass-shell quantities. Second, it is fully possible that the inclusion of nonzero mass term coupling

$$-m_0\bar\psi'\psi'[\exp(-\tfrac{1}{2}\kappa\phi) - 1]$$

may alter the situation. Thus, even though we have so far failed to demonstrate this, it is conceivable that a technique of summation over the major coupling constant (m_0 in this case) may be devised which regularizes the theory, though the prognosis for this happening does not appear too bright.

C. The Tensor Gravity Lagrangian and Electromagnetic Gauge Invariance

Fortunately, true gravity is tensor and cannot be decoupled. We shall attempt in this section to make field transformations which may assist in the task of preserving gauge invariance. One of the major difficulties we encountered in Ref. 9 was connected with the technical fact that whereas the Heisenberg electromagnetic current from (3.3) equals

$$\frac{e_0\bar\psi\gamma_a\psi L^{\mu a}}{(\det L)^{2w_e+1}}$$

and is conserved using the Heisenberg equations of motion, the conserved quantity in the interaction representation does not, however, coincide with this, being just $e_0\bar\psi\gamma_\mu\psi$. Stated differently, it is difficult to make gauge-independent computations because of the awkward factor $L^{\mu a}(\det L)^{-2w_e-1}$ which multiplies the interaction term $e_0\bar\psi\gamma_a\psi A_\mu$ in (3.3*). This factor can be removed by making a suitable choice of the basic field variables. To this end, we choose to assign the weight $w_e = -\frac{1}{2}$ to the electron and regard the combination $A^a = A_\mu L^{\mu a}$ as the photon field. Notice that this does not decouple the tensor gravity from the electron and, even more significantly, from the photon.

With these choices write the Lagrangian (3.3*) in the form

$$\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 , \tag{3.6}$$

where

$$\mathcal{L}_0 = \tfrac{1}{2} i \delta^{\mu a}(\bar\psi\gamma_a\partial_\mu\psi - \partial_\mu\bar\psi\gamma_a\psi) - m_0\bar\psi\psi - \tfrac{1}{4}\delta^{\mu\nu}\delta^{\kappa\lambda}F_{\mu\kappa}F_{\nu\lambda} , \tag{3.7}$$

$$\mathcal{L}_1 = e_0 : \bar\psi\gamma_a\psi A_a : , \tag{3.8}$$

$$\mathcal{L}_2 = \tfrac{1}{2} i : (L^{\mu a} - \delta^{\mu a})(\bar\psi\gamma_a\partial_\mu\psi - \partial_\mu\bar\psi\gamma_a\psi) : , \tag{3.9}$$

$$\mathcal{L}_3 = -\tfrac{1}{4} : \left(\frac{g^{\mu\kappa}g^{\nu\lambda}}{(\det L)} - \delta^{\mu\kappa}\delta^{\nu\lambda}\right) F_{\mu\nu}F_{\kappa\lambda} : , \tag{3.10}$$

$$\mathcal{L}_4 = -\tfrac{1}{2} : L^{\mu a}\bar\psi\gamma_a(\tfrac{1}{2} i B_\mu^{ab}\sigma_{ab})\psi : . \tag{3.11}$$

$\mathcal{L}_0$ is the conventional free Lagrangian for electrons and photons; $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$ are terms of order κ. It is important to remark that $\mathcal{L}_3$ and $\mathcal{L}_4$ are explicitly gauge invariant, while it is only the sum $\mathcal{L}_1 + \mathcal{L}_2$ which is explicitly so. In Ref. 9 we considered only the $\mathcal{L}_1$ part of the Lagrangian, ignoring $\mathcal{L}_2$. The results could not be expected to be gauge invariant.

First consider this Lagrangian for its infinity suppression role. $\mathcal{L}_4$ in (3.11) is a Lagrangian of the general form

$$\chi^n e^{\kappa\phi} \quad \text{or} \quad (\partial\chi)^2\chi^m e^{\kappa\phi},$$

where χ and ϕ are massless scalar particles. As has been shown elsewhere,[12] the exponential term is highly potent in its infinity-regularizing role. Heuristically one may see this as follows. The two-point superpropagator $\langle \chi^n e^{\kappa\phi}, \chi^n e^{\kappa\phi}\rangle$ is of the form $(1/x^2)^n e^{-\kappa^2/x^2}$. Approaching $x^2 \to 0$ from an appropriate direction in the x space (and using analytic continuation methods for the approach from other directions) the singularity of the superpropagator will be regularized to zero for all n. This will however, not be the case for Lagrangians of the generic variety,

$$\mathcal{L}_{\text{int}} = \chi^n(: e^{\kappa\phi} : -1) , \tag{3.12}$$

or of the type

$$\mathcal{L}_{\text{int}} = (\partial\chi)^2(: e^{\kappa\phi} : -1) . \tag{3.13}$$

The $\mathcal{L}_2$ and $\mathcal{L}_3$ pieces of the electromagnetic Lagrangian belong to this last category. Our major task in showing that gravity-modified electrodynamics does indeed possess an inbuilt regularizing nonpolynomiality lies in analyzing the potentially unregularized singularity produced, for example, by the $-(\partial\chi)^2$ term in $(: e^{\kappa\phi} : -1)(\partial\chi)^2$ and showing that such terms are harmless.

D. The Theory of Kinetic-Energy Kinks

It is a general feature of particle Lagrangians in gravity theory that the kinetic-energy terms of the free Lagrangians are mixed in with graviton-particle interactions. Examples are provided by the electron-graviton and photon-graviton interaction terms $\mathcal{L}_2$ and $\mathcal{L}_3$, which, together with $\mathcal{L}_0$, formed one whole, before the split into $\mathcal{L}_0$ and $\mathcal{L}_{\text{int}}$ was carried out. A generic example is the one discussed in the last subsection. If $\mathcal{L}_{\text{total}}$ equals $(\partial\chi)^2 : e^{\kappa\phi} :$, the split $\mathcal{L}_0 = (\partial\chi)^2$, $\mathcal{L}_{\text{int}} = (: e^{\kappa\phi} : -1)(\partial\chi)^2$ represents the situation presented by the $\mathcal{L}_2$ and the $\mathcal{L}_3$ terms. This nonpolynomial interaction Lagrangian would, acting by itself, give finite matrix elements were it not for the possible infinities which its "kinking" part $-(\partial\chi)^2$ might produce. The "kinking" terms in $\mathcal{L}_{\text{int}}$ are so called because inside any χ line the operation of $(\partial\chi)^2$ acts simply as minus the unit operator. In momentum space, for example, the χ propagator $1/p^2$ may be written in the form

$$\frac{1}{p^2} \equiv \frac{1}{p^2}p^2\frac{1}{p^2} \equiv \frac{1}{p^2}p^2\frac{1}{p^2}p^2\frac{1}{p^2} \equiv \cdots \tag{3.14}$$

corresponding to the successively kinked lines shown graphically in Fig. 1.

FIG. 1. Insertion of unit kinks into a massless free-boson propagator.

In what follows we shall use the words "single kinking" to refer to the act of inserting such a unit operator (which corresponds physically to minus the emission of one zero-graviton) into a free propagator. "Double kinking" will correspond to the amplitude of two zero-graviton emission, "triple kinking" to minus the amplitude for three zero-graviton emission and so on.

To illustrate the manner in which kinking is used consider the computation of the propagator $\langle T(\chi(x)\chi(y)\exp(i\int\mathcal{L}_{\text{int}}))\rangle_0$, where $\mathcal{L}_{\text{int}}$ is given by Eq. (3.13). This is shown graphically in Fig. 2, up to the second order in $\mathcal{L}_{\text{int}}$. The first graph is the free χ propagator, the second represents the one-graviton modification due to $\mathcal{L}_{\text{int}}$, the third the two-graviton modification, and so on. Our basic contention is that the first graph should be regarded as part of the series formed by the rest by inserting two kinks, at the space-time points x_1 and x_2, and using the graphical identity of Fig. 3. The sum of all the graphs in Fig. 2 may be written in terms of the one supergraph shown in Fig. 4 in which the effective Lagrangian operating at

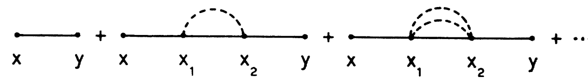

FIG. 2. Series of gravitational self-energy graphs.

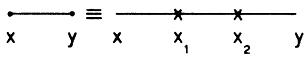

FIG. 3. Insertion of two unit kinks.

x_1 and x_2 is $(\partial\chi)^2 e^{\kappa\phi}$ which will produce finite answers. We shall refer to this absorption of the free propagator into a superpropagator as "cradling." Such a procedure is always possible in any theory in which $\mathcal{L}_{\text{int}}$ has the generic form $[v(\varphi)-1](\partial\chi)^2$ since there will always exist in the theory graphs which (by suitable kinking) can be cradled as part of a nonpolynomial chain representing zero ϕ-particle exchange. The effective interaction in such situations is therefore $:v(\varphi):$ and not $[:v(\phi):-1]$.

Similarly, if we were considering terms up to third order in $\mathcal{L}_2'$ in $\langle T\chi\chi\exp(i\int\mathcal{L}_{\text{int}})\rangle_0$, the graphs (i) and (ii) shown in Fig. 5 could be kinked as shown in Fig. 6 and then cradled (when proper account is taken of the sequence of plus and minus signs corresponding to even and odd numbers of kinks) as the sum of the supergraph shown in Fig. 7 plus four times the supergraph shown in Fig. 4 minus twice the free χ propagator which corresponds to $\mathcal{L}_{\text{eff}}=e^{\kappa\phi}(\partial\chi)^2$. (For the general kinking and cradling formula see Appendix B.) It is important to note that "kinking" and "cradling" are possible only when free Lagrangians are hewn out from a total Lagrangian which is finite [i.e., $(\partial\chi)^2$ separated out from $e^{\kappa\phi}(\partial\chi)^2$]. This is of course always the case for gravity theory where $\mathcal{L}_0$ for matter fields is obtained from $\mathcal{L}_{\text{matter}}$ by replacing $L^{\mu a}$ by $\delta^{\mu a}$.

To consider a really complicated "kinking" situation take the interaction Lagrangian

$$\mathcal{L}_{\text{int}}=:\bar\psi\psi A:+(\partial_\mu A)^2(:e^{\kappa\phi}:-1)\,. \tag{3.15}$$

This is a prototype of gravity-modified electrodynamics with ψ the (zero-mass) electron field and A and ϕ the (scalar) photon and graviton fields. The photon self-energy graphs are shown in Fig. 8.

Clearly the first graph (with no gravitons) *identically equals* $(-1)^2$ times the kinked graph of Fig. 9 [with $(\partial A)^2$ operating at the two kink points] and, as such, forms part of the graviton-exchanged chain representing zero-graviton exchange. With the inclusion of this graph, $\mathcal{L}_{\text{int}}$ behaves as if the effective photon-graviton Lagrangian for this particular situation is the (manifestly regularized)

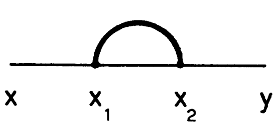

FIG. 4. Supergraph sum of graphs in Fig. 2.

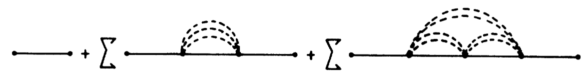

FIG. 5. Higher-order corrections to a self-energy graph.

Lagrangian $(\partial_\mu A)^2:e^{\kappa\phi}:$ rather than $(\partial_\mu A)^2(:e^{\kappa\phi}:-1)$. Practical applications of kinking and cradling will be found in the photon and electron self-energy calculations in Sec. IV.

E. Kinking, Cradling, and the Calculus of Derivatives

Analytically, the graphs of Fig. 2 or, equivalently, of Fig. 4 correspond to the expression

$$\int dx_1 dx_2 \frac{\partial D(x-x_1)}{\partial x_\mu} F_{\mu\nu}(x_1-x_2)\frac{\partial D(x_2-y)}{\partial y_\nu}\,, \tag{3.16}$$

where $F_{\mu\nu}$ is given by

$$F_{\mu\nu}(x)=e^{\kappa^2 D(x)}\partial_\mu\partial_\nu D(x)$$
$$=\sum_0^\infty \frac{1}{n!}\kappa^{2n}[D(x)]^n\partial_\mu\partial_\nu D(x)\,. \tag{3.17}$$

The zero-mass causal propagator $D(x)$ is given by $(-4\pi^2x^2)^{-1}$.

The problem is to define the Fourier transform of (3.17). This could be done by the method of Lehmann and Pohlmeyer[4] or by the following, less rigorous, method. Consider the integral

$$F_{\mu\nu}(x,\lambda)=\frac{1}{2\pi i}\int_C dz\,\Gamma(-z)(-\lambda)^z[\kappa^2 D(x)]^z\partial_\mu\partial_\nu D(x)\,, \tag{3.18}$$

where the contour C comes from positive infinity, encircles the origin in the clockwise sense and returns to infinity. This integral evidently reproduces the sum (3.17) if $\lambda=1$. On the other hand, if $|\arg(-\lambda)|<\frac{1}{2}\pi$ then it is possible to replace the contour C by one running parallel to the imaginary axis with $\text{Re}\,z<0$. Disregarding for the moment the problems caused by the derivatives in (3.18), one could follow the Gel'fand-Shilov prescription for obtaining the Fourier transform of $D^z\partial_\mu\partial_\mu D$ since it is now possible to arrange the contour such that $0<\text{Re}\,(z+2)<2$, a necessary condition for the convergence of the Fourier integral. It

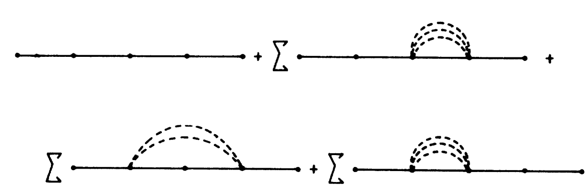

FIG. 6. Insertion of kinks into Fig. 5.

FIG. 7. Supergraph resulting from the kinking procedure of Fig. 6.

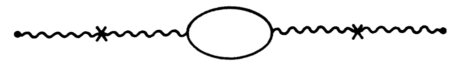

FIG. 9. Insertion of main kinks into Fig. 8.

must be emphasized that if the kinked graph of Fig. 3 had not been included in the sum then the contour would have been confined to the strip $0<\mathrm{Re}z<1$ and the Gel'fand-Shilov requirement could not have been met – signalling the presence of an unregularized infinity.

The derivative problem is dealt with in the following way. First, combine the factors D^z and $\partial_\mu\partial_\nu D^{z_1}$ into the form

$$D(x)^z\partial_\mu\partial_\nu D^{z_1}=\frac{z_1}{(z+z_1)(z+z_1+1)}\times\left[(1+z_1)\partial_\mu\partial_\nu-\frac{z}{2(z+z_1-1)}\delta_{\mu\nu}\partial^2\right]D^{z+z_1}, \tag{3.19}$$

which in the case $z_1=1$ becomes

$$D(x)^z\partial_\mu\partial_\nu D(x)=\frac{2}{(z+1)(z+2)}\times(\partial_\mu\partial_\nu-\tfrac{1}{4}\delta_{\mu\nu}\partial^2)D^{z+1}, \tag{3.20}$$

which is an identity except in the neighborhood of $x_\mu=0$, where it becomes ambiguous. We shall adopt this formula as a definition for all x_μ except in the neighborhood of $z=0$ where it needs to be elaborated. It is clear that (3.20) cannot be a satisfactory definition at $z=0$ since the left-hand side assumes the well-defined form, $\partial_\mu\partial_\nu D(x)$, while the right-hand side assumes the equally well-defined form, $\partial_\mu\partial_\nu D(x)+(i/4)\delta(x)$, which is different.

To meet this difficulty and also to render the formula useful for computing the Fourier transform of integrals like

$$\int_{\mathrm{Re}z<0}dz f(z)[D(x)]^z\partial_\mu\partial_\nu D(x),$$

where $f(z)$ has a pole of order $r\geq 1$ at $z=0$, we shall adopt the definition

$$[D(x)]^z\partial_\mu\partial_\nu D(x)=\lim_{\epsilon\to 0}\frac{2}{(z+1)(z+2)}\times\left[\partial_\mu\partial_\nu-\tfrac{1}{4}\delta_{\mu\nu}\left(\frac{z}{z+\epsilon}\right)^N\partial^2\right][D(x)]^{z+1}, \tag{3.21}$$

where $N\geq r$ is an integer and ϵ is a positive number. It is to be understood that the singularity at $z=-\epsilon$ lies to the left of the z contour and that the limit $\epsilon\to 0$ is therefore to be taken after evaluating the Fourier transform and after translating the contour to the right of $z=0$. In this way one obtains a definition which is consistent at $z=0$ where (3.20) failed. For other values of z it coincides with (3.20).

Another feature of (3.21) may be noted. Contracting the indices μ, ν, one finds

$$[D(x)]^z\partial^2 D(x)=0 \quad\text{for}\quad z\neq 0.$$

This has the important consequence that all those tadpole-like graphs in the theory which arise from a consonance of terms like $D(x)^z\partial^2 D(x)=D(0)^z\delta(x)$ and which cannot be removed by the normal-ordering procedures, automatically vanish. Thus, in effect $D(0)=0$ everywhere.

Using (3.21) and taking the Fourier transform of (3.18), one obtains

$$\tilde{F}_{\mu\nu}(p,\lambda)=-i\frac{p_\mu p_\nu}{p^2}+O(\kappa^2).$$

The higher-order terms will depend on the auxiliary parameter λ. By taking an average of the limits $\lambda\to -e^{i\pi}$ and $\lambda\to -e^{-i\pi}$ one obtains the minimally singular solution of Lehmann and Pohlmeyer.

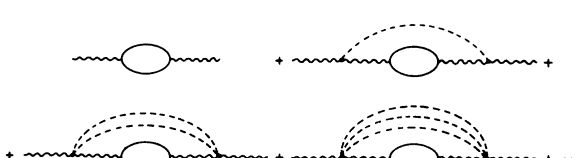

FIG. 8. Prototype of photon self-energy graphs.

IV. GRAVITATIONAL SELF-ENERGY OF THE PHOTON

The relevant graph is shown in Fig. 10, where the thick line represents the graviton superpropagator. The vertices come from the derivative interaction of the gravitational field with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ in the Lagrangian [cf. Eq. (3.10)]

$$\mathcal{L}_3=-\frac{1}{4}\left(\frac{1}{(-\det g)^{1/2}}g^{\mu\nu}g^{\kappa\lambda}-\delta^{\mu\nu}\delta^{\kappa\lambda}\right)F_{\mu\kappa}F_{\nu\lambda}. \tag{4.1}$$

This interaction is present in a tensor gravity theory and is the field-theoretic manifestation of the

FIG. 10. Gravitational self-energy of the photon.

way in which the light cone is directly affected by the gravitational field. The tensor structure of (4.1) is combinatorially very complicated. Since we wish to illustrate the principles of the calculation we approximate (4.1) by a model Lagrangian

$$-\frac{1}{4}\left(\frac{1}{(\det g)^{1/2}}-1\right)F_{\mu\kappa}F_{\mu\kappa} \tag{4.2}$$

[i.e., we replace $g^{\mu\nu}$ in (4.1) by $\delta^{\mu\nu}$]. We expect that the numerical answers obtained in the model (4.2) differ only very slightly from full tensor theory. The calculation of the superpropagator

$$\langle T((\det g^{\cdot\cdot})^{-1/2}(\det g^{\cdot\cdot})^{-1/2})\rangle \tag{4.3}$$

is particularly simple when the exponential parametrization is used. Indeed if $g^{\mu\nu}=(e^{\kappa\phi})^{\mu\nu}$ then

$$\langle T((\det g^{\cdot\cdot})^{-1/2}(\det g^{\cdot\cdot})^{-1/2})\rangle \equiv \langle T(e^{-(\kappa/2)\mathrm{Tr}(\phi)}e^{-(\kappa/2)\mathrm{Tr}(\phi)})\rangle . \tag{4.4}$$

In the DeDonder gauge which is characterized by the condition

$$[g^{\mu\nu}(\det g^{\cdot\cdot})^{1/2}]_{,\nu}=0 , \tag{4.5}$$

the Lagrangian for the tensor field $g^{\mu\nu}$ can be chosen so as to yield the bare propagator (Ref. 9):

$$\langle T\phi^{\mu\nu}, \phi^{\alpha\beta}\rangle = \tfrac{1}{2}(\delta^{\mu\alpha}\delta^{\nu\beta}+\delta^{\mu\beta}\delta^{\nu\alpha}-\delta^{\mu\nu}\delta^{\alpha\beta})D(x) . \tag{4.6}$$

Now it can be shown[15] that if $X^{\alpha\beta}$ and $Y^{\alpha\beta}$ are any 4×4 symmetric matrices, then

$$\langle T(e^{-\phi_{\alpha\beta}X^{\alpha\beta}}e^{-\phi_{\gamma\delta}Y^{\gamma\delta}})\rangle = \exp\{[\mathrm{Tr}(XY)-\tfrac{1}{2}\mathrm{Tr}X\,\mathrm{Tr}Y]D(x)\} .$$

In particular, putting $X^{\alpha\beta}=Y^{\alpha\beta}=(-\kappa/2)\delta^{\alpha\beta}$, we obtain

$$\langle T(e^{-(\kappa/2)\mathrm{Tr}(\phi)}e^{-(\kappa/2)\mathrm{Tr}(\phi)})\rangle = e^{-\kappa^2 D(x)} \tag{4.7}$$

for the superpropagator in Eq. (4.2). [Note that the sign of κ^2 in (4.7) is opposite to that one normally obtains for a superpropagator of a scalar free field. This simply signals the fact that the approximation $\langle T((\eta^{\mu a}/\det L)(\eta^{\nu b}/\det L))\rangle$ to the superpropagator $\langle T((L^{\mu a}/\det L)(L^{\nu b}/\det L))\rangle$ and the neglect of closed loops of Feynman-Faddeev-De-Witt fictitious particles is too much of an oversimplification if one is computing to an order higher than the lowest. No error is introduced so far as the present paper is concerned, but if ever higher orders are computed more care will be needed.]

The Green's function of interest, corresponding to Fig. 10, is

$$D'_{\mu\nu}(x)=\left\langle 0\left| T\left(A_\mu(x)A_\nu(0)\exp\left[-i\int \tfrac{1}{4}F_{\mu\nu}(y)F_{\mu\nu}(y)[v(\phi(y))-1]d^4y\right]\right)\right| 0\right\rangle_{\mathrm{con}}$$

$$=D_{\mu\nu}(x)-\tfrac{1}{2}\int d\xi d\eta\langle 0|T(A_\mu(x)A_\nu(0)[\tfrac{1}{4}F_{\alpha\beta}(\xi)F_{\alpha\beta}(\xi)][\tfrac{1}{4}F_{\kappa\lambda}(\eta)F_{\kappa\lambda}(\eta)])|0\rangle\,\mathcal{G}(\xi-\eta)+\text{higher orders}, \tag{4.8}$$

where $D_{\mu\nu}(x)$ is the free-photon propagator and $\mathcal{G}(\xi-\eta)$ denotes the superpropagator

$$\mathcal{G}(\xi-\eta)=\langle 0|T([v(\phi(\xi))-1][v(\phi(\eta))-1])|0\rangle , \tag{4.9}$$

where

$$v(\phi)=:(\det g)^{-1/2}:-1=\exp[(-\kappa/2)\varphi] .$$

Graphically we are computing Fig. 11 in which, because of the subtraction of "one" in Eqs. (4.7) and (4.9), the lowest term in the superpropagator corresponds to the propagation of one graviton. The fundamental "kink" technique discussed in Sec. III is that the free photon should be included as part of the infinite series defining the superpropagator by regarding it as being equal to itself plus two kinks. Thus Fig. 11 becomes modified to Fig. 12 in which the crosses denote the kink vertices arising from an effective derivative interaction $\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$. The kinked graph is clearly the first term of the series obtained by computing the superpropagator using $v(\phi)$ rather than $v(\phi)-1$. In this approach the required Green's function therefore becomes

$$D'_{\mu\nu}(x)=-\tfrac{1}{2}\int d\xi d\eta\langle 0|T(A_\mu(x)A_\nu(0)[\tfrac{1}{4}F_{\alpha\beta}(\xi)F_{\alpha\beta}(\xi)]\times[\tfrac{1}{4}F_{\kappa\lambda}(\eta)F_{\kappa\lambda}(\eta)])|0\rangle\,\mathcal{G}'(\xi-\eta), \tag{4.10}$$

where

$$\mathcal{G}'(\xi-\eta)=\langle 0|T(v(\phi(\xi))v(\phi(\eta)))|0\rangle . \tag{4.11}$$

Notice that the kink procedure only works if the photon is originally in the Landau gauge with the transverse propagator

$$D_{\mu\nu}(x)=\left(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)D(x) , \tag{4.12}$$

where $D(x)$ is the usual zero-mass boson propagator. If any other gauge is used, such as

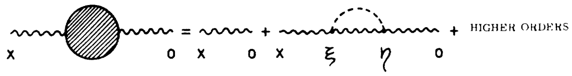

FIG. 11. Gravitationally modified photon Green's function.

$$D_{\mu\nu}(x)=\left(\delta_{\mu\nu}-\frac{\lambda\partial_\mu\partial_\nu}{\Box}\right)D(x)\,, \qquad (4.13)$$

then the transverse and longitudinal parts must be separated out as

$$D_{\mu\nu}(x)=\left(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)D(x)+(1-\lambda)\frac{\partial_\mu\partial_\nu}{\Box} \qquad (4.14)$$

and the transverse part treated in the above way. The longitudinal part is then added on separately to the resulting Green's function.

We are interested in the Fourier transform of the Green's function in Eq. (4.10) defined as

$$D'_{\mu\nu}(p^2)=\frac{1}{i}\int e^{ip\cdot x}D'_{\mu\nu}(x)d^4x\,. \qquad (4.15)$$

As is standard in the nonpolynomial procedures, this integral is first evaluated in the Symanzik region of the external momenta (in the present case this corresponds simply to $p^2<0$) and the result thus obtained is analytically continued (i.e., to timelike $p^2\geqslant 0$). The Symanzik procedure enables Wick rotations to be performed on the space-time integrals, converting them into Euclidean form. Assuming this has been done, we use the expressions

$$\langle 0|T(A_\mu(x)F_{\alpha\beta}(\xi))|0\rangle=(\delta_{\mu\alpha}\partial_\beta-\delta_{\mu\beta}\partial_\alpha)D(x-\xi) \qquad (4.16)$$

and

$$\langle 0|T(F_{\alpha\beta}(\xi)F_{\kappa\lambda}(\eta))|0\rangle =(\delta_{\alpha\kappa}\partial_\beta\partial_\lambda-\delta_{\beta\kappa}\partial_\alpha\partial_\lambda+\delta_{\beta\lambda}\partial_\alpha\partial_\kappa-\delta_{\alpha\lambda}\partial_\beta\partial_\kappa)D(\xi-\eta) \qquad (4.17)$$

in Eq. (4.10). After performing two integrations by parts, we arrive at

$$D'_{\mu\nu}(p^2)=-\frac{1}{i}\int e^{ip\cdot x}d^4x\int d\xi d\eta\, D(x-\xi)D(\eta) \times\partial_\beta\partial_\lambda\{\mathcal{G}'(\xi-\eta)(\delta_{\mu\nu}\partial_\beta\partial_\lambda-\delta_{\beta\nu}\partial_\mu\partial_\lambda+\delta_{\beta\lambda}\partial_\mu\partial_\nu-\delta_{\mu\lambda}\partial_\beta\partial_\nu)D(\xi-\eta)\} \qquad (4.18)$$

in which all the derivatives are with respect to ξ.

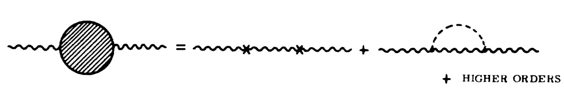

FIG. 12. Insertion of kinks in Fig. 11.

The superpropagator can be written in the usual way[8] as a contour integral:

$$\mathcal{G}'(r)=\frac{1}{2\pi i}\int_\Gamma dz\,\Gamma(-z)[v(z)]^2[D(r)]^z(-\lambda)^z \qquad (4.19)$$

with

$$D(r)=1/4\pi^2r^2\,,$$
$$r^2=(\xi_4-\eta_4)^2+(\vec{\xi}-\vec{\eta})\cdot(\vec{\xi}-\vec{\eta})\,.$$

The contour Γ passes through the real axis to the left of $z=0$ and, as shown in Ref. 8, $v(z)$ is the Laplace transform of the interaction potential $v(\phi)$.

It is useful to define $D'_{\mu\nu}(x,z)$ by

$$D'_{\mu\nu}(x)=\frac{1}{2\pi i}\int_\Gamma dz\,\Gamma(-z)[v(z)]^2D'_{\mu\nu}(x,z)(-\lambda)^z \qquad (4.20)$$

with, from (4.18),

$$D'_{\mu\nu}(x,z)=-\int d\xi d\eta\, D(x-\xi)D(\eta) \times\partial_\beta\partial_\lambda\{D^z(\xi-\eta)(\delta_{\mu\nu}\partial_\beta\partial_\lambda-\delta_{\beta\nu}\partial_\mu\partial_\lambda+\delta_{\beta\lambda}\partial_\mu\partial_\nu-\delta_{\mu\lambda}\partial_\beta\partial_\nu)D(\xi-\eta)\}\,, \qquad (4.21)$$

and similarly

$$D'_{\mu\nu}(p^2,z)=\frac{1}{i}\int d^4x\, e^{ip\cdot x}D'_{\mu\nu}(x,z)\,. \qquad (4.22)$$

To evaluate the integrals in Eq. (4.21) use must be made of the derivative formula discussed in Sec. III. Thus we write [cf. Eq. (3.19)]

$$D^z\partial_\beta\partial_\lambda D=\frac{2}{(z+1)(z+2)}[\partial_\beta\partial_\lambda-\tfrac{1}{4}\delta_{\beta\lambda}\chi^N(z,\epsilon)\partial^2]D^{z+1}\,, \qquad (4.23)$$

where $\chi(z,\epsilon)=z/(z+\epsilon)$ and the limit $\epsilon\to 0$ is to be taken only at the very end of the calculation.

Inserting Eq. (4.23) in Eq. (4.21) and using the relation

$$\Box D^z=-16\pi^2z(z-1)D^{z+1} \qquad (4.24)$$

leads to

$$D'_{\mu\nu}(x,z)=\frac{16\pi^2z}{2+z}[2-\chi^N(z,\epsilon)]\int d\xi d\eta\, D(x-\xi) \times D(\eta)(\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu)D^{z+2}(\xi-\eta)\,. \qquad (4.25)$$

To perform the Fourier transform in Eq. (4.22) we take as usual the basic Gel'fand-Shilov expression

$$\frac{1}{i}\int d^4x\, e^{ip\cdot x}D^z(x)=(4\pi)^{2-2z}(-p^2)^{z-2}\frac{\Gamma(2-z)}{\Gamma(z)}\,, \qquad (4.26)$$

which when used in Eq. (4.25) leads finally to the result

$$D'_{\mu\nu}(p^2) = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_\Gamma dz\,\Gamma(-z)[v(z)]^2\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)(-p^2)^{z-1}(4\pi)^{-2z}\frac{\Gamma(1-z)}{\Gamma(3+z)}[2-\chi^N(z,\epsilon)](-\lambda)^z \tag{4.27}$$

in which the limit $\epsilon\to 0$ is taken at the end, in accordance with the discussion of the derivative formula in Sec. III. As remarked earlier, if any photon gauge other than Landau's is used then the longitudinal part must be subtracted out and added in by hand to (4.27). Particularly, if

$$D_{\mu\nu}(x) = \left(\delta_{\mu\nu} - a\frac{\partial_\mu\partial_\nu}{\Box}\right)D(x),$$

then

$$D'_{\mu\nu}(p^2) = (1-a)\frac{p_\mu p_\nu}{(p^2)^2} + \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_\Gamma dz\,\Gamma(-z)[v(z)]^2\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)(-p^2)^{z-1}(-\lambda)^z(4\pi)^{-2z}\frac{\Gamma(1-z)}{\Gamma(3+z)}[2-\chi^N(z,\epsilon)]\,. \tag{4.28}$$

Only single and double poles occur along the positive real z axis in this expression and, in particular, there is just a single pole at $z=0$, which means that we can choose $N=1$. The actual form of $D'_{\mu\nu}(p^2)$ may easily be computed once the explicit form of $v(z)$ is known. In general, $v(z)$ can be written as $\kappa^z\omega(z)$ and it is always normalized so that $v(0)=1$. The zeroth order in κ is therefore the same for all choices of $v(z)$ (since it comes from the single pole at $z=0$) and we have

$$D'_{\mu\nu}(p^2) = (1-a)\frac{p_\mu p_\nu}{(p^2)^2} + \left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2} + O(\kappa^2) = \left(\delta_{\mu\nu} - a\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2} + O(\kappa^2)\,.$$

Note that kinking, cradling, and the derivative formula have correctly reproduced the free propagator.

V. THE ELECTRON SELF-ENERGY

The next process to be discussed is the electron's electromagnetic self-energy. The order-e^2 contribution to this is shown graphically in Fig. 13 which is simply the convolution of the free electron propagator with the gravitationally modified photon propagator computed in Sec. IV. The crucial result here is that the mass shift δm is finite, gauge invariant, and numerically the same as the analogous quantity computed in Ref. 9.

Suppose initially that the photon is in the Landau gauge. Then the electron self-energy insertion is (with $p^2<0$)

$$\Sigma(p^2) = e^2\int d\!\!{}^-{}^4k\,\gamma_\mu\frac{D'_{\mu\nu}(k^2)}{(\not p - \not k - m)}\gamma_\nu$$
$$= \lim_{\epsilon\to 0}\int_\Gamma dz\,\Gamma(-z)[v(z)]^2\Sigma(p^2,z,\epsilon)(-\lambda)^z\,, \tag{5.1}$$

where $\Sigma(p^2,z,\epsilon)$ is defined from Eq. (4.28) (with $a=1$) as

$$\Sigma(p^2,z,\epsilon) = e^2F(z,\epsilon)\int d\!\!{}^-{}^4k\,\gamma_\mu\frac{(\not p-\not k+m)(\delta_{\mu\nu}-k_\mu k_\nu/k^2)}{(p-k)^2-m^2}\gamma_\nu(-k^2)^{z-1}\,, \tag{5.2}$$

with

$$F(z,\epsilon) \equiv \frac{\Gamma(1-z)}{\Gamma(3+z)}(4\pi)^{-2z}[2-\chi^N(z,\epsilon)](-\lambda)^z \tag{5.3}$$

and $d\!\!{}^-{}^4k$ stands for $(2\pi)^{-4}d^4k$. Now

$$\gamma_\mu(\not p-\not k+m)\gamma_\mu = 4m - 2\not p + 2\not k\,,$$
$$\not k(\not p-\not k+m)\not k = \not k(2p\cdot k - k^2) - k^2(\not p - m)\,,$$

giving

$$\Sigma(p^2,z,\epsilon) = e^2F(z,\epsilon)\int d\!\!{}^-{}^4k\frac{3m-\not p+3\not k-\not k(2p\cdot k)/k^2}{(p-k)^2-m^2}(-k^2)^{z-1}$$
$$\equiv \not p A(p^2,z,\epsilon) + mB(p^2,z,\epsilon) \tag{5.4}$$

in which $A(p^2,z,\epsilon)$ and $B(p^2,z,\epsilon)$ are defined, respectively, as

$$A(p^2, z, \epsilon) = (1/4p^2)\,\mathrm{Tr}[\not{p}\Sigma(p^2, z, \epsilon)], \tag{5.5}$$

$$B(p^2, z, \epsilon) = (1/4m)\,\mathrm{Tr}\Sigma(p^2, z, \epsilon). \tag{5.6}$$

For convenience let us define the integral

$$I(p^2, m^2, z) = \int d\!\!\bar{}^4k \frac{(-k^2)^{z-1}}{(p-k)^2 - m^2}, \tag{5.7}$$

which occurs frequently in what follows. In Feynman's α-parameter form we have

$$I(p^2, m^2, z) = \frac{-1}{\Gamma(1-z)} \int d\!\!\bar{}^4k \int_0^\infty d\alpha_1 d\alpha_2 \exp\{\alpha_1[(p-k)^2 - m^2] + \alpha_2 k^2\}\alpha_2^{-z}. \tag{5.8}$$

Now diagonalize the quadratic form in the exponential by defining

$$l_\mu = k_\mu - \frac{p_\mu \alpha_1}{\alpha_1 + \alpha_2} = k_\mu - p_\mu + \frac{p_\mu \alpha_2}{\alpha_1 + \alpha_2} \tag{5.9}$$

so that (remember $p^2 < 0$)

$$I(p^2, m^2, z) = \frac{-1}{\Gamma(1-z)} \int d\!\!\bar{}^4l \int_0^\infty d\alpha_1 d\alpha_2 \exp\left(l^2(\alpha_1 + \alpha_2) + p^2 \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2} - m^2\alpha_1 \right) \alpha_2^{-z}.$$

On using

$$\int d\!\!\bar{}^4l \exp[l^2(\alpha_1 + \alpha_2)](1, l_\mu, l_\mu l_\nu) = \frac{1}{16\pi^2(\alpha_1 + \alpha_2)^2}\left(1, 0, -\frac{\delta_{\mu\nu}}{(\alpha_1 + \alpha_2)} \right), \tag{5.10}$$

this becomes

$$I(p^2, m^2, z) = \frac{-1}{16\pi^2\Gamma(1-z)} \int_0^\infty d\alpha_1 d\alpha_2 \frac{\alpha_2^{-z}}{(\alpha_1 + \alpha_2)^2} \exp\left(p^2 \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2} - m^2\alpha_1 \right). \tag{5.11}$$

Introduce the change of variables $\alpha_1 = rt$ and $\alpha_2 = r(1-t)$ with the Jacobian

$$\frac{\partial(\alpha_1, \alpha_2)}{\partial(r, t)} \equiv \begin{vmatrix} t & r \\ 1-t & -r \end{vmatrix} = -r;$$

therefore

$$\begin{aligned} I(p^2, m^2, z) &= \frac{1}{16\pi^2\Gamma(1-z)} \int_0^\infty dr \int_0^1 dt\, r^{-1-z}(1-t)^{-z} \exp[p^2 rt(1-t) - m^2 rt] \\ &= \frac{\Gamma(-z)}{16\pi^2\Gamma(1-z)} \int_0^1 dt (1-t)^{-z} t^z [(m^2 - p^2) + p^2 t]^z \\ &= \frac{\Gamma(-z)}{16\pi^2\Gamma(1-z)} (m^2 - p^2)^z \frac{\Gamma(1+z)\Gamma(1-z)}{\Gamma(2)} F\left(-z, 1+z; 2; \frac{-p^2}{m^2 - p^2} \right) \\ &= \frac{\Gamma(-z)\Gamma(1+z)}{16\pi^2} m^{2z} F\left(1-z, -z; 2; \frac{p^2}{m^2} \right). \end{aligned} \tag{5.12}$$

From Eqs. (5.4) and (5.7) it follows that

$$\begin{aligned} B(p^2, z, \epsilon) &= 3e^2 F(z, \epsilon) I(p^2, m^2, z) \\ &= \frac{3\alpha}{4\pi} \frac{\Gamma(1+z)\Gamma(-z)\Gamma(1-z)}{\Gamma(3+z)} \left(\frac{m}{4\pi} \right)^{2z} [2 - \chi^N(z, \epsilon)](-\lambda)^z F\left(1-z, -z; 2; \frac{p^2}{m^2} \right). \end{aligned} \tag{5.13}$$

The computation of $A(p^2, z, \epsilon)$ is slightly harder. From Eqs. (5.4) and (5.6) we obtain

$$A(p^2, z, \epsilon) = \frac{e^2}{p^2} F(z, \epsilon) \int d\!\!\bar{}^4k \frac{-p^2 + 3p\cdot k - 2(p\cdot k)^2/k^2}{(p-k)^2 - m^2} (-k^2)^{z-1}, \tag{5.14}$$

which in α-parameter form is

$$A(p^2,z,\epsilon)=-\frac{e^2F(z,\epsilon)}{p^2}\int d^4k\int_0^\infty d\alpha_1 d\alpha_2\exp\{\alpha_1[(p-k)^2-m^2]+\alpha_2k^2\}\left(\frac{(-p^2+3p\cdot k)\alpha_2{}^{-z}}{\Gamma(1-z)}+\frac{2(p\cdot k)^2\alpha_2{}^{1-z}}{\Gamma(2-z)}\right) \tag{5.15}$$

$$=-\frac{e^2F(z,\epsilon)}{16\pi^2}\int_0^\infty d\alpha_1 d\alpha_2\frac{\exp[p^2\alpha_1\alpha_2/(\alpha_1+\alpha_2)-m^2\alpha_1]}{(\alpha_1+\alpha_2)^2}$$
$$\times\left[\left(-1+\frac{3\alpha_1}{\alpha_1+\alpha_2}\right)\frac{\alpha_2{}^{-z}}{\Gamma(1-z)}+\left(-\frac{1}{\alpha_1+\alpha_2}+\frac{2p^2\alpha_1{}^2}{(\alpha_1+\alpha_2)^2}\right)\frac{\alpha_2{}^{1-z}}{\Gamma(2-z)}\right]. \tag{5.16}$$

Using Eq. (5.11) we can rewrite this as

$$A(p^2,z,\epsilon)=-e^2F(z,\epsilon)\left(I(p^2,m^2,z)-\frac{3}{(-z)}\frac{\partial}{\partial p^2}I(p^2,m^2,z+1)+\frac{\partial}{\partial p^2}\int_{m^2}^\infty dt\,\frac{I(p^2,t,z)}{1-z}\right.$$
$$\left.-\frac{2p^2}{(1-z)(-z)}\frac{\partial^2}{\partial(p^2)^2}I(p^2,m^2,z+1)\right) \tag{5.17}$$

$$=-\frac{e^2F(z,\epsilon)\Gamma(-z)\Gamma(1+z)}{16\pi^2}\left[m^{2z}F\left(1-z,-z;2;\frac{p^2}{m^2}\right)+\frac{3}{z}\frac{1+z}{(-1-z)}m^{2z+2}\frac{(-z)(-z-1)}{2}\frac{1}{m^2}F\left(1-z,-z;3;\frac{p^2}{m^2}\right)\right.$$
$$+\tfrac{1}{2}\,{}_3F_2\left(-z,2-z,1-z;1-z;3;\frac{p^2}{m^2}\right)m^{2z}$$
$$\left.-\frac{2p^2}{(1-z)(-z)}\frac{1+z}{(-1-z)}m^{2z+2}\frac{(-z)(-z-1)}{2}\frac{(1-z)(-z)}{3}\frac{1}{(m^2)^2}F\left(2-z,1-z;4;\frac{p^2}{m^2}\right)\right]$$

$$=-\frac{\alpha}{4\pi}\frac{\Gamma(1+z)\Gamma(-z)\Gamma(1-z)}{\Gamma(3+z)}\left(\frac{m}{4\pi}\right)^{2z}[2-\chi^N(z,\epsilon)](-\lambda)^z\left[F\left(1-z,-z;2;\frac{p^2}{m^2}\right)-\tfrac{3}{2}(1+z)F\left(1-z,-z;3;\frac{p^2}{m^2}\right)\right.$$
$$\left.+\tfrac{1}{2}F\left(-z,2-z;3;\frac{p^2}{m^2}\right)+\frac{p^2}{3m^2}z(1+z)F\left(2-z,1-z;4;\frac{p^2}{m^2}\right)\right]. \tag{5.18}$$

Equations (5.13) and (5.18) when integrated over z as in Eq. (5.1) give the scalar and the spin contributions to the electron self-energy. The highest singularity occuring in the z plane at $z=0$ is a double pole. According to the discussion in Sec. III, this would necessitate taking $N=2$ in Eq. (5.3).

The leading-order contribution in κ to the mass shift δm [evaluated from $\Sigma(\not p=m)$] comes from this double pole at $z=0$ and gives the result

$$\frac{\delta m}{m}=\frac{3\alpha}{4\pi}\ln\left(\frac{4\pi}{\kappa m}\right)^2. \tag{5.19}$$

This is numerically precisely the same as the result in Ref. 9 up to terms of order $\alpha\ln G_N m_e{}^2$ and is of course finite.

So far we have assumed that the photon is in the Landau gauge. If an arbitrary gauge is used then the longitudinal part is subtracted off separately, as explained before, and the propagator in Eq. (4.28) (now with $a\neq 1$) is used in the convolution integral in Eq. (5.1) to construct the electron self-energy. The additional contribution is

$$\Sigma(a,p^2)\equiv e^2(1-a)\int d^4k\,\frac{\gamma_\mu k_\mu}{(k^2)^2}\frac{1}{(\not p-\not k-m)}\gamma_\nu k_\nu\,, \tag{5.20}$$

which manifestly vanishes on the mass shell $\not p=m$, showing that the mass shift δm is indeed a gauge-invariant quantity. The off-shell value of this additional contribution is ultraviolet divergent (unless $a=1$). Thus in all but the Landau gauge our procedure does not regularize the electron's *off-shell* Green's function and needs to be extended. This is in contradistinction to the on-shell magnitudes which are finite and realistically regularized.

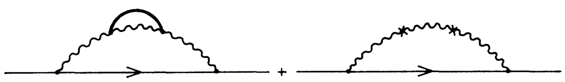

FIG. 13. Electron electromagnetic self-energy, gravitationally modified.

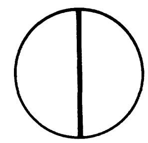

FIG. 15. Basic electron-graviton loop.

VI. PHOTON SELF-ENERGY

The simplest set of gravity-modified gauge-invariant graphs in our theory is the Ward-Feynman set of Fig. 14. To this, obtained by attaching photon lines to the electrons in the electron-graviton graph[16] (Fig. 15), we add the basic graph (also gauge invariant) shown in Fig. 16. It is our contention that when the graph of Fig. 16 is suitably cradled (and this needs kinking in six places) in the master graph shown in Fig. 17, the resulting graph (Fig. 18) is both finite and gauge-invariant, even though individually graphs of Fig. 14 and 16 are not finite. (If n_1, n_2, n_3 represent the number of graviton lines in Fig. 18, it is clear that Fig. 17 corresponds to $n_1 = n_2 = n_3 = 0$.) The computation of the graph in Fig. 18 involves a triple integral in z space. Even though we do not anticipate that any basic problem will arise, this calculation is unfortunately very complicated and has not yet been explicitly carried through.

VII. CONCLUDING REMARKS

We conclude with a number of general remarks:

(1) It is the main message of this paper that, from a good field-theoretic point of view, it is desirable to work with localizable rather than nonlocalizable nonpolynomial theories. When graviton-graviton interactions are neglected, an exponential parametrization of gravity does indeed provide such a localizable theory. It is important to state that when graviton-graviton couplings are added in, we expect that the theory would remain finite and no new infinities are likely to appear, though this has been verified on a power-counting basis only.

(2) The rational parametrization of gravity appears to define a nonlocalizable theory when graviton-electron interactions alone are considered. Nothing is known when graviton-graviton interactions are added in. It is perfectly possible that the full theory is localizable on account of cancellations of singular terms. In this case the rational and the exponential parametrizations may be fully equivalent in Borchers's sense. It is also possible that the two theories (in their quantized versions) are indeed inequivalent. More work is needed in elucidating these points.

(3) The major technical advance of this paper consists in kinking and cradling techniques and in the beginnings made towards a calculus of derivatives of singular functions. This development holds the promise of being able to treat properly tadpoles of the second kind and the related problems of ensuring gauge covariance.

(4) Quite generally it is important to realize that finiteness of self-charge or self-mass does not obviate the necessity of charge and mass renormalizations, which are operations that must be carried out in a Lagrangian field theory in any case. The S-matrix elements in a gravity-modified theory, in general, have the form

$$\alpha^p (\ln \kappa^2 m^2)^q (\kappa^2)^r \quad (p,\ q,\ r \text{ are non-negative integers and } q \leq p).$$

These are singular in the limit $\kappa \to 0$ only for $r = 0$. This clearly is the case for self-mass and self-charge matrix elements. For all other matrix elements, in the limit $\kappa \to 0$, we will inevitably recover the standard power series in α.

(5) One is frequently asked if gravity modifications will affect Lamb shift or other low-energy phenomena to any measurable extent. The answer in view of (2) above is clearly "no."

(6) We have been concerned in this paper with electromagnetic gauge invariance only. When graviton-graviton interactions are introduced, we will be faced with the problem of showing that (nonpolynomial) calculational techniques exist which preserve also the gravitational gauge invariance of S-matrix elements. We conjecture the following: If we draw a set of conventional perturbation graphs which are "correct" in the formal sense that their sum is both electromagnetically and gravitationally gauge-independent on the mass shell up to a given order in $\alpha^r \kappa^s$, then a replace-

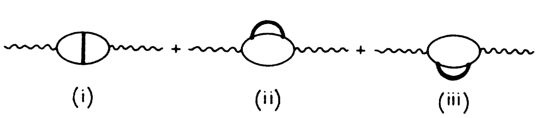

FIG. 14. Gauge-invariant set of photon self-energy graphs.

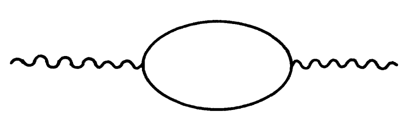

FIG. 16. Original photon self-energy graph.

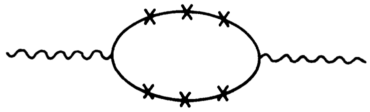

FIG. 17. Kinked version of Fig. 16.

ment of all graviton lines in such graphs by graviton superpropagators will yield finite expressions for the S-matrix elements whose gauge independence is maintained up to the order $\alpha^r\kappa^s$, and no higher. In other words, insertion of superpropagators in a "correct" set of graphs in perturbation theory renders them finite and preserves their "correctness" up to the order we started from.

Note added in proof. In a recent DESY preprint H. Lehmann and H. Trute have claimed that for chiral theories the demands of Jaffe localizability are so strong that only one parametrization can satisfy them – the exponential parametrization. We would conjecture that, following Lehmann and Trute, a similar proof can be constructed to show that the exponential parametrization of gravity discussed in this paper is the unique localizable one.

APPENDIX A

The infinity-suppression mechanism of localizable nonpolynomial theories is so transparent and so easily exhibited that we feel the following illustrative calculation could form part of first-year courses in quantum field theory.

Consider a model electrodynamic interaction $\mathcal{L} = e\bar{\psi}\psi A\exp(\kappa\phi)$ where ψ is the electron field, A is the (scalar) photon, and ϕ is the (scalar) gravitational field:

$$\alpha = e^2/4\pi = \tfrac{1}{137}\,,$$
$$m_e{}^2\kappa^2 = (16\pi)^{-1}G_N m_e{}^2 \approx 10^{-22}\,. \qquad \text{(A1)}$$

We wish to exhibit the realistic regularization of the otherwise infinite electron self-mass through gravity. Writing the most singular parts of the relevant propagators in the form (we have dropped some factors of 4π)

$$\langle\psi\bar{\psi}\rangle = -(i\gamma\cdot\partial + m)1/x^2$$
$$+\text{less singular terms}\,, \qquad \text{(A2)}$$
$$\langle AA\rangle = -1/x^2\,,$$
$$\langle\phi\phi\rangle = -1/x^2\,,$$

and noting that $\langle e^{\kappa\phi(x)}e^{\kappa\phi(0)}\rangle = \exp(-\kappa^2/x^2)$, the contribution to the electron self-mass from the sum of the chain of graphs in Fig. 2 (with $y=0$) is given by

$$F(x) = \alpha\sum_{n=0}^{\infty}\frac{1}{n!}\left[(i\gamma\cdot\partial + m)\left(-\frac{1}{x^2}\right)\right]\times\left(-\frac{\kappa^2}{x^2}\right)^n\left(-\frac{1}{x^2}\right). \qquad \text{(A3)}$$

Using a Sommerfeld-Watson transform, this equals

$$\frac{\alpha}{2\pi i}\int dz\frac{(\kappa^2)^z(-1)^z}{\Gamma(z+1)\sin\pi z}\left(-\frac{1}{x^2}\right)^{z+1}\times(i\gamma\cdot\partial + m)\left(-\frac{1}{x^2}\right), \qquad \text{(A4)}$$

where the contour lies round the positive real z axis. This contour may be rotated to lie parallel to the imaginary axis to give

$$F(x) = \frac{\alpha}{2\pi i}\int_{\mathrm{Re}\,z<0} dz\frac{(\kappa^2)^z(-\lambda)^z}{\Gamma(z+1)\sin\pi z}\times\left(\frac{i\gamma\cdot\partial}{z+2} + m\right)\left(-\frac{1}{x^2}\right)^{z+2}. \qquad \text{(A5)}$$

The advantage of doing this is that now we can use the unambiguous expression for the Fourier transform of the (classical) function $(-1/x^2)^z$ valid in the range $0<\mathrm{Re}\,z<2$, given by

$$-\frac{1}{(4\pi)^2}\left[\frac{-p^2}{(4\pi)^2}\right]\frac{\Gamma(2-z)}{\Gamma(z)}\,.$$

Thus

$$\tilde{F}(p) = -\frac{\alpha}{2\pi i(4\pi)^2}\int_{\mathrm{Re}\,z<0}\frac{dz(\kappa^2)^z(-\lambda)^z}{\sin\pi z\,\Gamma(z+1)}\times\left(\frac{\not{p}}{-z+2} + m\right)\left(-\frac{p^2}{(4\pi)^2}\right)^z\frac{\Gamma(-z)}{\Gamma(z+2)}\,. \qquad \text{(A6)}$$

Rotating the contour back to the real axis, the double pole in z space at $z=0$, gives regularized contributions to self-mass $\tilde{F}(p)|_{\not{p}-m=0}$ of the form $\alpha m\ln(\kappa^2m^2)$. This simple idealized example illustrates the basic technique used in this and earlier papers. The conventional infinity of $\delta m/m$ is instantly recovered by taking the limit $\kappa\to 0$. Since numerically $\ln(G_N m_e{}^2)\approx\ln(\kappa^2 m_e{}^2)\approx 100\approx\alpha^{-1}$, the magnitude of $\delta m/m$ is not outrageously different from unity – lending support to the Lorentz view

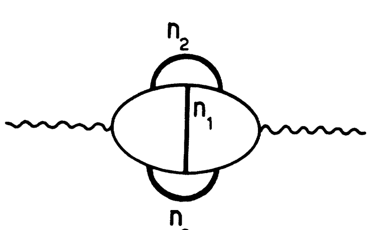

FIG. 18. Final finite and gauge-invariant photon self-energy graph.

that all electron self-mass may be (gravity-modified) electrodynamic in origin.

APPENDIX B

We have four remarks to make to clarify kinking and cradling in Sec. III D.

(1) Take as an example the Lagrangian (see Sec. III D)

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \chi \partial_\mu \chi) : e^{\kappa\phi} : + \tfrac{1}{2}\partial_\mu \phi \partial_\mu \phi ,$$

then the insertion of a χ kink corresponds to minus the unit operator when proper account is taken of factors of i, -1, etc. In general we define

$$\mathcal{G}(x, y) = \left\langle T^*\left(\chi(x)\chi(y) \exp\left[i \int \tfrac{1}{2}\partial_\mu \chi(\xi)\partial_\mu \chi(\xi) : (e^{\kappa\phi(\xi)} - 1) : d\xi \right]\right)\right\rangle = \sum_{n=0}^{\infty} \mathcal{G}^{(n)}(x, y) ,$$

where $\mathcal{G}^{(n)}$ is the nth term arising from the expansion of the exponential, whilst the kinked Green's functions are defined as

$$\mathcal{K}(x, y) = \left\langle T^*\left(\chi(x)\chi(y) \exp\left[i \int \tfrac{1}{2}\partial_\mu \chi(\xi)\partial_\mu \chi(\xi) : e^{\kappa\phi(\xi)} : d\xi \right]\right)\right\rangle = \sum_{n=0}^{\infty} \mathcal{K}^{(n)}(x, y) .$$

One can show by a lengthy but straightforward inductive proof that

$$\mathcal{G}^{(n)} = \sum_{r=0}^{n} \binom{n}{r} \mathcal{K}^{(r)} . \tag{B1}$$

This expression is the main "cradling" formula. It expresses the physical graphs on the left-hand side in terms of the kinked graphs on the right-hand side which are finite when nonpolynomial techniques are used and hence it serves as the definition of the left-hand side. Note that cradled graphs represent essentially a perturbation expansion in terms of $\mathcal{L}_{\text{tot}}$ in contrast to normal perturbation theory which is a series expansion in terms of $\mathcal{L}_{\text{int}}$.

(2) The sum of all graphs up to order n is

$$\mathcal{S}^{(n)} = \sum_{j=0}^{n} \mathcal{G}^{(j)}$$

and can be expressed, using (B1), as

$$\mathcal{S}^{(n)} = \sum_{r=0}^{n} \binom{n+1}{r+1} \mathcal{K}^{(r)} . \tag{B2}$$

(3) In using Eqs. (B1) and (B2) it should be remembered that because of the normal ordering of the Lagrangian we have

$$\mathcal{G}^{(1)} = 0$$

and so, by (B1),

$$\mathcal{K}^{(1)} + \mathcal{K}^{(0)} = 0 . \tag{B3}$$

(4) As a particular application consider the Green's function $\mathcal{S}^{(3)}$. From (B2) and (B3) we have

$$\mathcal{S}^{(3)} = \mathcal{K}^{(3)} + 4\mathcal{K}^{(2)} - 2\mathcal{K}^{(0)} . \tag{B4}$$

---

[1]Many physicists in our experience have so despaired of ever coping with this problem that even the need to compute finite values for renormalization constants some day is no longer felt. Those who entertain such hopes are often considered "irrational." We feel that the following quotation from Bertrand Russell's postscript to the third and final volume of his autobiography *The Final Years 1944-1967* (George Allen & Unwin, London, 1969), p. 221 (though somewhat harsh) adequately comments on the current attitude to the "misery" of the infinity problem:

"In the modern world, if communities are unhappy, it is often because they have ignorances, . . . , beliefs, . . . , which are dearer to them than happiness or even life. I find many men in our dangerous age who seem to be in love with misery and death, and who grow angry when hopes are suggested to them. They think hope is irrational and that, in sitting down to lazy despair, they are merely facing facts."

[2]For a recent survey of results see the articles of J. Glimm and A. Jaffe, in *Proceedings of the International School of Physics "Enrico Fermi," Local Quantum Theory*, edited by R. Jost (Academic, New York, 1969).

[3]W. Pauli, *Theory of Relativity* (Pergamon, London, 1967); S. Deser, talk given at the Austin Conference on Particle Physics, 1970 (unpublished).

[4]H. Lehmann and K. Pohlmeyer, in *Nonpolynomial Lagrangians, Renormalization and Gravity*, 1971 Coral Gables Conference on Fundamental Interactions at High Energy, Vol. 1 (Gordon and Breach, New York, 1971), p. 60.

[5]J. G. Taylor, in *Nonpolynomial Lagrangians, Renormalization and Gravity*, 1971 Coral Gables Conference on Fundamental Interactions at High Energy, Vol. 1 (Ref. 4), p. 42.

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[13]Such a parametrization has been used by C. Misner in the discussion of the "mixmaster" solution of Einstein's equations. We are indebted to Professor Bryce DeWitt for the remark that the virtue of exponential parametrization (so far as general relativists are concerned) is that the sign of $\det g$ is automatically preserved and the problem of the metric $+---$ changing, for example, to $++--$ does not arise.

[14]J. Ashmore and R. Delbourgo (unpublished) have recently completed the explicit construction of the superpropagator in an exponential parametrization.

[15]R. Delbourgo and A. Hunt, Imperial College, London, Report No. ICTP/69/18 (unpublished).

[16]One can see the virtue of the field transformations which carried us to the Lagrangian form of (3.8)–(3.11). The fact that the direct photon-electron interaction is completely given by $\bar{\psi}\gamma_a\psi A_a$ permits us to use the Feynman-Ward classification of gauge-invariant sets of photon self-energy graphs. All one does is to draw loops of charged lines with (neutral) gravitons interchanged between them and then attach photon lines to the loops in all possible ways.