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Dynamics Underlying Duality and Gauge Invariance in the Dual-Resonance Models*

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Making use of the similarity between the factorized spectrum of the generalized beta function and the spectrum of a vibrating string in Minkowski space, we construct a formalism within which we give a geometrical interpretation to the gauge symmetries inherent in the theory. The key point in the construction is the identification of the harmonic-oscillator amplitude with the components of actual coordinates in space-time. It turns out that these coordinates must satisfy certain constraints arising from purely geometrical considerations, and these constraint equations are then shown to be the familiar gauge conditions on the underlying states. Analogies with more familiar gauge theories are pointed out.

INTRODUCTION

The dynamical concept of duality[1] is most concretely realized within a pure resonance framework.[2] The resulting elastic scattering amplitude can be written as ratios of Γ functions and embodies duality in that it may be expressed either as a sum over s-channel resonances or over t-channel resonances. This particular realization of duality has been generalized to scattering amplitudes involving an arbitrary number of external scalar particles.[3] When such an amplitude is factorized[4] in a given channel, an extremely rich spectrum of resonances is revealed, which decay into scalar particles in a highly correlated (sequential) manner.

Straightforward factorization of the dual-resonance model requires the introduction of many states not all of which can represent resonances. The states that do represent resonances[5] are specified by their mass and spin and are supposed to carry[6] timelike unitary irreducible representations of the Poincaré group. This means that they must have positive norms. Many of the states that arise from straightforward factorization of the dual-resonance model (DRM) have negative norms, however, so that they are unphysical in the sense described above.

A remarkable feature of the DRM is that it appears to have an intrinsic symmetry which allows for the freedom of imposing certain constraint equations on the factorizing states.[7] This in turn raises the possibility that one might be able to partition these states into two equivalence classes, on one of which the Poincaré group action could be unitarily implemented. Since the result of such constraints is compatibility with Poincaré invariance, one may inquire whether such constraint equations express the consistency conditions between Poincaré invariance and the dynamical realization of duality. The relevance of such an inquiry can be substantiated by recalling that in quantum electrodynamics the compatibility between Poincaré invariance and the coupling of photons to matter to yield long-range interactions, a dynamical requirement, yield the gauge identities which eliminate the ghosts in the photon field.

In order to see whether such analogies provide a clue to the solution of the problems encountered in the DRM, one must have a formalism in which the realization of the conventional form of duality, the nature of the gauge symmetry involved, and the origin of the unphysical features of the model are clearly brought to the foreground. We report in this paper a possible framework which is internally consistent and within which such questions can be examined in detail.

Our considerations are based to a large extent on the observation that the dual-resonance spectrum bears a striking resemblance to that of a vibrating string[8-10] in four-dimensional space-time. The amplitude of vibration is taken to be a Minkowski coordinate that expresses the internal degrees of freedom of the hadronic system. The equation of motion for this string is now reinterpreted as a geometric constraint in the Minkowski space. Such a space-time interpretation leads to a formalism which closely parallels that of the known gauge theories of physics. The subsidiary conditions are then seen to emerge from the geometrical invariances inherent in our considerations.

The theory which we develop along these lines also imposes severe restrictions on the possible forms of interaction. In particular, if one is to attach any significance to the gauge invariance implied by such geometrical considerations, one must ensure that the gauge invariance is not disturbed in the presence of interactions. We shall show that the appearance of tachyons in the ground state is the result of preserving this gauge invariance in first-order perturbation theory.

We are unable at present to offer a general solution to the possible forms of interaction compatible with our gauge-invariance requirement. We suspect, however, that it is indeed possible to relax the unphysical mass-shell constraint. We shall come back to this and the problems related to spin in a later work. We only mention here in passing that our geometrical approach allows us to discuss these problems in a fairly direct way.

The plan of the paper is as follows: In Sec. I we discuss briefly those aspects of the DRM which provide the motivation for introducing the string variables. In Sec. II we construct an action from geometrical considerations and go on to deduce its basic consequences. In Sec. III we quantize our system and obtain a wave equation together with constraints imposed on its solution. The interaction is introduced in Sec. IV. The Conclusion is devoted to a summary of the results and conclusions.

I. THE STRING VARIABLES

It is well known by now that the factorization of the DRM can best be carried out [9,10] in terms of the four-dimensional annihilation and creation operators, $a^\mu(n)$ and $a^{\mu\dagger}(n)$, satisfying the commutation relations

$$[a^\mu(n), a^{\nu\dagger}(m)] = -g^{\mu\nu}\delta_{mn} . \tag{1.1}$$

Although the introduction of these operators is, strictly speaking, not necessary for factorization, their use definitely simplifies the description of the factorizing states.

The conventional dual amplitude [11] for the scattering of N scalar particles may then be written in the form

$$B_N(p_1, \ldots, p_N) = {}_N\langle 0|\Gamma(p_{N-1})\Delta(N-1, N-2)\cdots\Gamma(p_2)|0\rangle_1 , \tag{1.2}$$

where $\Gamma(p_r)$ is the vertex function for the particle r defined by

$$\Gamma(p_r) \equiv \exp\left[ip_r\cdot\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^{1/2} a^\dagger(n)\right] \times\exp\left[ip_r\cdot\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^{1/2} a(n)\right]. \tag{1.3}$$

The states $|0\rangle_1$ and $|0\rangle_N$ are ground states which are annihilated by $a^\mu(n)$ for all n. The external momenta satisfy the mass-shell condition $p_r^2 = -1$ which fixes the intercept of the corresponding trajectory at $\alpha_0 = +1$. The propagation function $\Delta(i,j)$ is given by

$$\Delta = H^{-1}, \quad \alpha_0 = +1, \tag{1.4}$$

where

$$H = P^2 + \alpha_0 + \sum n a^\dagger(n)\cdot a(n). \tag{1.5}$$

The operators $a^\mu(n)$ transform like four-vectors under homogeneous Lorentz transformations. However, they do *not* carry a momentum label and at this stage serve only to label the degeneracy of the states at each pole that appears in B_N. The physical states of the system are generated by the operators $\Gamma(p_r)$ acting on proper eigenvectors of H, and these states carry momentum only through the momentum dependence of the vertex. We thus require, by momentum conservation, that

$$[P^\mu, \Gamma(p_r)] = p_r^\mu\Gamma(p_r). \tag{1.6}$$

A general solution of the Eq. (1.6) is

$$\Gamma(p_r) \propto \exp(ip_r\cdot X), \tag{1.7}$$

with

$$[P^\mu, X^\nu] \propto +ig^{\mu\nu}. \tag{1.8}$$

We shall now recast the vertex (1.3) in a form compatible with (1.7) and (1.8). To do so we define the quantity $\tilde{Y}^\mu$ as

$$\tilde{Y}^\mu = X^\mu + \sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^{1/2}[a^\mu(n) + a^{\mu\dagger}(n)], \tag{1.9a}$$

such that

$$[P^\mu, X^\nu] = ig^{\mu\nu}. \tag{1.9b}$$

The operator $\Gamma(p)$ in (1.3) may now be written as

$$\Gamma(p_r) \equiv :e^{ip_r\cdot\tilde{Y}}:, \tag{1.10}$$

where the dots indicate normal ordering of the operators involved. By virtue of (1.9) it clearly satisfies the Eq. (1.6). Moreover, since the poles of the DRM are controlled by the operator H which describes the spectrum of a vibrating string in four-dimensional space-time, we may regard the quantity $\tilde{Y}^\mu$ as the end point of the vibrating amplitude, i.e.,

$$\tilde{Y}^\mu \equiv Y^\mu(\xi, \tau)|_{\tau=\xi=0},$$

where

$$Y^\mu(\tau, \xi) \equiv X^\mu + 2P^\mu\tau + \sum\left(\frac{2}{n}\right)^{1/2}[a^\mu(n)e^{-in(\tau-\xi)} + a^{\mu\dagger}(n)e^{in(\tau-\xi)}]. \tag{1.11}$$

For fixed τ, the quantities $Y^\mu(\tau, \xi)$ describe the extension of the string in space-time, and as τ varies the points on the string sweep out a two-dimensional surface in space-time, the equation of which is given by $Y^\mu(\tau, \xi)$. The exact nature of the surface naturally depends upon the boundary conditions, but the normal mode decomposition of the form given by (1.11) can always be carried out.

Observe that from the point of view of constructing scattering amplitudes one does not need the string variables $Y^\mu(\tau, \xi)$. However, as we shall see, their introduction greatly facilitates the discussion of the possible symmetries of the dual-resonance model.[12]

We can reexpress the basic commutation relations (1.1) in terms of $Y^\mu(\tau, \xi)$. To this end we define

$$\beta^\mu(\tau, \xi) = \frac{\partial Y^\mu}{\partial \tau} \ . \tag{1.12}$$

Then using (1.11), Eq. (1.1) becomes

$$[\beta^\mu(\tau, \xi), Y^\nu(\tau, \xi')] = +ig^{\mu\nu}\delta(\xi - \xi'). \tag{1.13}$$

It is to be noted, however, that this commutator contains more information than that in (1.1), since it carries information about Poincaré transformations due to the presence of the operator P^μ in Y^μ and β^μ.

We finally note that the quantity $Y^\mu(\tau, \xi)$ as defined by (1.11) satisfies the wave equation

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \xi^2}\right) Y^\mu(\tau, \xi) = 0. \tag{1.14}$$

With these preliminaries, we now turn to the formulation of the gauge symmetries inherent in DRM. It is well known that the canonical formalism offers the most convenient framework for the discussion of such symmetries. We shall therefore regard Y^μ as a dynamical variable and, in view of (1.13), β^μ as its canonical momentum, and go on to introduce an action which will generate the equations of this section.

II. CONSTRUCTION OF THE ACTION

The action that we want to construct must satisfy certain requirements. Firstly, it must of course be Poincaré-invariant. This is to ensure that our dynamical equations generate states of definite mass and spin which have positive norms. (There is some question as to whether unstable resonances should be required to carry timelike irreducible representations of the Poincaré group. The resonances which we deal with here[5] are regarded as stable so that the corresponding states must be so characterized.)

A second requirement on the action is that it be a function of the two internal variables we introduced in Sec. I. We suppose that the composite nature of hadrons can be adequately described by such variables, so that the underlying Hilbert space of states realizes the algebra (1.13). In short, the action must be a function of the dynamical string variables of Sec. I, and the (τ, ξ) dependence of the action must come through the string variables $Y^\mu(\tau, \xi)$ and its derivatives. In addition, care must be taken to ensure that dependence on τ and ξ does not affect the definition of physical states. In this respect the procedure that we follow is analogous to that used by Dirac[13] in his description of magnetic monopoles. The vector potential in the presence of monopoles loses its manifest isotropy, and its definition depends on a direction specified by the "Dirac strings." Such an anisotropy is, of course, spurious and consistency demands that transformations involving such variables not affect the physical states.

An analogous requirement must be imposed with respect to τ and ξ parameters, so that transformations on τ and ξ do not affect the physical states. This is the statement of gauge invariance in our formalism. The easiest way to incorporate this is to require that the action be invariant under such transformations.

The action will be constructed from the geometrical quantities characterizing the surface Σ generated by Y^μ. Since the surface Σ is embedded in Minkowski space, the differential of length on the surface is given by

$$ds^2 = g_{ab}d\xi^a d\xi^b, \quad a, b = 0, 1 \tag{2.1}$$

where

$$g_{ab} = g_{\mu\nu}\frac{\partial Y^\mu}{\partial \xi^a}\frac{\partial Y^\nu}{\partial \xi^b}, \tag{2.2}$$

$g_{\mu\nu}$ being the metric tensor in Minkowski space ($g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$). The intrinsic geometry of a surface is completely determined[14] by the metric tensor g_{ab}. The outer geometry, on the other hand, depends[14] on the quantities

$$b_{ab}^\alpha = \left(\frac{\partial \hat{n}_\alpha^\mu}{\partial \xi^a}\right)\left(\frac{\partial Y^\nu}{\partial \xi^b}\right) g_{\mu\nu}, \quad \alpha = 1, 2 \tag{2.3}$$

where $\hat{n}_\alpha^\mu$ ($\alpha = 1, 2$) are unit vectors normal to the surface

$$\hat{n}_\alpha \cdot \frac{\partial Y}{\partial \xi^a} = 0, \quad \alpha = 1, 2; \ a = 0, 1. \tag{2.4}$$

The action that we wish to construct must be invariant under the coordinate transformations on the surface Σ. It must therefore be a scalar quantity constructed from the two fundamental forms defined above. If one further requires that the equation of motion for Y^μ be not higher than second order, one finds that the simplest action consistent with these requirements is

$$I=\int d^2\xi\sqrt{-g}\equiv\int d^2\xi\mathcal{L}, \tag{2.5}$$

where

$$g=\det|g_{ab}| \ . \tag{2.6}$$

The equations of motion are obtained by extremizing (2.5). Before doing this, we shall utilize the freedom of the choice of coordinates implied by the action integral to linearize the Lagrangian density. A very convenient choice of parameters are obtained by defining

$$du^\pm=(f_1\pm i f_2)\times[\sqrt{g_{00}}d\xi^0+(g_{00})^{-1/2}(g_{01}\pm i\sqrt{g}\,)d\xi^1]. \tag{2.7}$$

Then the differential form (2.1) will take the form

$$ds^2=(f_1{}^2+f_2{}^2)^{-1}du^+du^-\equiv g_{+-}du^+du^-. \tag{2.8}$$

The significance of the new parameters $u^\pm$ is that the lines of $u^\pm$ =constant form light paths on the surface Σ. In terms of these coordinates the metric tensor (2.2) takes a particularly simple form:

$$g_{ab}=g_{\mu\nu}\frac{\partial Y^\mu}{\partial u^a}\frac{\partial Y^\nu}{\partial u^b}=g_{ba}=\sqrt{g}\,(\sigma^1)_{ab}, \quad a, b=+\text{or}- \tag{2.9}$$

where σ^1 is the Pauli matrix

$$\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{2.10}$$

The Lagrangian density now becomes

$$\mathcal{L}=g_{\mu\nu}\frac{\partial Y^\mu}{\partial u^+}\frac{\partial Y^\nu}{\partial u^-} \tag{2.11}$$

subject to the constraint equations

$$g_{\mu\nu}\frac{\partial Y^\mu}{\partial u^+}\frac{\partial Y^\nu}{\partial u^+}=g_{\mu\nu}\frac{\partial Y^\mu}{\partial u^-}\frac{\partial Y^\nu}{\partial u^-}=0. \tag{2.12}$$

The Euler-Lagrange equations which follow from (2.11) are

$$\frac{\partial^2}{\partial u^+\partial u^-}Y^\mu(u^+,u^-)=0. \tag{2.13}$$

A general solution of this equation can be written as

$$Y^\mu(u^+,u^-)=Y^\mu_+(u^+)+cY^\mu_-(u^-), \tag{2.14}$$

where c is a constant and

$$Y^\mu_\pm=X^\mu+P^\mu u^\pm+\sum_{n=1}^{\infty}[A^\mu_\pm(n)e^{-inu^\pm}+A^{\mu\dagger}_\pm(n)e^{inu^\pm}]. \tag{2.14$'$}$$

The reality of the solution requires that

$$A^{\mu\dagger}_\pm(n)=A^\mu_\pm(-n).$$

The equations of motion and the constraint equations follow from the same action (2.5), so that they are (at least at the classical level) internally consistent.

It should perhaps be pointed out that the constraint equations came about when we made a specific choice of coordinates by utilizing the coordinate independence of the action. In the more suggestive language of gauge theories, this means that the gauge invariance of our formalism *requires* that we impose these gauge conditions when we make a specific choice of gauge.

For many purposes, in particular to make contact with the results of Sec. I, it is convenient to introduce a different set of coordinates related to $u^\pm$ by

$$u^\pm=\frac{1}{\sqrt{2}}(\tau\pm\xi). \tag{2.15}$$

Then the equations (2.11)–(2.13) will take the form

$$\mathcal{L}=\frac{1}{2}\left(\frac{\partial Y^\mu(\tau,\xi)}{\partial\tau}\right)^2-\frac{1}{2}\left(\frac{\partial Y^\mu(\tau,\xi)}{\partial\xi}\right)^2, \tag{2.16}$$

$$\left(\frac{\partial Y^\mu}{\partial\tau}\pm\frac{\partial Y^\mu}{\partial\xi}\right)^2=0, \tag{2.17}$$

$$\left(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial\xi^2}\right)Y^\mu(\tau,\xi)=0. \tag{2.18}$$

The constraint equations (2.17) are equivalent to the set

$$\left(\frac{\partial Y^\mu}{\partial\tau}\right)^2+\left(\frac{\partial Y^\mu}{\partial\xi}\right)^2=0,\qquad \frac{\partial Y^\mu}{\partial\tau}\frac{\partial Y^\mu}{\partial\xi}=0. \tag{2.19}$$

From the Lagrangian (2.11) we can obtain the Hamiltonian in the usual way:

$$\mathcal{H}(\xi)=\left(\frac{\partial Y^\mu}{\partial\tau}\right)^2+\left(\frac{\partial Y^\mu}{\partial\xi}\right)^2. \tag{2.20}$$

Comparison with Eq. (2.19) shows that the Hamiltonian density is zero. This is not surprising since the Lagrangian density is homogeneous of first degree.

We wish to conclude this section by making a few comments on the role of the *proper time* in our formalism. We have emphasized throughout that as a result of gauge invariance, the choice of the two invariant parameters has no bearing on our conclusions. However, in special circumstances, such as in giving a physical interpretation, the choice of the parameter τ as proper time may turn out to be advantageous. In such cases we must require

$$g_{\mu\nu}\frac{\partial Y^\mu}{\partial \tau}\frac{\partial Y^\nu}{\partial \tau}=1. \tag{2.21}$$

Then to still satisfy our constraint equations (2.9) we must also set

$$\left(\frac{\partial Y^\mu}{\partial \xi}\right)^2=-1. \tag{2.22}$$

These relations are now regarded as special cases of the constraint equations (2.19). They can be satisfied by a suitable choice of the boundary conditions imposed on Y^μ. These boundary conditions are, however, not gauge-invariant in the sense defined above.

III. QUANTIZATION

We now quantize our system by requiring that the dynamical variable $Y^\mu(\tau, \xi)$ and its canonically conjugate momentum $\Pi^\mu(\tau, \xi)$ satisfy the commutation relations

$$[\Pi^\mu(\tau, \xi), Y^\nu(\tau, \xi')]=+ig^{\mu\nu}\delta(\xi-\xi'). \tag{3.1}$$

Then the Hamiltonian and the constraint equations become operators acting on states.[15] Because of the homogeneity conditions on the Lagrangian density (and consequently the vanishing of the Hamiltonian density) the determination of quantum expressions for the conjugate momentum and the Hamiltonian is a nontrivial task. We may ensure the correct expression for these by making use of the Lagrange multipliers in setting up the variational problem and requiring that the Heisenberg equations of motion be satisfied, i.e.,

$$i[Y^\mu(\tau, \xi), H]=\frac{\partial Y^\mu}{\partial \tau}. \tag{3.2}$$

The resulting action is

$$I=\int d\tau\int d\xi\,\frac{1}{2}\left[\left(\frac{\partial Y^\mu}{\partial \tau}\right)^2-\left(\frac{\partial Y^\mu}{\partial \xi}\right)^2+\frac{\partial Y^\mu}{\partial \tau}\frac{\partial Y_\mu}{\partial \xi}+\frac{\partial Y^\mu}{\partial \xi}\frac{\partial Y_\mu}{\partial \tau}\right]. \tag{3.3}$$

Then

$$\Pi^\mu(\tau, \xi)=\frac{\delta \mathcal{L}}{\delta\left(\dfrac{\partial Y^\mu}{\partial \tau}\right)}=\frac{\partial Y^\mu}{\partial \tau}+\frac{\partial Y^\mu}{\partial \xi} \tag{3.4}$$

and

$$\mathcal{H}(\xi)=\frac{\partial Y^\mu}{\partial \tau}\frac{\delta \mathcal{L}}{\delta\left(\dfrac{\partial Y^\mu}{\partial \tau}\right)}-\mathcal{L}$$

$$=\frac{1}{2}\left(\Pi^\mu-\frac{\partial Y^\mu}{\partial \xi}\right)^2+\frac{1}{2}\left(\frac{\partial Y^\mu}{\partial \xi}\right)^2. \tag{3.5}$$

If we now *choose* to regard τ as the proper time we can write

$$\mathcal{H}(\xi)=\frac{1}{2}\left[\left(\Pi^\mu-\frac{\partial Y^\mu}{\partial \xi}\right)^2-1\right]. \tag{3.6}$$

Then the Hamiltonian, H, is given by

$$H=\int d\xi\,\mathcal{H}(\xi)$$

$$=\frac{1}{2}\int d\xi\left[\left(\Pi^\mu-\frac{\partial Y^\mu}{\partial \xi}\right)^2-1\right]. \tag{3.7}$$

Since the classical Hamiltonian was zero,[15] the physical states must satisfy the constraint equation

$$H|\psi\rangle=0. \tag{3.8}$$

Written out in full, Eq. (3.8) takes the form

$$\int d\xi\left[\left(\Pi^\mu-\frac{\partial Y^\mu}{\partial \xi}\right)^2-1\right]\bigg|\psi\bigg\rangle=0. \tag{3.8$'$}$$

The integrand has a striking resemblance to the Klein-Gordon operator in the presence of minimal electromagnetic coupling, the *self-coupling* $\partial Y^\mu/\partial\xi$ playing the role of the vector potential. In fact, one can stretch the analogy a bit further and try to linearize the quadratic operator in the integrand in the same manner that Dirac obtained his operator from the Klein-Gordon operator. One will then get the following wave equation[16]:

$$\int d\xi\left[\gamma_\mu(\xi)\left(\Pi^\mu-\frac{\partial Y^\mu}{\partial \xi}\right)-1\right]\bigg|\psi\bigg\rangle=0. \tag{3.9}$$

Since we will take up the fermion problem in a subsequent paper, we defer the discussion of this equation to the latter work. We only mention here that it is possible to introduce space-time spinors consistent with our geometrical considerations *without* making use of the above equation.

The states (3.8) are subject to additional constraints. The most general equation satisfied by $|\psi\rangle$ is, according to (2.12),

$$\left(\frac{\partial Y^\mu}{\partial u^\pm}\right)^2\bigg|\psi\bigg\rangle=0 \tag{3.10a}$$

or (2.17)

$$\left(\frac{\partial Y^\mu}{\partial \tau}\pm\frac{\partial Y^\mu}{\partial \xi}\right)^2\bigg|\psi\bigg\rangle. \tag{3.10b}$$

In these expressions symmetrization with respect to noncommuting operators is understood. It can be seen that for each τ, the states $|\psi\rangle$ must satisfy an infinite number of constraints since (3.10) is valid for each value of the parameter ξ. In fact, the wave equation (3.8) is also contained in (3.10).

The two sets of constraints in (3.10) correspond to the two linearly independent solutions $Y^\mu_\pm$ given in (2.14′). In obtaining the conventional DRM the usual procedure is to eliminate one of the solutions by a suitable boundary condition, so that only one set of excitations remain. This need not be the case, however. One can, for example, impose periodic boundary conditions so that both solutions are retained. Then to obtain the conventional DRM the interaction Hamiltonian must depend either on Y_+^μ or Y_-^μ but not both. To allow for the latter possibility, we shall retain both sets of constraint equations with the understanding that one of them becomes trivially satisfied when one deals with the conventional DRM.

The classical equations of motion and the corresponding constraints were consistent because they were all obtained from the same action. We must now check that the *quantum* constraint equations (a) are compatible with each other, and (b) do not lead to any inconsistencies in the sense of being too stringent. To investigate these questions, it is convenient to define the Fourier coefficients

$$T_\pm^m = \frac{1}{2\pi}\int_0^{2\pi} du^\pm \left(\frac{\partial Y^\mu}{\partial u^\pm}\right)^2 e^{imu^\pm}, \quad m=0,\pm1,\ldots . \tag{3.11}$$

They satisfy the commutation relations

$$[T_\pm^m, T_\pm^n] = (m-n)T_\pm^{m+n} + \tfrac{1}{3}m(m-1)(m+1)\delta_{m+n,0}\,, \tag{3.12}$$

$$[T_+^m, T_-^n] = 0, \quad m,n=0,\pm1,\ldots .$$

It will be shown in Sec. IV that in the conventional DRM, where the physical states can be constructed in the manner described in Sec. I, the required gauge operators can be constructed from the T^m's.

IV. INTERACTIONS AND GAUGE INVARIANCE

In the developments of the previous sections we have worked in the Heisenberg picture. Since the results are true for all values of the parameter τ, they are true also for $\tau=0$, at which value the Heisenberg and interaction pictures coincide. We shall now regard the τ dependence of the state $|\psi\rangle$ to arise because of an interaction Hamiltonian V. Then, the τ development of the state is given by

$$i\frac{\partial}{\partial\tau}|\psi\rangle = V|\psi\rangle. \tag{4.1}$$

The time dependence of the interaction picture operators are determined by the unperturbed Hamiltonian:

$$-i\frac{\partial V}{\partial\tau} = [V, H]. \tag{4.2}$$

To discuss the restrictions that gauge invariance imposes on the possible form of interaction Hamiltonians, we consider the interaction Hamiltonian[17]

$$V_-(k) = :e^{ik\cdot Y_-(0)}:, \tag{4.3}$$

which is the simplest potential consistent with momentum conservation. It reproduces the conventional DRM in the tree approximation. Here the dots refer to normal ordering, and the interaction with the external field occurs at the point $\xi=0$. Since $V_-(k)$ is defined at a specific point on the surface Σ, the manifest homogeneity of the Lagrangian is destroyed, and the total action will no more remain invariant under all the transformations generated by $T_\pm^m$. We must therefore look for a *reduced* set of gauge operators formed from the $T_\pm^m$'s which leave the point $\xi=0$ invariant. These are given by[8]

$$G_\pm^m = T_\pm^m - T_\pm^0. \tag{4.4}$$

The $G_\pm^m$'s satisfy the algebra

$$[G_\pm^m, G_\pm^n] = (m-n)G_\pm^{m+n} - mG_\pm^m - nG_\pm^n. \tag{4.5}$$

In the absence of interaction we have

$$[G_\pm^m, H_\pm] = mG_\pm^m + mH_\pm. \tag{4.6}$$

Therefore, the consistency of the formalism requires that when an interaction Hamiltonian is introduced, i.e., as $H_\pm \to H_\pm + V_\pm$, then (4.6) must be replaced by

$$[G_\pm^m, H_\pm + V_\pm] = mG_\pm^m + m(H_\pm + V_\pm). \tag{4.7}$$

These are effectively the integrability conditions on our equations of motion. It is easy to verify that when the interaction is given by (4.3), the conditions (4.7) hold only if the condition

$$k^2 = -1 \tag{4.8}$$

is satisfied. This is the price one has to pay to preserve gauge invariance.

Next let us consider the effect of the operators G^m on the physical state $|\psi_\Gamma\rangle$ obtained by the application of the operator (1.7) to the vacuum. Clearly if

$$G_\pm^m|\psi_\Gamma\rangle \overset{?}{=} 0, \quad m=0,\pm1,\ldots \tag{4.9}$$

then

$$[G_\pm^m, G_\pm^n]|\psi_\Gamma\rangle = 0 \text{ for all } |\psi_\Gamma\rangle,$$

so that the quantization procedure does not lead to any additional constraints. One must check, however, whether the gauge conditions (4.9) are too stringent. A similar situation arises in quantum electrodynamics in which the imposition of the classical Lorentz gauge requirement $\partial^\mu A_\mu = 0$ as an operator relation leads to inconsistencies and must be replaced by the condition

$$\partial^\mu A_\mu^{(+)}|\phi\rangle = 0.$$

Here $A_\mu^{(+)}$ is the positive-frequency part of A_μ.

Since we know the explicit form of the state $|\psi_\Gamma\rangle$, it is easy to verify directly that Eq. (4.9) holds only if $m>0$. The correct quantum gauge conditions are therefore given by

$$G_\pm^m|\psi_\Gamma\rangle=0,\ \ m>0 \tag{4.10}$$

so that

$$[G_\pm^m, G_\pm^n]|\psi_\Gamma\rangle=0,\quad m,n>0. \tag{4.11}$$

Since the mass-shell condition (4.8) is the direct result of assuming the interaction Hamiltonian (4.3), it may at first be thought that by a suitable modification of (4.3) or a different choice of interaction Hamiltonian this condition could be relaxed completely. It appears, however, that this is highly unlikely to occur without a nontrivial modification of the vertex V. The best one can hope for by multiplicative modifications of the interaction Hamiltonian (4.3) would be to obtain a more physical mass-shell condition. In this respect one must also bear in mind the possibility that renormalization effects from higher-order diagrams may shift the external masses from the value −1.

CONCLUSION

In this work we have attempted to present a formalism within which the connection between the gauge symmetries of the conventional DRM and Poincaré invariance can be examined in detail. We have chosen to carry out our analysis in the canonical framework which is very convenient for the discussion of symmetries and gauge invariance. In the development of the formalism, we introduced "stringlike" variables, reinterpreted them as points on a surface embedded in Minkowski space, and constructed an action which was invariant under the general coordinate transformations on the surface. It was from the invariance of action that we obtained a realization of gauge symmetries in terms of the string variables.

We have not offered a direct proof of the Poincaré invariance of the DRM, but we are optimistic that the concepts introduced in this paper will provide the means of understanding how the Poincaré group is realized in the dual-resonance model. The problem of dealing effectively with spin degrees of freedom is, of course, intimately related to this important question. Within our formalism there is no problem in introducing two component or four component objects which transform as spin-$\frac{1}{2}$ representations of the homogeneous Lorentz group. In fact, two attractive possibilities immediately suggest themselves. One is to introduce spin by a suitable modification of the "tetrad" (or *Vierbeine*) method as in the general theory of relativity. The idea here is to define spin degrees of freedom at each point on the surface, the spins at different points being correlated by the field Y^μ. The other possibility is to build up the spin degrees of freedom by using the twistor formalism of Penrose[18] appropriately modified for our surface. In the latter approach, the lightlike coordinates $u^\pm$ are regarded as components of a spinor covariant under the group SU(1, 1)⊗SU(1, 1). We will discuss these in a subsequent paper.

Clearly, a lot more work needs to be done before the usefulness of our formalism can be justified.

Note added in proof. We have learned that the gauge conditions in the conventional DRM have also been studied in the context of a "detailed wave equation" by Takabayashi. See T. Takabayashi, Nagoya University Report No. DPNU-14, 1971 (unpublished).

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Generalized Goldstone Theorems in an Effective-Lagrangian Framework

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A set of sum rules among coupling tensors of *different* rank is derived for spontaneously broken continuous symmetries in an effective-Lagrangian framework. The sum rules, among which is the Goldstone theorem, constitute in this framework the physical content of that part of the spontaneously broken symmetry not contained in the conventionally realized subgroup which leaves the vacuum invariant, the latter subsymmetry manifesting itself in Wigner-Eckart relations among coupling tensors of the *same* rank.

In a spontaneously broken[1] symmetry (SBS) group G of a Hamiltonian H, the vacuum state is invariant only against a lower symmetry G', a subgroup of G, and hence zero-mass excitations or Goldstone-Nambu particles[1,2] (GNP's) are required to support the noninvariance of the vacuum against group operations in G but not in G' which leave H invariant. The single-particle states form irreducible G' multiplets, the mass matrix is invariant against G' but not G, and group operations in G but not in G' take single-particle states into multiparticle states which have zero-mass particles in them and take the vacuum into zero-mass particle states.[1,2] Multiplets irreducible under G exist[3] but do not correspond to particle states with fixed particle number.[1,2] Wigner-Eckart selection rules arising from the G' symmetry result in the usual relations among scattering amplitudes for fixed n-particle processes.

It is interesting to inquire into the nature and physical content of the residual symmetry of H under operations not in G', that is, Wigner-Eckart selection rules involving these operations. If experience with broken chiral symmetry is a guide, and if the soft-meson limits endemic in chiral-symmetry calculations[4] take one to an underlying spontaneously broken theory with a multiplet of mesons playing the role of GNP's,[5] then the invariance of the theory in this limit against group operations not in G' results not in the usual Wigner-Eckart relations among n-particle amplitudes but in theorems for the emission or absorption of GNP's,[4] that is, relations among amplitudes having different numbers of particles in external states. This is not surprising since the fixed-particle-number states[6] cannot diagonalize the conserved charges which are generators of group operations in G but not in G'. That the results of an SBS can be extracted by Wigner-Eckart relations among states not having fixed-particle number, but diagonal in the spontaneously broken generators, has been shown for the particular case of theories invariant against c-number translations of scalar boson fields.[7] This method, while complete and straightforward, is also awkward and tedious, involving first the diagonalization process and then the restatement of the resultant Wigner-Eckart relation in terms of fixed-particle-number states. More importantly, the method does not easily generalize to the physical case in which the vacuum degeneracy is lifted by the addition of explicit symmetry-breaking terms which remove the vacuum degeneracy and give the GNP's a nonzero mass.

Apart from the practical purpose of establishing *all* physical consequences of SBS over and above what is well known[1] [namely (i) conventional invariance under G', the symmetry of the vacuum,