and c_L can be chosen so that $|e_L - z_{i,L}| < 3^L E$ implies that $e_L - z_{i,L}$ is real. This procedure can then be carried out inductively, routing c_r in such a way that $|e_r - z_{i,r}| < 3^r E$ implies that $e_r - z_{i,r}$ is real. Such a choice of c_r might be impeded by two singularities $z_{r,i}$ and $z_{r,j}$ with $\mathrm{Im}(z_{r,i} - z_{r,j}) \neq 0$ and $|z_{r,i} - z_{r,j}| \leq 2 \times 3^r E$. However, the choice of the contour c_{r+1} rules out this possibility: $|z_{r,i} - z_{r,j}| < 3^{r+1} E$ implies that $z_{r,i} - z_{r,j}$ is real.

[25] For a slightly more complete discussion of this model, see N. Christ, in *Nonpolynomial Lagrangians, Renormalisation and Gravity,* Proceedings of the 1971 Coral Gables Conference on Fundamental Interactions at High Energy, Vol. 1 (Gordon and Breach, New York, 1971), p. 69.

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Application to the Scattering Problem of a Higher-Order Modified WKB Approximation Due to Miller and Good

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The modified WKB method due to Miller and Good is used to derive the scattering phase-shift formula including $\hbar^4$ terms for the radial equation of the three-dimensional scattering problem.

I. INTRODUCTION

Previous papers have developed a derivation, in the spirit of Miller and Good,[1] which yields the scattering phase shifts in terms of an expansion in powers of $\hbar$. A previous paper[2] presents the terms to the order $\hbar^2$.

The present paper is an extension of the previous result to terms of the order of $\hbar^4$. Section II of the present paper recapitulates the rigorous derivation of Ref. 2. A rigorous derivation of the scattering phase shifts to order $\hbar^4$ by the method of Ref. 2 would, however, require a very great expenditure of labor.

Therefore, in Sec. III we present a less rigorous derivation which can serve the purpose nicely. This same procedure can be easily followed for obtaining terms of higher order than $\hbar^4$ for the method developed is a straightforward expansion. This is, however, not so if we follow the rigorous derivation and try to obtain some higher-order terms by that process.

The divergence of the perturbation terms to $\hbar^2$ was successfully avoided in the scattering cases.[3,4] The main idea there is to replace the divergent integrals by contour integrals. As a result, we were able to obtain some meaningful results. In the spirit of the modified WKB method of Miller and Good, we can express the phase shifts which we want in terms of the known phase shifts of a known potential. In the specific example chosen in Ref. 4, the terms of second order in $\hbar$, in general, contribute to the phase shifts to the second decimal place while the zeroth-order terms in some places cannot give the result even to the first decimal place. An important question to be answered in this note is the extent to which improvement is possible. When the potential of the known part is sufficiently different from the potential of the unknown part, we may need to go to terms of higher order than $\hbar^2$ in order to get results of higher precision. We, therefore, investigate the contribution due to $\hbar^4$ terms.

Throughout this paper we avoid Langer's substitution, which replaces the three-dimensional distance r by e^x with x being the one-dimensional distance. Breit[5] raised the point that it is difficult to give a physical justification for the lower limit used in the integration if Langer's substitution is introduced. Here we simply consider the case where the angular momentum quantum number $L \neq 0$ for expansion in $1/L$ will be made later.

In the derivation of the formula in Secs. II and III the basic assumption is that $\psi(r)$ has the form $\psi(r) = T(r)\phi(S(r))$, where $\psi(r)$ is the unknown wave function and $\phi(S)$ is the known wave function that we want to make use of. Care should be exercised in the choice of the known part, $\phi(S)$. For example, in solving to order $\hbar^4$, we find that it is inadequate to represent the Coulomb scattering problem by the fractional-order Bessel-function formula as was possible when solving to order $\hbar^2$. This situation may be changed if we set, for example, $\psi(r) = T_1\phi(S(r)) + T_2\phi'(S(r))$ where T_1 and T_2 are functions dependent on r and $\phi'(S(r)) = d\phi/dS$. However, if short-range forces are the main concern here, we will see that good results will be obtained as we take $\psi(r)=T(r)\phi(S(r))$ up to order $\hbar^4$.

II. DERIVATION OF THE SCATTERING PHASE-SHIFT FORMULA TO ORDER $\hbar^2$

The Schrödinger equation we are going to solve for the phase shifts is

$$\frac{d^2}{dr^2}\psi(r)+\frac{p_1^2}{\hbar^2}\psi(r)=0\,, \tag{1}$$

with

$$p_1^2(r)=2m[E-V_1(r)]-\frac{L(L+1)\hbar^2}{r^2}\,.$$

An approximate solution is obtained by means of a solved equation

$$\left(\frac{d^2}{dS^2}+\frac{p_2^2}{\hbar^2}\right)\phi(S)=0\,, \tag{2}$$

with

$$p_2^2(S)=2m[E-V_2(S)]-\frac{L(L+1)\hbar^2}{S^2}\,.$$

Here we use the assumption

$$\psi(r)=T(r)\phi(S(r))\,. \tag{3}$$

After substituting this relation in Eq. (1), as in Miller and Good, we get

$$T=S'^{-1/2} \tag{4}$$

and

$$\hbar^2\frac{T''}{T}-S'^2p_2^2+p_1^2=0\,. \tag{5}$$

We use then the WKB expansion

$$S=S_0+\hbar^2S_2+\hbar^4S_4+\cdots\,, \tag{6}$$

and let

$$p_2^2(S)=p_{2,0}^2(S_0)+\hbar^2p_{2,2}^2(S_0,S_2)+\cdots\,. \tag{7}$$

Here the first term $p_{2,0}^2$ depends on S_0 only. We will treat the $L(L+1)\hbar^2$ term as if it is of the zeroth order in $\hbar^2$ as was done before.[6] Or we will set

$$p_{2,0}^2(S_0)=2m[E-V_2(S_0)]-\frac{L(L+1)\hbar^2}{S_0^2}$$

in the latter part of this note. So to the zero order in $\hbar^2$, we have

$$S_0'=\frac{p_1(r)}{p_{2,0}(S_0)}\,, \tag{8}$$

or

$$\int_{S_1}^{S}p_{2,0}(\sigma)d\sigma=\int_{r_1}^{r}p_1(\xi)d\xi\,. \tag{9}$$

Here we take $p_{2,0}(S_1)=0$ and $p_1(r_1)=0$ or the lower limits of integration are taken to be the classical turning points for both sides. And hence we make the assumption that there is only one real and positive classical turning point for $p_{2,0}$ or p_1.

Equation (9) will give us the relationship among phase shifts. For short range forces, we have at large r and S,

$$\psi(r)\sim\sin(kr-\tfrac{1}{2}L\pi+\delta_r)\,, \tag{10}$$

and

$$\phi(S)\sim\sin(kS-\tfrac{1}{2}L\pi+\delta_S)\,, \tag{11}$$

with $\hbar^2k^2=2mE$. By our assumption [Eq. (3)], those two expressions should give

$$\lim_{\substack{r\to\infty\\ S\to\infty}}k(r-S)=\delta_S-\delta_r\,. \tag{12}$$

δ_S is assumed known, hence δ_r can be determined from Eqs. (9) and (12). Notice here Eq. (9) is valid to the lowest order of approximation in expansion against $\hbar$. This part of the presentation follows from the work of Rosen and Yennie.[7] Our purpose in this paper is to give the expansion to $\hbar^4$ terms. In this section, we will limit the derivation to order $\hbar^2$. The increase in complication will be seen to be very rapid indeed. Here we present the formulas to order $\hbar^2$ without introducing any other assumptions.

The next order term in $\hbar$ is

$$S_2'=\frac{1}{8p_{2,0}^2}\left(\frac{3S_0''^2}{S_0'^3}-\frac{2S_0'''}{S_0'^2}-4S_0'p_{2,2}^2\right)\,. \tag{13}$$

If we denote

$$p_{2,0}'(S_0)=\frac{dp_{2,0}(S_0)}{dS_0}\,,$$

we find from Eq. (8)

$$\begin{aligned}S_0''&=\frac{d}{dx}\left(\frac{p_1(x)}{p_{2,0}(S_0)}\right)\\&=\frac{p_1'(x)}{p_{2,0}(S_0)}-\frac{p_1(x)}{p_{2,0}^2(S_0)}S_0'(x)\frac{dp_{2,0}(S_0)}{dS_0}\\&=\frac{p_1'(x)}{p_{2,0}(S_0)}-\frac{p_1^2(x)}{p_{2,0}^3(S_0)}p_{2,0}'(S_0)\,,\end{aligned} \tag{14}$$

and

$$\begin{aligned}S_0'''&=\frac{p_1''}{p_{2,0}}-\frac{3p_1'p_{2,0}'p_1}{p_{2,0}^3}\\&\quad-\frac{p_1^3p_{2,0}''}{p_{2,0}^4}+\frac{3p_1^3p_{2,0}'^2}{p_{2,0}^5}\,.\end{aligned} \tag{15}$$

After substituting Eqs. (14) and (15) into Eq. (13), also using Eq. (8), we have the following expression

$$p_{2,0}(dS_0+\hbar^2 dS_2)+\hbar^2\frac{p_{2,2}{}^2}{2p_{2,0}}dS_0 + \tfrac{1}{8}\hbar^2\left(\frac{3p_{2,0}'{}^2-2p_{2,0}''p_{2,0}}{p_{2,0}{}^3}\right)dS_0 = p_1 dx+\tfrac{1}{8}\hbar^2\left(\frac{3p_1'{}^2-2p_1''p_1}{p_1{}^3}\right)dx. \tag{16}$$

We compare terms in the above equation with

$$p_2=(p_{2,0}{}^2+\hbar^2 p_{2,2}{}^2)^{1/2}=p_{2,0}+\tfrac{1}{2}\hbar^2\frac{p_{2,2}{}^2}{p_{2,0}}. \tag{17}$$

Therefore, to the order of $\hbar^2$, we get

$$\int_{S_1}^{S} p_2 dS-\tfrac{1}{8}\hbar^2\int_{S_1}^{S}\left(\frac{2p_2''}{p_2^2}-\frac{3p_2'^2}{p_2^3}\right)dS = \int_{r_1}^{r} p_1 dr-\tfrac{1}{8}\hbar^2\int_{r_1}^{r}\left(\frac{2p_1''}{p_1^2}-\frac{3p_1'^2}{p_1^3}\right)dr. \tag{18}$$

However, we notice that the perturbed terms become divergent, respectively, at S_1 and r_1, for then we have $p_2(S_1)=0$ and $p_1(r_1)=0$ or these are defined as turning points. We can avoid such difficulty by following the work of Bertocchi, Fubini, and Furlan[8] as before[3] replacing the divergent integrals by line integrals which are evaluated under and over the real axis around the turning point with open ends at positive infinity. We get, therefore, instead of Eq. (18) which is divergent, the following convergent expression:

$$\int_{r_1}^{r} p_1 dr-\oint\tfrac{1}{16}\hbar^2\left(\frac{2p_1''}{p_1^2}-\frac{3p_1'^2}{p_1^3}\right)dr = \int p_2 dS-\oint\tfrac{1}{16}\hbar^2\left(\frac{2p_2''}{p_2^2}-\frac{3p_2'^2}{p_2^3}\right)dS. \tag{19}$$

The method of writing of the above equation is not unique for it is connected by integration by parts or

$$\oint u\,dv=-\oint v\,du. \tag{20}$$

One of the expressions is

$$\int_{S_1}^{S} p_2 dS-\tfrac{1}{16}\hbar^2\oint\frac{p_2'^2 dS}{p_2^3}=\int_{r_1}^{r} p_1 dr-\tfrac{1}{16}\hbar^2\oint\frac{p_1'^2}{p_1^3}dr. \tag{21}$$

Because the above equation is simple we will use it in Sec. III. We will transform it further for a specific potential as given in Sec. IV.

By defining $t_{1,2}=p_{1,2}{}^2$, respectively, we can obtain the following expression by several partial integrations:

$$\int_{S_1}^{S} p_2 dS-\tfrac{1}{24}\hbar^2\left(\int_{S_1}^{S}\frac{t_2''^2}{t_2^{1/2}t_2'^2}dS-\int_{S_1}^{S}\frac{t_2'''}{t_2^{1/2}t_2'}dS\right) = \int_{r_1}^{r} p_1 dr-\tfrac{1}{24}\hbar^2\left(\int_{r_1}^{r}\frac{t_1''^2}{t_1^{1/2}t_1'^2}dr-\int_{r_1}^{r}\frac{t_1'''}{t_1^{1/2}t_1'}dr\right). \tag{22}$$

Here notice that in the terms in $\hbar^2$ we use the integration from S_1 or r_1 to S or r again for there are no longer any divergences involved. This is another way of writing the relation of Eq. (19) and gives the solution of the problem to order $\hbar^2$.

III. DERIVATION OF THE SCATTERING PHASE-SHIFT FORMULA TO ORDER $\hbar^4$

We are going to derive a formula of the following form:

$$I_1+I_2=J_1+J_2, \tag{23}$$

with

$$I_1\equiv\int_{r_1}^{r} p_1 dr-\tfrac{1}{16}\hbar^2\oint p_1'^2 p_1^{-3}dr,$$

$$I_2\equiv-\tfrac{1}{64}\hbar^4\oint p_1''^2 p_1^{-5}dr+\hbar^4\tfrac{15}{256}\oint p_1'^4 p_1^{-7}dr,$$

$$J_1\equiv\int_{S_1}^{S} p_2 dS-\tfrac{1}{16}\hbar^2\oint p_2'^2 p_2^{-3}dS,$$

and

$$J_2\equiv-\tfrac{1}{64}\hbar^4\oint p_2''^2 p_2^{-5}dS+\hbar^4\tfrac{15}{256}\oint p_2'^4 p_2^{-7}dS,$$

solving the problem to order $\hbar^4$. If we just take terms to the order of $\hbar^2$, we get $I_1=J_1$ or Eq. (21) from the above equation. The upper limits r and S in Eq. (23) will be set equal to infinity. The contour will be taken above and below the real axis around the turning point to the left and going to infinity to the right as in Sec. II. Of course, terms in Eq. (23) cannot be uniquely given as they are connected with other possible integrations by parts. It is precisely this property that we shall use to get rid of the divergent terms as illustrated in Sec. IV.

We observe that in the case $p_2(S)=\text{constant}$, the derivation is much simplified. We will derive Eq. (23) for the case of a general $p_1(r)$ and $p_2(S)=\text{constant}$. If we are successful in this, we can replace $p_1(r)$ by $p_3(r)$ with $p_3(r)$ having another form of dependence on r. Eq. (23) becomes, to order $\hbar^4$

$$I_1 + I_2 = p_2(S - S_1) = K_1 + K_2\,, \tag{24}$$

with

$$K_1 = \int_{r_1}^{r} p_3 dr - \tfrac{1}{16}\hbar^2 \oint p_3{'}^2 p_3{}^{-3} dr$$

and

$$K_2 = -\tfrac{1}{64}\hbar^4 \oint p_3{''}^2 p_3{}^{-5} + \hbar^4 \tfrac{15}{256} \oint p_3{'}^4 p_3{}^{-7} dr\,.$$

Then we say that Eq. (23) is valid in general because each side, $I_1 + I_2$ or $K_1 + K_2$, has to be equal to $p_2(S - S_1)$ in Eq. (24) and therefore they have to be equal to themselves.

The following is the derivation of Eq. (23) with $p_2 =$ constant. Since we make the assumption

$$\psi(r) = T(r)\phi(S(r))\,, \tag{25}$$

we get, as before in Eq. (4) and (5),

$$T = S'^{-1/2}\,, \tag{26}$$

$$\hbar^2 \frac{T''}{T} - S'^2 p_2{}^2 + p_1{}^2 = 0\,. \tag{27}$$

With the WKB approximation

$$S = S_0 + \hbar^2 S_2 + \hbar^4 S_4 + \cdots\,, \tag{28}$$

we can obtain

$$T = T_0 + \hbar^2 T_2 + \hbar^4 T_4 + \cdots\,, \tag{29}$$

with

$$\begin{aligned} &T_0 = \frac{1}{S_0{'}^{1/2}}\,, \quad T_2 = -\frac{S_2{'}}{2S_0{'}^{3/2}}\,, \quad S_0{'} = \frac{p_1}{p_2}\,, \\ &S_2{'} = \frac{T_0{''}}{2S_0{'} p_2{}^2 T_0}\,, \quad \text{and} \quad S_4{'} = \frac{T_2{''}}{2S_0{'} p_2{}^2 T_0}\,. \end{aligned} \tag{30}$$

Since p_2 is taken here to be constant, we get, after some algebraic substitutions,

$$\begin{aligned} p_2 dS &= p_2(S_0{'} + \hbar^2 S_2{'} + \hbar^4 S_4{'})dr \\ &= p_1 dr + \hbar^2\left(-\frac{p_1{''}}{4p_1{}^2} + \frac{3p_1{'}^2}{8p_1{}^3}\right)dr \\ &\quad + \tfrac{1}{16}\hbar^4\left(\frac{p_1{''''}}{p_1{}^4} - 10\frac{p_1{'}p_1{'''}}{p_1{}^5} - \frac{13}{2}\frac{p_1{''}^2}{p_1{}^5}\right. \\ &\quad \left. - \frac{99}{2}\frac{p_1{'}^2 p_1{''}}{p_1{}^6} - \frac{297}{8}\frac{p_1{'}^4}{p_1{}^7}\right)dr\,. \end{aligned} \tag{31}$$

As before, we get,

$$\tfrac{1}{2}\oint\left(-\frac{p_1{''}}{4p_1{}^2} + \frac{3p_1{'}^2}{8p_1{}^3}\right)dr = -\tfrac{1}{16}\oint p_1{'}^2 p_1{}^{-3} dr\,, \tag{32}$$

and

$$\begin{aligned} &\tfrac{1}{32}\oint\left(\frac{p_1{''''}}{p_1{}^4} - \frac{10p_1{'}p_1{'''}}{p_1{}^5} - \frac{13}{2}\frac{p_1{''}^2}{p_1{}^5}\right. \\ &\quad \left. + \frac{99}{2}\frac{p_1{'}^2 p_1{''}}{p_1{}^6} - \frac{297}{8}\frac{p_1{'}^4}{p_1{}^7}\right)dr \\ &\qquad = -\tfrac{1}{64}\oint\frac{p_1{''}^2}{p_1{}^5}dr + \tfrac{15}{256}\oint\frac{p_1{'}^4}{p_1{}^7}dr \\ &\qquad = \frac{I_2}{\hbar^4}\,. \end{aligned} \tag{33}$$

Collecting terms, we get the first part of Eq. (24) as

$$\int_{S_1}^{S} p_2 dS = I_1 + I_2\,. \tag{34}$$

This completes the derivation of Eq. (23). Then the connection it gives between S and r give the phase shifts through Eq. (12).

IV. AN EXAMPLE: THE POTENTIAL γr^{-2}

To illustrate the method of evaluation we choose the potential γr^{-2}, where γ is a positive constant, to represent a repulsive potential. The reasons for this choice are threefold: First, it is of short range in comparison with the Coulomb potential. If we accept the fractional-order Bessel function as the solved part $\phi(S)$, we think that $\psi(r) = T(r)\phi(S(r))$ is valid here to order $\hbar^4$. Second, the integrations involved in the WKB approximation to order $\hbar^4$ are much simplified. And third, the exact phase shifts are given in Mott and Massey,[9] hence comparison can be readily made.

The exact phase shifts for the Schrödinger equation

$$\frac{d^2}{dr^2}(rR_L) + \left(k^2 - \frac{L(L+1)+\beta}{r^2}\right)(rR_L) = 0\,, \tag{35}$$

as given on page 40 of Ref. 8, are

$$\delta_L = \tfrac{1}{2}\pi(\nu - L)\,, \tag{36}$$

with $\nu(\nu+1) = L(L+1) + \beta$ and $\beta = (2m/\hbar^2)\gamma$. We are going to obtain the corresponding WKB results and then compare these results with the exact ones.

The corresponding solved equation, the fractional-order Bessel equation, was used previously.[10] We need only change one side from $L(L+1)$ to $L(L+1)+\beta$ to get the other side with the integrals remaining the same. It is seen that the following integral is valid:

$$\lim_{S\to\infty}\int_{[L(L+1)]^{1/2}}^{S}\frac{1}{\sigma}[\sigma^2-L(L+1)]^{1/2}d\sigma$$

$$=\lim_{S\to\infty}\left\{[S^2-L(L+1)]^{1/2}-[L(L+1)]^{1/2}\tan^{-1}\frac{[S^2-L(L+1)]^{1/2}}{[L(L+1)]^{1/2}}\right\}$$

$$=\lim_{S\to\infty}\{S-\tfrac{1}{2}\pi[(L+1)L]^{1/2}\}. \qquad (37)$$

We will make use of the following reduction formulas:

$$\oint\frac{dx}{x(a+bx+cx^2)^{7/2}}=\frac{1}{a}\oint\frac{dx}{x(a+bx+cx^2)^{5/2}}=\frac{1}{a^2}\oint\frac{dx}{x(a+bx+cx^2)^{3/2}}=\frac{1}{a^3}\oint\frac{dx}{x(a+bx+cx^2)^{1/2}}, \qquad (38)$$

and

$$\oint\frac{dx}{x(a+bx+cx^2)^{1/2}}=\frac{2\pi}{(-a)^{1/2}}\quad\text{for}\quad a<0. \qquad (39)$$

To order $\hbar^2$, we have, as before,[9] the phase shifts, $(\delta_S=0)$,

$$\delta_L^{(2)}=\lim_{\substack{S\to\infty\\ r\to\infty}}k(r-S)$$

$$=\tfrac{1}{2}\pi\{[L(L+1)+\beta]^{1/2}-[L(L+1)]+\tfrac{1}{8}[L(L+1)+\beta]^{-1/2}-\tfrac{1}{8}[L(L+1)]^{-1/2}\}. \qquad (40)$$

Remember

$$p_2{}^2=\hbar^2T_2=\frac{1}{\sigma^2}[\sigma^2-L(L+1)]\hbar^2,$$

$$T_2{}'=\frac{2L(L+1)}{\sigma^3},\quad T_2{}''=-\frac{6L(L+1)}{\sigma^4},$$

and (41)

$$T_2{}'''=\frac{24L(L+1)}{\sigma^5}.$$

We can prove, after several algebraic steps,

$$-\tfrac{1}{32}\hbar^4\oint p_2{}''^2p_2{}^{-5}d\sigma+\hbar^4\tfrac{15}{128}\oint p_2{}'^4p_2{}^{-7}d\sigma$$

$$=\hbar\left(-\tfrac{7}{1536}\oint T_2{}^{-7/2}T_2{}''^2d\sigma+\tfrac{5}{1536}\oint T_2{}^{-7/2}T_2{}'T_2{}'''d\sigma\right)$$

$$=\tfrac{1}{128}\hbar L^2(L+1)^2\left(-\oint\frac{d\sigma}{\sigma[\sigma^2-L(L+1)]^{7/2}}\right)$$

$$=\frac{\hbar}{128L(L+1)}\oint\frac{d\sigma}{\sigma[\sigma^2-L(L+1)]^{1/2}}$$

$$=\frac{\hbar}{128}\frac{2\pi}{[L(L+1)]^{3/2}}. \qquad (42)$$

Therefore, in accordance with the modified WKB method, the phase shifts are, to order $\hbar^4$,

$$\delta_L^{(4)}=\tfrac{1}{2}\pi\{[(L+1)L+\beta]^{1/2}-[(L+1)L]^{1/2}+\tfrac{1}{8}[L(L+1)+\beta]^{-1/2}-\tfrac{1}{8}[L(L+1)]^{-1/2}-\tfrac{1}{128}[L(L+1)+\beta]^{-3/2}+\tfrac{1}{128}[L(L+1)]^{-3/2}\}. \qquad (43)$$

We are going to identify the results given in Eq. (43) by writing Eq. (36) in the following way, where we let $L(L+1)=A$ so $L=\frac{1}{2}[-1+(1+4A)^{1/2}]$:

$$\delta_L=\tfrac{1}{2}\pi\{\tfrac{1}{2}[-1+(1+4L+4L^2+4\beta)^{1/2}]-L\}$$

$$=\tfrac{1}{2}\pi\{\tfrac{1}{2}[-1+(1+4A+4\beta)^{1/2}]-\tfrac{1}{2}[-1+(1+4A)^{1/2}]\}$$

$$=\tfrac{1}{2}\pi[\tfrac{1}{2}(1+4A+4\beta)^{1/2}-\tfrac{1}{2}(1+4A)^{1/2}]$$

$$=\tfrac{1}{2}\pi\left\{(A+\beta)^{1/2}\left(1+\frac{1}{4A+4\beta}\right)^{1/2}-A^{1/2}\left(1+\frac{1}{4A}\right)^{1/2}\right\}$$

$$=\tfrac{1}{2}\pi\left\{(A+\beta)^{1/2}\left[1+\frac{1}{8(A+\beta)}-\frac{1}{128(A+\beta)^2}+\cdots\right]-A^{1/2}\left[1+\frac{1}{8A}-\frac{1}{128A^2}+\cdots\right]\right\}$$

$$=\delta_L^{(4)}+\cdots. \qquad (44)$$

This can easily be identified as the result given in Eq. (43) given by the modified WKB perturbation theory.

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