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Finite Predictions from a Simple Nonrenormalizable Field Theory*

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A simple, nonrenormalizable quantum field theory containing a scalar field derivative-coupled to a nonconserved vector current of massive fermions is investigated. A method is presented which permits the calculation of S-matrix elements as a power series in a symmetry-violating mass difference Δm_0 instead of the usual series expansion in the coupling strength g: $\langle\alpha|S|\beta\rangle=\sum a_n(\Delta m_0)^n$. It is shown that the resulting scattering matrix is unitary and that for $n\leq 4$ all of the coefficients in this new expansion are explicitly finite. It is not yet known whether the proposed method yields finite coefficients for all n. Finally, it is noted that these techniques can be applied to a number of other nonrenormalizable theories.

I. INTRODUCTION

In the following we study a simple nonrenormalizable field theory describing a coupled system of scalar and spin-$\frac{1}{2}$ particles specified by the Lagrangian density[1]

$$\mathcal{L}(x)=-\frac{1}{2}\left(\frac{\partial\phi}{\partial x_\mu}(x)\right)^2-\bar{l}'(x)\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+m_0+\Delta m_0'\tau_3\right)l'(x)-ig\frac{\partial\phi}{\partial x_\mu}(x)\bar{l}'(x)\gamma_\mu\tau_1 l'(x),\tag{1}$$

where

$$l'(x)=\begin{pmatrix}l_1'(x)\\ l_2'(x)\end{pmatrix}$$

is an eight-component spinor field representing an isotopic doublet of spin-$\frac{1}{2}$ particles, $\phi(x)$ is a scalar field, and τ_i are the usual 2×2 Pauli matrices.[2] Our investigation is motivated by the possibility that the weak interactions may obey a similar (although considerably more complicated) field theory.

If the amplitude for a particular process predicted by the Lagrangian (1) is computed as a power series in the coupling constant g, divergent integrals are encountered in all but the lowest-order terms. Because of the derivative coupling appearing in $\mathcal{L}(x)$ these infinities are not as manageable as those found, for example, in quantum electrodynamics. In particular, the theory cannot be made finite by the introduction of a finite number of renormalization constants. In this paper we attempt to avoid these difficulties by carrying out systematically, a partial summation of the usual perturbation series.[3] The result of this process is an essentially unambiguous, unitary scattering matrix expressed as a power series in the mass difference Δm_0,

$$\langle\alpha|S|\beta\rangle=\sum_{n=0}^{\infty}a_n(\Delta m_0)^n\ .\tag{2}$$

The coefficients in this expansion are explicitly found to be finite for $n\leq 4$. The finiteness of higher-order terms has not yet been established.

We begin by making a canonical transformation[4] on the field variables appearing in Eq. (1). In terms of

$$l(x)=e^{ig\tau_1\phi(x)}l'(x),\tag{3}$$

the Lagrangian density becomes

$$\mathcal{L}(x)=-\frac{1}{2}\left(\frac{\partial\phi}{\partial x_\mu}(x)\right)^2-\bar{l}(x)\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+m_0\right)l(x)-\Delta m_0'\bar{l}(x)e^{ig\tau_1\phi(x)}\tau_3e^{-ig\tau_1\phi(x)}l(x)\ .\tag{4}$$

Let us further simplify this Lagrangian by normal-ordering the product of exponentials of the scalar field using the formula

$$e^{\pm ig\phi(x)}=e^{\pm ig\phi^-(x)}e^{\pm ig\phi^+(x)}e^{-g^2[\phi^+(x),\phi^-(x)]/2},\tag{5}$$

where

$$\phi(x)=\phi^+(x)+\phi^-(x),$$

$\phi^+(x)$ and $\phi^-(x)$ being the usual annihilation and creation operators. If the divergent constant $\exp\{-\frac{1}{2}g^2[\phi^+(x),\phi^-(x)]\}$ is absorbed by the mass renormalization

$$\Delta m_0=\Delta m_0'e^{-2g^2[\phi^+(x),\phi^-(x)]},\tag{6}$$

our Lagrangian becomes

$$\mathcal{L}(x)=-\frac{1}{2}\left(\frac{\partial\phi}{\partial x_\mu}\right)^2-\bar{l}(x)\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+M\right)l(x)+\bar{l}(x)\delta M l(x)-\Delta m_0\bar{l}(x):(e^{ig\phi(x)\tau_1}\tau_3e^{-ig\phi(x)\tau_1}-\tau_3):l(x).\tag{7}$$

The physical mass matrix $M=m+\Delta m\tau_3$ retains the original diagonal form because of the symmetry of our Lagrangian under the discrete operation

$$\phi(x)\to-\phi(x),\qquad l(x)\to e^{i\pi\tau_3/2}l(x).\tag{8}$$

The standard mass counterterm $\bar{l}(x)\delta M l(x)$ follows from

$$\delta M = (m - m_0) + (\Delta m - \Delta m_0)\tau_3 . \tag{9}$$

The physical mass of the scalar particle is zero because of symmetry under the operation

$$\phi(x) \to \phi(x) + c , \quad l(x) \to e^{ig\tau_1 c} l(x) \tag{10}$$

for any real constant c, possessed by the Lagrangian (4). If we treat the final two terms in Eq. (7) as the interaction Lagrangian and transform to the interaction picture, we obtain the following Dyson-Wick expansion for the scattering matrix S:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} T\left\{ \prod_{l=1}^{n} \int d^4x_l \mathcal{H}_I(x_l) \right\}, \tag{11}$$

where

$$\mathcal{H}_I(x) = \Delta m_0 \bar{l}(x) : (e^{ig\tau_1\phi(x)}\tau_3 e^{-ig\tau_1\phi(x)} - \tau_3) : l(x) - \bar{l}(x)\delta M l(x). \tag{12}$$

The fields $\phi(x)$ and $l(x)$ are now fields in the interaction representation having the time dependence of free fields,

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + M\right) l(x) = 0, \quad \Box\phi = 0. \tag{13}$$

In Sec. II a graphical analysis of the time-ordered product of n operators $\mathcal{H}_I(x)$ found in the expansion (11) is presented. Modified Feynman rules are deduced which allow an amplitude expanded to a finite order in Δm_0 (but to all orders in g) to be represented by a finite number of graphs. Each term in the amplitude corresponding to a given graph is written as a sum of products of the singular functions [5]

$$e^{\pm i4g^2\Delta_F(x)}, \quad S_F(x;m)e^{\pm i4g^2\Delta_F(x)}, \quad \text{and} \quad \mathrm{tr}[S_F(x;m)S_F(-x;m')]\, e^{\pm i4g^2\Delta_F(x)} . \tag{14}$$

Such an nth-order amplitude is an explicit function of n coordinates $x_1, \ldots, x_n$ and possesses essential singularities when $(x_i - x_j)^2 = 0$. Determination of the physical scattering amplitude requires integration over these coordinates and a method for dealing with the singularities.

The expansion in Δm_0 can now be discussed in detail using the framework established in Sec. II. To first order in Δm_0 the scattering amplitudes, given by matrix elements of $\int d^4x \mathcal{H}_I(x)$, are quite simple since no loop integrations occur. However, the second-order amplitudes are proportional to the Fourier transforms of the singular functions (14) and, therefore, require careful consideration. These second-order amplitudes are studied in Sec. III. There it is found that the singular functions (14) can be Fourier-transformed provided the factor $\pm g^2$ in the exponent has a negative real part. Thus all the second-order scattering amplitudes fall into one of two categories: (a) those defined for real values of g, and (b) those defined for unphysical, imaginary values of g. The amplitudes of type (a) are unitary and except for exponential high-energy growth are quite satisfactory. However, the amplitudes of type (b) require additional attention. Simple analytic continuation of these amplitudes from imaginary to real values of g is complicated by the presence of a logarithmic branch point at $g = 0$. Consequently, each amplitude defined for imaginary g is a multivalued function of g being properly defined on a Riemann surface possessing many sheets. For real g we must then choose among a number of independent amplitudes – each corresponding to the original amplitude evaluated on a different Riemann sheet. We will assume that each physical amplitude for real g of the type (b) can be written as a linear combination of these various continued amplitudes. This assumption [6] and the requirements of unitarity determine the entire second-order scattering matrix as a function of the parameters m_0, Δm_0, g, and six additional real constants. It should be noted that the methods and results of Secs. II and III are similar to the previous work of many authors, in particular that of Volkov.

In Sec. IV, these techniques are extended to higher-order processes. Following the graphical analysis of Sec. II, we attempt to construct high-order amplitudes as the integral of products of lower-order amplitudes. It is found that again, to both third and fourth order in Δm_0, all amplitudes fall into one of two groups: (a) those amplitudes that can be constructed as convergent integrals of products of lower-order amplitudes for real values of g and (b) those which can be so defined only for imaginary g. The possibility of such a classification to arbitrary order in Δm_0 has not yet been established. Nevertheless, we will assume that such a division can be made. Again both types of amplitudes are found to possess logarithmic singularities in g^2 – this time of the type $\ln^N g^2$. Those amplitudes of the type (a) are shown to be unitary and, hence, satisfactory predictions of the theory. We propose to treat amplitudes of the type (b) just as was done in second order. Each physical amplitude of the type (b) is written as a linear combination of the independent amplitudes found by continuing from imaginary to real g along various paths. It is shown that the imposition of unitarity essentially eliminates the ambiguity inherent in such a procedure so that the entire S matrix is determined by this technique as a function of the parameters m_0, Δm_0, g, and a single additional real number b.

The proposed method of calculation is not a simple one; it requires summation in position space before integration and intricate analytic continuation in the coupling constant. However, the result is a complete prescription for calculation in a nonrenormalizable field theory which does not require the introduction of an infinite number of parameters. The techniques developed are quite general and can be applied to many theories in which the nonrenormalizable part of the interaction can be written in exponential form. In particular, the application of these techniques to a theory of exponentially coupled scalar fields and to a simple nonrenormalizable vector-boson theory is discussed in Appendix E.

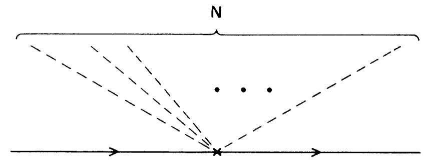

FIG. 1. A diagram representing the amplitude for a general scattering process involving N scalar and two spin-$\frac{1}{2}$ particles computed to first order in Δm_0. The dashed lines represent incident or scattered scalar particles and the solid lines represent spin-$\frac{1}{2}$ particles.

II. GRAPHICAL ANALYSIS

The perturbation series in Δm_0 specified by Eq. (11) has a surprisingly simple graphical representation which is developed below. We postpone a discussion of the ultraviolet divergences present in this expansion by omitting the integrals $\int d^4x_i$. Consequently the amplitudes investigated in this section will be explicit functions of n coordinates $x_1, \ldots, x_n$. A physical scattering amplitude is obtained from these amplitudes only if the coordinates $x_1, \ldots, x_n$ are integrated over and the difficulties raised by the singularities at $(x_i - x_j)^2 = 0$ overcome.

To first order in Δm_0 the scattering amplitude for a particular process is given directly by the matrix elements of $\mathcal{H}_I(x)$. The scattering amplitude A_N for a process involving N scalar particles and two fermions is

$$A_N = \Delta m_0 (2g)^N \overline{U}_2 U_1 [(\tau_3)_{\alpha_2\alpha_1} \cos(N\tfrac{1}{2}\pi) + (\tau_2)_{\alpha_2\alpha_1} \sin(N\tfrac{1}{2}\pi)], \tag{15}$$

where U_i is the four-component Dirac spinor and α_i the isospin index for the ith fermion. Such an amplitude is represented by the graph in Fig. 1. Reactions without two external fermions are possible to first order in the expansion (11), but are proportional to Δm and are therefore treated with the second-order amplitudes; see Ref. 11 in Sec. III.

Let us begin the discussion of higher-order processes by examining the time-ordered product of the scalar field $\phi(y)$ with $\mathcal{H}_I(x)$.[7] It is easy to see that

$$\begin{aligned} T\{\phi(y)\bar{l}(x)(:e^{ig\phi(x)\tau_1}\tau_3 e^{-ig\phi(x)\tau_1}: - \tau_3) l(x)\} &= 2gi\Delta_F(x-y)\bar{l}(x) : e^{ig\phi(x)\tau_1}\tau_2 e^{-ig\phi(x)\tau_1} : l(x) \\ &\quad + \bar{l}(x) : \phi(y)(e^{ig\phi(x)\tau_1}\tau_3 e^{-ig\phi(x)\tau_1} - \tau_3) : l(x)\,. \end{aligned} \tag{16}$$

Likewise the contraction of N fields $\phi(y_i)$ with $\mathcal{H}_I(x)$ will contain N factors of $2gi\Delta_F(y_i - x)$ and the 2×2 matrix $\tau_3 \cos(\pi\frac{1}{2}N) + \tau_2 \sin(\pi\frac{1}{2}N)$. Thus, if we consider the time-ordered product of $\mathcal{H}_I(x_1)$ with $\mathcal{H}_I(x_2)$ and apply Wick's theorem, the term with N scalar fields contracted is given by

$$[4g^2 i\Delta_F(x_1 - x_2)]^N \frac{1}{N!} \bar{l}(x_1)_{\alpha_1} l(x_1)_{\beta_1} \bar{l}(x_2)_{\alpha_2} l(x_2)_{\beta_2} : V_j(x_1)_{\alpha_1\beta_1} V_j(x_2)_{\alpha_2\beta_2} : ,$$

where

$$V_j(x)_{\alpha\beta} = (e^{ig\phi(x)\tau_1}\tau_j e^{-ig\phi(x)\tau_1})_{\alpha\beta} \tag{17}$$

and

$$i = \tfrac{5}{2} + \tfrac{1}{2}(-1)^N.$$

The factor $(N!)^{-1}$ is introduced to eliminate double counting. Summing over N, we obtain

$$\begin{aligned} T\{\mathcal{H}_I(x_1)\mathcal{H}_I(x_2)\} = \bar{l}(x_1)_{\alpha_1} l(x_1)_{\beta_1} \bar{l}(x_2)_{\alpha_2} l(x_2)_{\beta_2} (\Delta m_0)^2 \big(&: [V_3(x_1)_{\alpha_1\beta_1} - (\tau_3)_{\alpha_1\beta_1}][V_3(x_2)_{\alpha_2\beta_2} - (\tau_3)_{\alpha_2\beta_2}] : \\ &+ \{\cosh[4ig^2\Delta_F(x_1 - x_2)] - 1\} : V_3(x_1)_{\alpha_1\beta_1} V_3(x_2)_{\alpha_2\beta_2} : \\ &+ \sinh[4ig^2\Delta_F(x_1 - x_2)] : V_2(x_1)_{\alpha_1\beta_1} V_2(x_2)_{\alpha_2\beta_2} : \big)\,. \end{aligned} \tag{18}$$

Hence, a general second-order amplitude can be represented by a graph with two vertices. These graphs are of three types as shown in Figs. 2(a), 2(b), and 2(c) corresponding to processes with a total number of zero, two, or four fermions in the initial of final states. The three graphs in each category represent the separation of the right-hand side of Eq. (18) into terms depending on the boson propagator $\Delta_F(x_1-x_2)$ as 1, $\cosh[4ig^2 \times\Delta_F(x_1-x_2)]-1$, and $\sinh[4ig^2\Delta(x-y)]$, where the cosh −1 and sinh factors are represented by lines labeled e and o, respectively. The factor corresponding to each vertex in Fig. 2 is $A_N(2g)^{N'-N}$ where A_N is defined by Eq. (15), N' is the number of external scalar lines attached to that vertex and N is the number of external scalar lines plus the number of o-type internal lines attached to that vertex. Of course, multiplication by the spinors U_i and $\bar{U}_j$ is replaced by the appropriate spin-$\frac{1}{2}$ propagator $iS_F(x_i-x_j)$ if the corresponding fermion line happens to be internal.

A general amplitude of nth order in Δm_0 is obtained from the time-ordered product of n operators $\mathcal{H}_I(x_i)$. Such an amplitude can be determined by the same procedure as was applied above in the second-order case. One must systematically sum all possible contractions of scalar fields between each pair of operators $\mathcal{H}_I(x_i)$ and $\mathcal{H}_I(x_j)$. Thus, a general nth-order amplitude will be represented by a graph with n vertices where each pair of vertices is connected by (a) no boson propagator, (b) an e line, or (c) an o line. Two fermion lines, either internal or external, must be joined to each vertex. Again to each vertex there corresponds

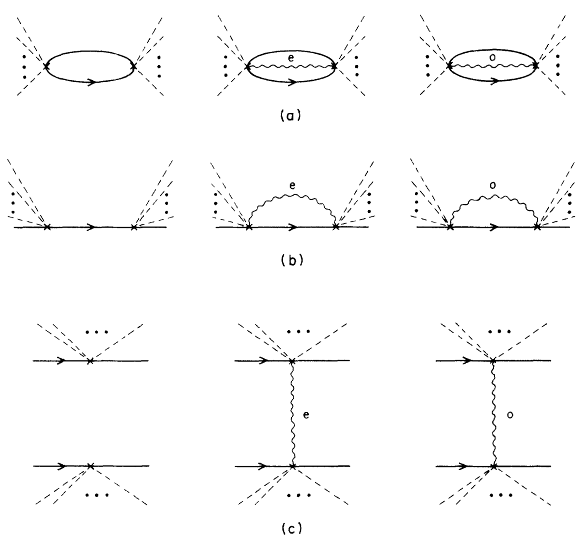

FIG. 2. Diagrams corresponding to the amplitude for a general second-order scattering process involving the interaction of scalar particles with (a) zero, (b) two, and (c) four spin-$\frac{1}{2}$ particles.

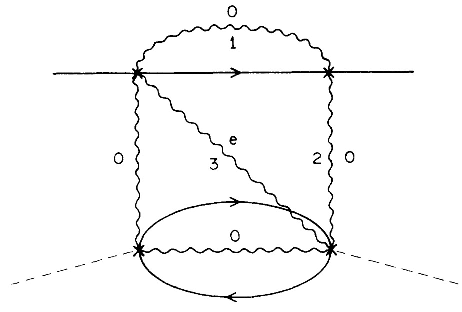

FIG. 3. A typical fourth-order graph representing one term in the scalar-fermion elastic scattering amplitude.

the factor $A_N(2g)^{N'-N}$ described above; while to each wavy line connecting vertex i with vertex j there corresponds the factor $\cosh[4g^2 i\Delta_F(x_i-x_j)]-1$ or $\sinh[4g^2 i\Delta_F(x_i-x_j)]$ for cases (b) and (c), respectively. A typical fourth-order scalar-fermion elastic scattering graph is shown in Fig. 3.

An alternative classification of these amplitudes which will be useful later decomposes a particular amplitude into separate terms according to the arrangement of the functions $\exp[\pm 4ig^2\Delta_F(x_i-x_j)]-1$ rather than the occurrence of even or odd powers of the quantity $4ig^2\Delta_F(x_i-x_j)$. To obtain this representation for a particular nth-order amplitude, we need only take all the terms corresponding to graphs with the same topology but different assignments of e and o to the internal lines, split each hyperbolic cosine or sine into the usual sum of two exponentials, and group together all terms containing a given combination of the functions $\exp[\pm 4ig^2\Delta_F(x_i-x_j)]-1$. This new arrangement of our amplitude has a graphical representation identical to that just described – we need only label each internal line connecting the vertices i and j with the symbol plus (+) or minus (−) to specify whether the particular term represented by the graph contains the factor $\exp[+i4g^2\Delta_F(x_i-x_j)]-1$ or $\exp[-4ig^2\Delta_F(x_i-x_j)]-1$. Thus, the particular graph for second-order fermion-fermion elastic scattering shown in Fig. 4(a) represents the position-space amplitude

$$-i(\Delta m_0)^2\bar{U}_3U_1\bar{U}_4U_2(e^{-4g^2 i\Delta_F(x_1-x_2)}-1)\times[(\tau_3)_{\alpha_3\alpha_1}(\tau_3)_{\alpha_4\alpha_2}-(\tau_2)_{\alpha_3\alpha_1}(\tau_2)_{\alpha_4\alpha_2}], \quad (19)$$

where U_i is the four-component Dirac spinor and α_i the isotopic spin index for the ith fermion.

Although the arrangement of Pauli matrices corresponding to a single graph of the plus, minus type is considerably more complicated than that for the first graphical representation discussed, the second description does obey a simple rule which limits the possible distributions of plus and

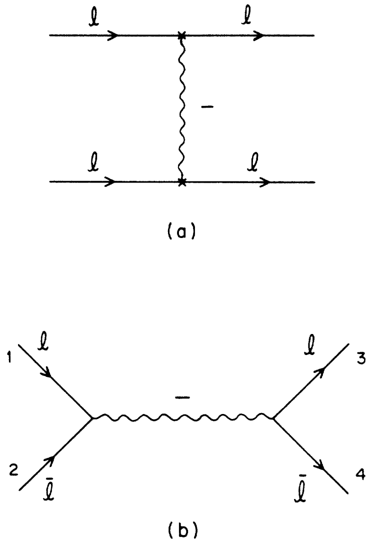

FIG. 4. Diagrams representing (a) fermion-fermion elastic scattering and (b) fermion-antifermion elastic scattering to second order in Δm_0.

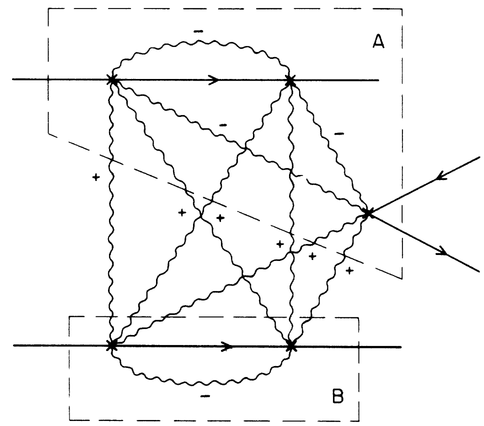

FIG. 5. A fifth-order graph containing the maximum number of wavy lines illustrating the division of vertices into two sets A and B as described in Sec. II.

minus lines in graphs of a given topology. First, let us consider any subset of wavy lines in a particular graph which form a closed polygon; for example, lines 1, 2, and 3 in Fig. 3. In the representation specifying sinh and cosh −1 factors we can exchange all the o lines with e lines in the polygon and leave the Pauli matrix structure of the vertices unchanged. In fact, the amplitudes represented by the two graphs would be identical except for the exchange of the factors −sinh and cosh − 1 corresponding to the lines in the polygon. [The extra minus sign preceding the hyperbolic sign factor is necessitated by the trigonometric functions in Eq. (15).] Thus, when all the amplitudes corresponding to graphs of such a topology are decomposed and added to form amplitudes arranged according to the distribution of the factors $\exp[\pm 4g^2 \times \Delta_F(x_i - x_j)] - 1$, the only amplitudes which occur will be those containing an even number of plus lines in that polygon. Hence, in our second graphical representation, all closed polygons of wavy lines must contain an even number of plus lines. This rule allows a simple enumeration of all graphs with a given topology. First consider a graph with n vertices connected in all possible ways by wavy lines [there must be $\frac{1}{2}n(n-1)$ such lines] with plus and minus lines distributed in some particular fashion. Pick one vertex, a, and let A be the set of all vertices connected to a by an unbroken chain of minus lines (A includes a). Clearly, all the vertices in A are connected to each other by minus lines. Choose a second vertex, b, not in A and let B be the set of all vertices connected to b by an unbroken chain of minus lines (B includes b). Again all the vertices of B must be connected to each other by minus lines while any line connecting a vertex in A with a vertex in B must be of the plus type. The two sets A and B must include all vertices since if there existed a third vertex c not in A or B the triangle with vertices a, b, and c would contain an odd number of plus lines. Thus, the vertices of a general graph containing the maximum number of wavy lines naturally divide into two sets A and B with the above properties. Such a grouping is illustrated for the fifth-order graph shown in Fig. 5. It is not difficult to see that an arbitrary nonvanishing graph can be obtained from a nonvanishing graph with the maximum number of wavy lines by the deletion of a subset of those lines.

In summary, the results of this section demonstrate that the explicit summation of a general scattering amplitude to all orders in g but nth order in Δm_0 is quite easy provided the coordinates $x_1, \ldots, x_n$ in Eq. (11) are not integrated over. Subsequent integration over these variables and investigation of the difficulties associated with the singularities at $(x_i - x_j)^2 = 0$ is carried out in Secs. III and IV.

III. SECOND-ORDER PROCESSES

In this section we will complete the determination of the second-order scattering matrix begun in Sec. II, by performing the coordinate-space integrations over x_1 and x_2. Carrying out these integrations amounts to evaluating the Fourier transforms of the singular functions (14). Let us begin by considering the functions which correspond to the plus and minus wavy lines in momentum space,

$$\frac{ig^4}{\pi^2} B_{\pm}\left(\frac{g^2p^2}{4\pi^2}\right) = \int d^4x\, e^{-ip\cdot x}\left(e^{\pm 4ig^2\Delta_F(x^2)} - 1\right), \tag{20}$$

for spacelike p^2 in a reference frame where $p_0 = 0$. We will first examine the x_0 integration, choosing the usual contour passing above the $-|\vec{x}|$ singularity and below the $+|\vec{x}|$ singularity. The large-x_0 behavior of the integrand in Eq. (20) allows the rotation of the x_0 contour to the imaginary axis. The angular part of the resulting four-dimensional Euclidean integration can be performed with the result

$$\frac{ig^4}{\pi^2} B_{\pm}\left(\frac{g^2p^2}{4\pi^2}\right) = -\frac{2\pi^2}{p^2} i \int_0^\infty dx^2 (x^2p^2)^{1/2} J_1((x^2p^2)^{1/2})\left(e^{\pm(g^2/\pi^2)(1/x^2)} - 1\right). \tag{21}$$

The function $B_-(g^2p^2/4\pi^2)$ is clearly well defined by Eq. (21), provided g^2 is positive – the initial summation in coordinate space having damped the singularities of the individual terms. The function $B_+(g^2p^2/4\pi^2)$ is not determined by Eq. (21) unless we consider unphysical, imaginary g. If we define $\overline{B}_+(g^2p^2/4\pi^2)$ as the function determined by Eq. (21) with the plus sign but negative g^2, then

$$\overline{B}_+(g^2p^2/4\pi^2) = B_-(-g^2p^2/4\pi^2) \tag{22}$$

for negative g^2.

The properties of the function $B_-(z)$ are determined in Appendix A. There it is shown that $B_-(z)$ has the following series expansion:

$$\begin{aligned} B_-(z) &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(z)^n}{(n+2)!(n+1)!n!} \tfrac{1}{2}(\ln z + 3\gamma - \tfrac{5}{2}) - \sum_{n=1}^{\infty} \frac{z^n}{(n+2)!(n+1)!n!}\left(\sum_{j=1}^{n} \frac{3j^2+6j+2}{(j+2)(j+1)j}\right) \\ &= \frac{1}{z} + f_1(z)\ln z + f_2(z), \end{aligned} \tag{23}$$

where $\gamma = 0.5772\ldots$ is Euler's constant, $f_1(z)$ and $f_2(z)$ are analytic functions of z. The asymptotic behavior for large positive z is found to be

$$B_-(z) \underset{z\to+\infty}{\sim} -\frac{e^{-3z^{1/3}/2}}{\sqrt{3}\, z^{4/3}}\left[\left(1 + \frac{8}{9}\frac{1}{z^{1/3}} + \cdots\right)\sqrt{3}\cos\left(\frac{3\sqrt{3}\, z^{1/3}}{2}\right) + \left(1 - \frac{8}{9}\frac{1}{z^{1/3}} + \cdots\right)\sin\left(\frac{3\sqrt{3}\, z^{1/3}}{2}\right)\right]. \tag{24}$$

If z is continued to large negative values by passing above/below the logarithmic branch point at the origin, we find

$$B_-(z) \underset{z\to-\infty\pm i\epsilon}{\sim} \frac{e^{\mp i\pi/6}}{\sqrt{3}\,|z|^{4/3}} e^{+3|z|^{1/3}/2} e^{\mp i|z|^{1/3}3\sqrt{3}/2}. \tag{25}$$

Thus, for real g and spacelike p^2, $B_-(g^2p^2/4\pi^2)$ is real and hence diagrams of the type shown in Fig. 4(a) will represent unitary amplitudes. As in the conventional case we can obtain amplitudes for timelike p^2 by continuing p^2 from spacelike to timelike values through the lower half-plane.[8] The resulting imaginary part of B_- is

$$\operatorname{Im} B_-\left(\frac{g^2(p^2 - i\epsilon)}{4\pi^2}\right) = -\pi \sum_{n=0}^{\infty} \frac{(g^2p^2/4\pi^2)^n}{(n+2)!(n+1)!n!}, \tag{26}$$

which is consistent with unitarity for processes of the sort

$$l + \bar{l} \to l + \bar{l} \tag{27}$$

illustrated in Fig. 4(b). In terms of $B_-(z)$ the scattering amplitude[9] represented by this graph is

$$T_{l_1\bar{l}_2 - l_3\bar{l}_4} = \overline{V}_2 U_1 \overline{U}_3 V_4 [(\tau_3)_{\alpha_2\alpha_1}(\tau_3)_{\alpha_3\alpha_4} - (\tau_2)_{\alpha_2\alpha_1}(\tau_2)_{\alpha_3\alpha_4}](\Delta m_0)^2 \frac{2g^4}{(2\pi)^4} B_-\left(\frac{g^2(p^2 - i\epsilon)}{4\pi^2}\right)\delta^4(p_3 + p_4 - p_1 - p_2), \tag{28}$$

where $-p^2$ is the square of the center-of-mass energy, U_1, U_3, V_2, V_4 and p_1, p_2, p_3, p_4 are the appropriate particle and antiparticle spinors or four-momenta and the α_i their isospin indices. The product of the lowest-order amplitude for the annihilation

$$l_1+\bar{l}_2\to n\ \phi\text{'s} \tag{29}$$

with the complex conjugate of the amplitude for

$$l_3+\bar{l}_4\to n\ \phi\text{'s} \tag{30}$$

integrated over n-body phase space yields

$$\int d\Omega_n T_{l_1\bar{l}_2\to n\phi}T^*_{l_3\bar{l}_4\to n\phi}=(\Delta m_0)^2\frac{4g^4}{(2\pi)^3}\delta^4(p_3+p_4-p_1-p_2)\frac{(-g^2p^2/4\pi^2)^{n-2}}{n!(n-1)!(n-2)!}$$
$$\times\bar{V}_2U_1\bar{U}_3V_4[(\tau_3)_{\alpha_2\alpha_1}(\tau_3)_{\alpha_4\alpha_3}\cos^2(\tfrac{1}{2}n\pi)-(\tau_2)_{\alpha_2\alpha_1}(\tau_2)_{\alpha_4\alpha_3}\sin^2(\tfrac{1}{2}n\pi)]. \tag{31}$$

By comparing Eqs. (26), (28), and (31), it can be seen that the amplitude proportional to the isospin matrix $(\tau_3)_{\alpha_2\alpha_1}(\tau_3)_{\alpha_3\alpha_4}-(\tau_2)_{\alpha_2\alpha_1}(\tau_2)_{\alpha_3\alpha_4}$ satisfies

$$T_{l_1\bar{l}_2\to l_3\bar{l}_4}-T^*_{l_3\bar{l}_4\to l_1\bar{l}_2}=-i\sum_{n=2}^{\infty}\int d\Omega_n T_{l_1\bar{l}_2\to n\phi}T^*_{l_3\bar{l}_4\to n\phi} \tag{32}$$

as required by unitarity.

Now let us turn to the problem of defining $B_+(g^2p^2/4\pi^2)$ for spacelike p^2 and $g^2>0$. Simple analytic continuation of $\bar{B}_+(g^2p^2/4\pi^2)$, defined for negative g^2 by Eq. (22), from negative to positive values of g^2 is made difficult by the logarithmic dependence on g^2 revealed in Eq. (23). If we define $[\bar{B}_+(g^2p^2/4\pi^2)]_n$ for n, a positive or negative integer, by continuing $\bar{B}_+(g^2p^2/4\pi^2)$ from negative g^2 counterclockwise n times about the $g^2=0$ branch point, even an arbitrary choice

$$B_+(g^2p^2/4\pi^2)=[\bar{B}_+(g^2p^2/4\pi^2)]_n \tag{33}$$

will not be real and hence inconsistent with unitarity. In order to define a unitary $B_+(g^2p^2/4\pi^2)$ we will generalize the continuation procedure of Eq. (33), postulating

$$B_+\left(\frac{g^2p^2}{4\pi^2}\right)=\sum_{n=-\infty}^{\infty}a_n[\bar{B}_+(g^2p^2/4\pi^2)]_n$$
$$=a\left[-\frac{4\pi^2}{g^2p^2}+f_2\left(-\frac{g^2p^2}{4\pi^2}\right)+\ln\left(\frac{g^2p^2}{4\pi^2}\right)f_1\left(-\frac{g^2p^2}{4\pi^2}\right)\right]+bf_1\left(-\frac{g^2p^2}{4\pi^2}\right), \tag{34}$$

where

$$a=\sum_{n=-\infty}^{\infty}a_n\,,\qquad b=\sum_{n=-\infty}^{\infty}2\pi(n+\tfrac{1}{2})ia_n\,.$$

Unitarity requires that $a=1$ and that b is real. This method for defining B_+, one among many alternatives, is of particular interest because of its general applicability to similar situations occurring in higher-order processes – as will be shown in Sec. IV.

The Fourier transforms of the other functions listed in (14) can be obtained in the same manner that the functions $B_\pm$ were determined. Let

$$F_\pm(g^2p^2,m^2/p^2)\not{p}+F'_\pm(g^2p^2,m^2/p^2)m=\frac{1}{g^2}\int d^4x\,e^{-ip\cdot x}S_F(x;m)e^{\pm 4ig^2\Delta_F(x)}$$

and

$$A_\pm(g^2p^2,m^2/p^2,m'^2/p^2)=-ig^2\int d^4x\,e^{-ip\cdot x}\mathrm{tr}[S_F(x;m)S_F(-x;m')]e^{\pm 4ig^2\Delta_F(x)}. \tag{35}$$

The functions F_-, F'_- enter, for example, the Compton amplitude represented in Fig. 6(a), while A_+ is required for the boson-boson elastic scattering amplitude corresponding to the graphs[10] shown in Figs. 6(b) and 6(c).[11] The functions F_-, F'_-, and A_- are defined directly by Eq. (35) for positive g^2 and the same choice of x_0 contour made in the derivation of Eq. (21). The resulting amplitudes are unitary and possess logarithmic singularities at the point $g^2=0$. In particular, the functions F_- and F'_- have the form $\mathfrak{F}_2(g^2)\ln^2g^2+\mathfrak{F}_1(g^2)\ln g^2+\mathfrak{F}_0(g^2)-1/g^2(p^2+m^2)$ where $\mathfrak{F}_0$, $\mathfrak{F}_1$, and $\mathfrak{F}_2$ are analytic functions of g^2 while A_- contains an additional term proportional to $\ln^3g^2$. The functions F_+, F'_+, and A_+, involving the exponent $+g^2/\pi^2x^2$, are determined by a continuation procedure identical to that summarized in Eq. (34). In this way five more constants are introduced into our sec-

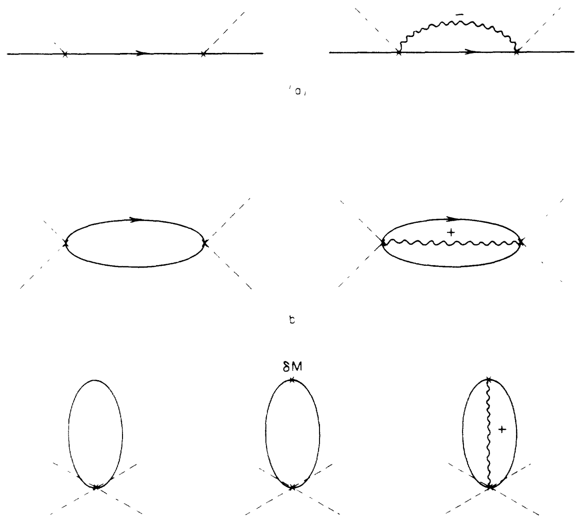

FIG. 6. Diagrams representing amplitudes determined by the functions (a) $F_-(g^2p^2, m^2/p^2)$, $F_-'(g^2p^2, m^2/p^2)$ and (b), (c) $A_\pm(g^2p^2, m^2/p^2)$ defined in Sec. III.

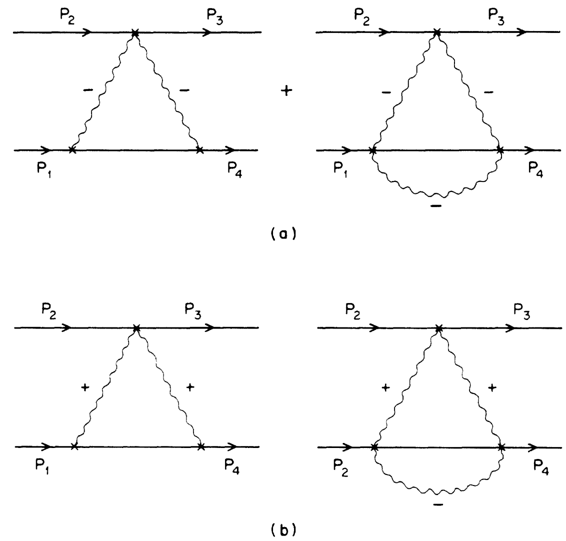

FIG. 7. Graphs representing two third-order amplitudes discussed in Sec. IV.

ond-order theory if F_+ and F_+' are treated identically. The functions $F_\pm$, $F_\pm'$, and $A_\pm$ are studied briefly in Appendix A where the single logarithmic singularity in g^2 is established and the asymptotic behavior for large p^2 or g^2 shown to be similar to that of $B_\pm(g^2p^2/4\pi^2)$.

Thus, by adopting the analytic-continuation procedure displayed in Eq. (34) we have determined all scattering amplitudes to order $(\Delta m_0)^2$ as a function of m_0, Δm_0, g, and six additional parameters.

IV. HIGHER-ORDER PROCESSES

We will now try to extend the methods introduced in Sec. III to the calculation of matrix elements of higher order in Δm_0. The analysis of Sec. II shows that such amplitudes can be written, at least formally, as the integral of products of lower-order amplitudes. For example, the third-order graphs of Fig. 7 represent two amplitudes which can be written in terms of the integrals

$$\int d^4k\,\overline{U}_5[F_-(g^2k^2, m_i^2/k^2)\not{k} + F_-'(g^2k^2, m_i^2/k^2)m_i]U_1 \times B_\sigma\!\left(\frac{g^2(k-p_1)^2}{4\pi^2}\right) B_\sigma\!\left(\frac{g^2(k+p_3-p_2-p_1)^2}{4\pi^2}\right) \qquad (36)$$

for $i=1$ and 2, $\sigma=\pm$, $m_1 = m+\Delta m$, and $m_2 = m-\Delta m$. In most cases the expressions obtained this way contain exponentially diverging integrals so that only particular amplitudes can be directly defined in terms of lower-order amplitudes. This situation is illustrated in third order by Eq. (36). For the case of Fig. 7(a) and $\sigma = -$, the asymptotic forms shown in Eqs. (24) and (A3) indicate that the Euclidean[8] integration $\int d^4k$ converges so that Eq. (36) defines the required amplitude for positive g^2. On the other hand, for Fig. 7(b) and $\sigma=+$ the asymptotic behavior of B_+ for positive argument, as indicated by Eqs. (22), (25), and (34), is so strong that the integral in Eq. (36) diverges for positive g^2. We may attempt to obtain finite values for these diverging amplitudes by considering imaginary values of the coupling constant, as was done in Sec. III. If a particular amplitude can be defined for imaginary g, then the physical amplitude might be obtained by the continuation procedure adopted in Sec. III.

Let us examine this possibility in third order by investigating the amplitude represented in Fig. 7(b) for negative g^2. In fact, it is possible to write this amplitude as a convergent integral of products of second-order amplitudes defined for negative g^2. If g^2 is negative, the two lines labeled plus represent the function $\overline{B}_+$ defined directly for negative g^2 in Sec. III. Although the amplitude represented by the combination of the third wavy line and the fermion line is defined directly by Eq. (35) only for a positive real part of g^2, we can try to obtain the appropriate amplitudes $\overline{F}_i$ and $\overline{F}_i'$ for negative g^2 by a continuation procedure of the sort specified in Eq. (34). Thus we can determine the amplitude represented in Fig. 7(b) for negative g^2 from the integrals

$$\int d^4k\,\overline{U}_5[\overline{F}_-(g^2k^2, m_i^2/k^2)\not{k} + \overline{F}_-'(g^2k^2, m_i^2/k^2)m_i]U_1 \times \overline{B}_+\!\left(\frac{g^2(k-p_1)^2}{4\pi^2}\right) \overline{B}_+\!\left(\frac{g^2(k+p_3-p_2-p_1)^2}{4\pi^2}\right) \qquad (37)$$

for

$$\bar{F}'_-(g^2k^2, m^2/k^2) = \sum_{n=-\infty}^{\infty} C_n[F'_-(g^2k^2, m^2/k^2)]_n ,$$

where $[F'_-(g^2k^2, m^2/k^2)]_n$ is obtained from the function $F'_-(g^2k^2, m^2/k^2)$, defined for positive g^2 in Eq. (35), by continuing g^2 from positive to negative values along a path circling the origin counterclockwise n times. The integral over k in Eq. (37) converges. Using these techniques it is not difficult to see that all third-order amplitudes fall into one of two classes: (a) those defined directly as integrals of products of lower-order amplitudes for positive values of g^2, and (b) those amplitudes so defined for negative values of g^2. This third-order discussion suggests that the amplitude corresponding to each nth-order graph can be similarly defined in terms of lower-order amplitudes for either real or imaginary values of g.

Thus, our investigation of the expansion in Δm_0 leads us to consider two operators: (i) the physical scattering matrix $S(g)$ to be evaluated for real coupling constant g, and (ii) a second, unphysical scattering matrix $U(g)$ defined for imaginary g. Each is to be determined as a power series in Δm_0,

$$S(g) = \sum_{n=0}^{\infty} (\Delta m_0)^n S_n(g),$$
$$U(g) = \sum_{n=0}^{\infty} (\Delta m_0)^n U_n(g), \tag{38}$$

where both $S_n(g)$ and $U_n(g)$ can be written as a finite sum of r_n terms corresponding to our various possible nth-order graphs,

$$S_n(g) = \sum_{i=1}^{r_n} S_{n,i}(g),$$
$$U_n(g) = \sum_{i=1}^{r_n} U_{n,i}(g). \tag{39}$$

In defining the operators $S_{n,i}(g)$ and $U_{n,i}(g)$ we have grouped all nth-order diagrams into a finite number of classes. The amplitudes which enter $S_{n,i}(g)$ and $U_{n,i}(g)$ correspond to graphs in the ith class. These classes of graphs are chosen so that two graphs belong to the same class if they differ only in the number or arrangement of external boson lines, the direction of a fermion line, or if they are both connected and can each be obtained by deleting zero or more wavy lines from the same more complicated graph. Finally, two graphs are to be placed in the same class if they can both be divided into the same number of disjoint connected subgraphs and these subgraphs can be put in one-to-one correspondence in such a way that corresponding subgraphs belong to the same class. Members of such a class of third-order graphs are shown in Fig. 8. If nth-order graphs of the type i have $2\rho_i$ external fermion lines joined to ρ'_i different vertices $(\rho_i \leq \rho'_i \leq 2\rho_i)$, the operator $S_{n,i}$ will have the form

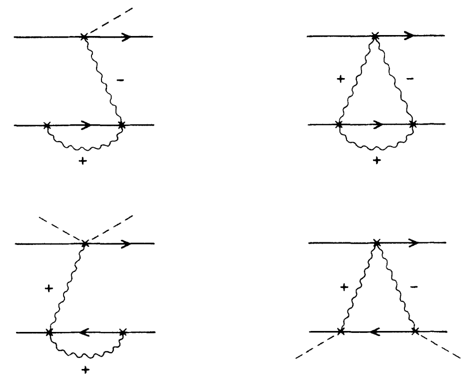

FIG. 8. Third-order graphs belonging to same class of graphs as defined in Sec. IV.

$$S_{n,i}(g) = \prod_{j=1}^{n}\left(\int d^4p_j \int d^4x_j\, e^{-ip_j\cdot x_j}(e^{+ig\tau_1\phi(x_j)})_{\delta_j\alpha_j}(e^{-ig\tau_1\phi(x_j)})_{\beta_j\gamma_j}\right)$$
$$\times \prod_{k=1}^{2\rho_i-\rho'_i}[\bar{l}(x_k)_{\delta_k} l(x_k)_{\gamma_k}] \prod_{k=2\rho_i-\rho'_i+1}^{\rho_i}[\bar{l}_{\sigma_k}(x_k)_{\delta_k}] \prod_{k=\rho_i+1}^{\rho'_i}[l_{\sigma_k}(x_k)_{\gamma_k}]$$
$$\times M_{n,i}(g^2, p_1, \ldots, p_n)^{\beta_1,\ldots,\beta_n;\ 2\rho i-\rho' i+1,\ldots,\gamma_{\rho i},\gamma_{\rho' i+1},\ldots,\gamma_n}_{\alpha_1,\ldots,\alpha_n;\ \delta_{\rho i+1},\ldots,\delta_n;\ \sigma_{2\rho i-\rho' i+1},\ldots,\sigma_{\rho' i}} . \tag{40}$$

Likewise, $U_{n,i}$ can be written in terms of a function $\overline{M}_{n,i}(g^2)$. In the discussion to follow, we will deal simultaneously with all those amplitudes which have been added together to form the function $M_{n,i}(g^2)$ or $\overline{M}_{n,i}(g^2)$. This is natural and convenient because (i) Any procedure which makes finite an nth-order amplitude containing the maximum of $\frac{1}{2}n(n-1)$ plus and minus wavy lines should be expected to render finite those amplitudes corresponding to graphs obtained by deleting some of those lines. (ii) As was noted for a particular case in Ref. 10, there may be significant cancellations between members of the same class of amplitudes. (iii) Given a sum of functions $M_{n,i}(g^2)$ over all classes i of graphs with the same ρ_i and ρ'_i, it is possible to project out a particular function $M_{n,i}(g^2)$ by acting only on the isotopic spin indices $\alpha, \beta, \gamma, \delta$, and the Dirac indices σ.

As indicated above, we will hypothesize that for each n and i either

(a) $M_{n,i}(g^2)$ can be obtained as a convergent integral of products of lower-order functions $M_{n',i'}(g^2)$ for real g, or
(b) $\overline{M}_{n,i}(g^2)$ can be obtained as a convergent integral of products of lower-order functions $\overline{M}_{n',i'}(g^2)$ for imaginary g.

This assumption is verified through fourth order in Appendix B. As was discussed in the second-order case, we will assume that those matrix elements of either S or U which are not directly defined above can be obtained by analytic continuation. If case (b) above is valid we will assume

$$M_{n,i}(g^2) = \sum_{m=-\infty}^{\infty} c_m(n,i)[\overline{M}_{n,i}(g^2)]_m \tag{41}$$

for positive g^2, while if case (a) holds,

$$\overline{M}_{n,i}(g^2) = \sum_{m=-\infty}^{\infty} d_m(n,i)[M_{n,i}(g^2)]_m \tag{42}$$

for negative g^2. As before, $[\overline{M}_{n,i}(g^2)]_m$ is the function obtained from $\overline{M}_{n,i}(g^2)$ by continuing g^2 from negative to positive values along a path circling the origin counterclockwise m times. Likewise, $[M_{n,i}(g^2)]_m$ is obtained from $M_{n,i}(g^2)$ by a similar continuation in g^2 from positive to negative values. These two assumptions allow us to proceed inductively, determining the functions $M_{n,i}(g^2)$ and $\overline{M}_{n,i}(g^2)$ for arbitrary n in terms of the constants $c_m(n',i')$ and $d_m(n',i')$ for $n' \leq n$. It is shown in Appendix C that the resulting functions have the form

$$\begin{aligned} M_{n,i}(g^2) &= \sum_{l=0}^{\Delta(n,i)} [M_{n,i}^{(l)}(g^2)\ln^l g^2 \\ &\qquad + M_{n,i}^{\prime(l)}(g^2)\sqrt{g^2}\,\ln^l g^2], \\ \overline{M}_{n,i}(-g^2) &= \sum_{l=0}^{\Delta(n,i)} [\overline{M}_{n,i}^{(l)}(-g^2)\ln^l(g^2) \\ &\qquad + \overline{M}_{n,i}^{\prime(l)}(-g^2)\sqrt{g^2}\ln^l(g^2)], \end{aligned} \tag{43}$$

for g^2 positive, where $M_{n,i}^{(l)}(g^2)$, $M_{n,i}^{\prime(l)}(g^2)$, $\overline{M}_{n,i}^{(l)}(g^2)$, and $\overline{M}_{n,i}^{\prime(l)}(g^2)$ are single-valued functions of g^2, analytic in the entire g^2 plane except for the point $g^2=0$. The functions $M_{n,i}^{\prime(l)}(g^2)$ and $\overline{M}_{n,i}^{\prime(l)}(g^2)$ are known to vanish for $n \leq 3$ and may well be zero for larger n; for simplicity we will neglect them in the following discussion – their inclusion somewhat complicates the arguments presented below but does not alter the conclusions drawn at the end of this section.

Equations (40) and (43) imply a similar decomposition of the operators $S_{n,i}(g)$ and $U_{n,i}(g)$:

$$\begin{aligned} S_{n,i}(g) &= \sum_{l=0}^{\Delta(n,i)} S_{n,i}^{(l)}(g)\ln^l g^2, \\ U_{n,i}(ig) &= \sum_{l=0}^{\Delta(n,i)} U_{n,i}^{(l)}(ig)\ln^l g^2, \end{aligned} \tag{44}$$

for real g, where the operators $S_{n,i}^{(l)}(g)$ and $U_{n,i}^{(l)}(g)$ depend analytically on g. If case (b) holds, we can use Eq. (43) to simplify (41), providing a direct relation between $M_{n,i}$ and $\overline{M}_{n,i}^{(l)}$,

$$M_{n,i}(g^2) = \sum_{l=0}^{\Delta(n,i)} f_{n,i}^{(l)}(g^2)\overline{M}_{n,i}^{(l)}(g^2),$$

for

$$f_{n,i}^{(l)}(g^2) = \sum_{m=-\infty}^{\infty} c_m(n,i)[\ln g^2 + (2n+1)\pi i]^l\,, \tag{45}$$

while for case (a) Eqs. (42) and (43) imply

$$\overline{M}_{n,i}(-g^2) = \sum_{l=0}^{\Delta(n,i)} h_{n,i}^{(l)}(g^2)M_{n,i}^{(l)}(-g^2)\,,$$

for

$$h_{n,i}^{(l)}(g^2) = \sum_{m=-\infty}^{\infty} d_m(n,i)[\ln g^2 + (2n+1)\pi i]^l\,. \tag{46}$$

In both Eqs. (45) and (46), g is real. We can write identical equations for the operators $S_{n,i}(g)$ and $U_{n,i}(g)$:

$$S_{n,i}(g) = \sum_{l=0}^{\Delta(n,i)} f_{n,i}^{(l)}(g^2)U_{n,i}^{(l)}(g) \tag{47}$$

for g real and if case (b) applies, while for case (a) and imaginary g

$$U_{n,i}(g) = \sum_{l=0}^{\Delta(n,i)} h_{n,i}^{(l)}(-g^2)S_{n,i}^{(l)}(g). \tag{48}$$

All of these equations, (41), (42), (45)–(48) can be extended to hold for both the (a)- and (b)-type graphs by inverting Eq. (41) in case (b) and Eq. (42) in case (a). This is not difficult because of the simple logarithmic dependence of $M_{n,i}$ and $\overline{M}_{n,i}$ on g^2. In each case we find the same set of $\Delta(n,i)$ equations for the $f_{n,i}^{(l)}$ and $h_{n,i}^{(l)}$,

$$\delta_{l,0} = \sum_{l'=0}^{l} \frac{f_{n,i}^{(l')}}{l'!}\left(\sum_{l''=0}^{l-l'} \frac{h_{n,i}^{(l'')}(-2\ln g^2)^{l-l'-l''}}{l''!(l-l'-l'')!}\right)\,, \tag{49}$$

which can be solved for the $\Delta(n,i)$ $f_{n,i}^{(l)}$'s in case (a) or the $h_{n,i}^{(l)}$'s in case (b). Thus the hypothesis described at the beginning of this paragraph, when coupled to the procedure specified in Eqs. (41) and (42), determines the nth-order scattering matrix in terms of a finite number of arbitrary parameters.

Just as in the second-order case studied in Sec. III, we should expect the requirement of unitarity to reduce the number of parameters entering the calculation of the nth-order scattering matrix. We will now show that the imposition of unitarity reduces the number of new parameters in this theory to one. For nth order in Δm_0 unitarity requires

$$S_n(g)+S_n(g)^\dagger=-\sum_{n'=1}^{n-1} S_{n'}(g)S_{n-n'}(g)^\dagger \tag{50}$$

for real g. Similarly, we will construct the operator $U_{n,i}(g)$ so that it obeys a "pseudo-unitarity" condition

$$U_n(g)+U_n(-g)^\dagger=-\sum_{n'=1}^{n-1} U_{n'}(g)U_{n-n'}(-g)^\dagger \tag{51}$$

for imaginary g. These equations can be made more specific if we project out the contribution of a particular class of graphs (i) and equate equal powers of $\ln g^2$:

$$S_{n,i}^{(l)}(g)+S_{n,i}^{(l)}(g)^\dagger=-\sum_{n'=1}^{n-1}\sum_{l'=0}^{l}\sum_{j,k}[S_{n',j}^{(l')}(g)S_{n-n',k}^{(l-l')}(g)^\dagger]_i \tag{52}$$

for real g and

$$U_{n,i}^{(l)}(g)+U_{n,i}^{(l)}(-g)^\dagger=-\sum_{n'=1}^{n}\sum_{l'=0}^{l}\sum_{j,k}[U_{n',j}^{(l')}(g)U_{n-n',k}^{(l-l')}(-g)^\dagger]_i \tag{53}$$

for imaginary g. The product operators appearing on the right-hand side of Eqs. (52) and (53) can be written as a sum of operators of the form displayed by the right-hand side of Eq. (40). The symbol $[\;]_i$ indicates that the operators corresponding to graphs containing $2\rho_i$ external fermion lines connected to ρ_i' vertices should be retained and from the multiplicative isospin and Dirac tensor [corresponding to the function $M_{n,i}(g)$ in Eq. (40)] one should project that tensor represented by graphs of the type i.

Let us proceed inductively assuming that Eqs. (52) and (53) hold for all orders $n'<n$ for some choice of $f_{n',i'}^{(l)}$ and $h_{n',i'}^{(l)}$. It seems reasonable to expect that Eq. (52) will be satisfied automatically in nth order for classes of graphs i of the type (a) above, for which the corresponding scattering-matrix elements $M_{n,i}(g^2)$ are constructed directly from lower-order, unitary amplitudes. Similarly, we expect Eq. (53) to be valid for classes of graphs of type (b). That this is in fact the case is demonstrated in Appendix D. Thus, for graphs i of type (b) Eq. (53) will be assumed automatically valid and we need only impose Eq. (52), which can be written

$$\begin{aligned}S_{n,i}(g)+S_{n,i}(g)^\dagger&=\sum_{l=0}^{\Delta(n,i)}[f_{n,i}^{(l)}U_{n,i}^{(l)}(g)+f_{n,i}^{(l)*}U_{n,i}^{(l)}(g)^\dagger]_i\\&=-\sum_{n'=1}^{n-1}\sum_{j,k}[S_{n',j}(g)S_{n-n',k}(g)^\dagger]_i\\&=-\sum_{n'=1}^{n-1}\sum_{l,l',j,k}\{[f_{n',j}^{(l)}U_{n',j}^{(l)}(g)][f_{n-n',k}^{(l')*}U_{n-n',k}^{(l')}(g)^\dagger]\}\,.\end{aligned} \tag{54}$$

If Eq. (53) is continued to real g and used to eliminate the operator $U_{n,i}^{(l)\dagger}$ in Eq. (54), we obtain

$$\sum_{l=0}^{\Delta(n,i)}(f_{n,i}^{(l)}-f_{n,i}^{(l)*})U_{n,i}^{(l)}=\sum_{l=0}^{\Delta(n,i)}f_l^*\sum_{n'=1}^{n-1}\sum_{l'=0}^{l}\sum_{j,k}[U_{n',j}^{(l')}(g)U_{n-n',k}^{(l-l')\dagger}]_i-\sum_{n'=1}^{n-1}\sum_{l,l'}\sum_{j,k}\{(f_{n',j}^{(l)}U_{n',j}^{(l)})[f_{n-n',k}^{(l')*}U_{n-n',k}^{(l')}(g)^\dagger]\}_i\,. \tag{55}$$

Equating coefficients of the independent operators[12] $U_{n,i}^{(l)}$, $[U_{n',j}^{(l')}U_{n-n',k}^{(l'')\dagger}]_i$, we obtain

$$f_{n,i}^{(l)}=f_{n,i}^{(l)*} \tag{56}$$

and

$$f_{n,i}^{(l)}=f_{n',j}^{(l')}f_{n-n',k}^{(l-l')*} \tag{57}$$

provided

$$[U_{n',j}^{(l')}U_{n-n',k}^{(l-l')}]_i\neq 0. \tag{58}$$

Although derived for i of type (b), Eqs. (56)–(58) are also valid for case (a) since both Eqs. (52) and (53) were used in their derivation. Because the first-order scattering matrix is directly defined by the interaction Hamiltonian (12) for both real and imaginary g, $f_1^{(0)}=1$. From Sec. III, Eq. (34), we find

$$f_{2,1}^{(1)}(g^2)=\ln g^2+b\,, \tag{59}$$

where we have labeled the class of second-order graphs containing the graph shown in Fig. 9(a) as $i=1$. Beginning with an nth-order graph of the j type, we can use Eq. (57) and $f_1^{(0)}=1$ to determine $f_{n,j}^{(l)}$ by sequentially joining N vertices to our graph. Each additional vertex is to be connected to every other by all possible wavy lines, producing a $(n+N)$th-order graph of type i for which

$$f_{n+N,i}^{(l)}=f_{n,j}^{(l)}\,. \tag{60}$$

We can then split this i-type graph into a $(n+N-2)$th-order graph of type k and a second-order graph of the type $j=2$, where this $j=2$ class

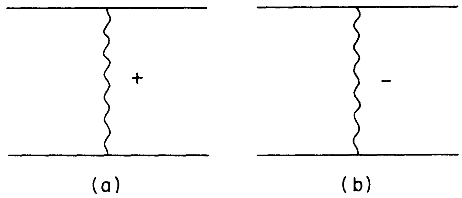

FIG. 9. (a) A member of the $i=1$ class of second-order graphs and (b) a member of the $i=2$ class.

of second-order graphs contains the graph shown in Fig. 9(b). (It follows from Sec. II that for $N>2$ this division is always possible.) If N is sufficiently large so that condition (58) holds, then

$$f^{(l)}_{n+N,i} = f^{(l-1)}_{n+N-2,k} f^{(1)}_{2,2}. \tag{61}$$

For large N this can be repeated until we obtain

$$f^{(l)}_{n,j} = f^{(l)}_{n+N,i} = [f^{(1)}_{2,2}]^l f^{(0)}_{n+N-2l,k} = [f^{(1)}_{2,2}]^l. \tag{62}$$

Consequently, all the parameters $f^{(l)}_{n,i}$ can be written in terms of the single real constant b,

$$f^{(l)}_{n,i} = (\ln g^2 + b)^l, \tag{63}$$

and Eq. (49) can be solved for the constants $h^{(l)}_{n,i}$,

$$h^{(l)}_{n,i} = (\ln g^2 - b)^l. \tag{64}$$

Thus, the parameters $f^{(l)}_{n,i}$ and $h^{(l)}_{n,i}$ which through Eqs. (47) and (48) completely determine the scattering matrix, are themselves fixed in terms of the single real constant b by the requirement of unitarity.[13] We have therefore shown that with the methods and assumptions developed above, the nth-order scattering matrix can be computed as a function of the parameters m_0, Δm_0, g, and a new real constant b.

V. CONCLUSION

In summary, we have attempted to solve the simple nonrenormalizable theory specified by the Lagrangian (1). The techniques developed determine partial summations of the usual perturbation series, organized into a power series in the symmetry-breaking mass difference Δm_0. Although a definite method of calculation to arbitrary order in Δm_0 has been deduced, its implementation in nth order depends upon a property of the lower-order amplitudes which has not been established for $n \geq 5$.

Finally, it should be noted that the techniques discussed in this paper can be applied to a large class of models in which the nonrenormalizable part of the interaction can be written in an exponential form of the sort shown in Eq. (4). Thus, for example, our methods can be successfully applied to the theory of two exponentially coupled scalar fields and to the theory of a neutral vector meson W_μ coupled to an axial-vector current of massive fermions, which are discussed in Appendix E.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix a derivation of the properties of the functions B_-, F_-, F'_-, and A_-, defined by Eqs. (20) and (35), is presented. Many of the results derived below have been obtained previously by other authors,[14] but are included here for completeness. Let us first consider the expression for B_- given in Eq. (21). If this equation is written in terms of $z = g^2 p^2/4\pi^2$, it becomes

$$B_-(z) = -\frac{1}{8z^2}\int_0^\infty dy\,\sqrt{y}\,J_1(\sqrt{y})(e^{-4z/y}-1). \tag{A1}$$

By using the Bessel differential equation and integration by parts, it is not difficult to show that Eq. (A1) implies

$$\frac{d^3}{dz^3}[z^2 B_-(z)] = B_-(z). \tag{A2}$$

Series expressions for the three solutions to Eq. (A2) can be easily obtained. The proper combination of these series solutions, as determined by the small-z behavior required by Eq. (A1) is dis-

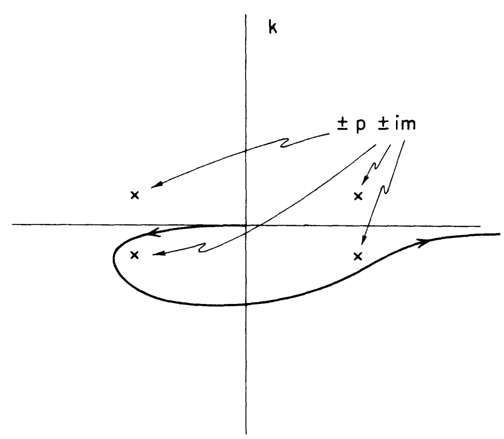

FIG. 10. The k contour that results if the integral in either Eq. (A4) or (A5) is written in polar coordinates, the angular integration performed and g^2 continued from a real value through a full circle about the origin in the complex plane. The crosses locate the singularities of the integrand in k.

played in Eq. (23). In a similar way, the behavior of $B_-(z)$ for large complex z can be found from Eq. (A2) in terms of three constants. These constants are then determined by examining the large-positive-z limit of Eq. (A1). The results of this procedure are given in Eq. (24).

The other well-defined functions F_-, F'_-, and A_- have quite similar properties. Their asymptotic behavior for large positive p^2 or g^2 follows readily from Eq. (35):

$$F_-(g^2p^2, m^2/p^2) \sim -\frac{1}{(2\pi)^2} B_-\left(\frac{g^2p^2}{4\pi^2}\right),$$

$$F'_-(q^2p^2, m^2/p^2) \sim \frac{2}{g^2p^2\sqrt{3}} \exp\left[-\frac{3}{2}\left(\frac{g^2p^2}{4\pi^2}\right)^{1/3}\right] \sin\left[\frac{3\sqrt{3}}{2}\left(\frac{g^2p^2}{4\pi^2}\right)^{1/3}\right], \tag{A3}$$

$$A_-(g^2p^2, m^2/p^2, m'^2/p^2) \sim -\frac{g^2p^2}{4\pi^2} B_-\left(\frac{g^2p^2}{4\pi^2}\right).$$

The analytic structure in g^2 of these functions can be studied by rewriting Eq. (35) in momentum space:

$$F_-(g^2p^2, m^2/p^2) = \frac{g^2}{\pi^2p^2}\int \frac{d^4k}{(2\pi)^4} \frac{p\cdot(p+k)}{(p+k)^2+m^2} B_-\left(\frac{g^2k^2}{4\pi^2}\right) - \frac{1}{g^2(p^2+m^2)}, \tag{A4}$$

$$F'_-(g^2p^2, m^2/p^2) = \frac{g^2}{\pi^2}\int \frac{d^4k}{(2\pi)^4} \frac{1}{(p+k)^2+m^2} B_-\left(\frac{g^2k^2}{4\pi^2}\right) - \frac{1}{g^2(p^2+m^2)}, \tag{A5}$$

$$A_-(g^2p^2, m^2/p^2, m'^2/p^2) = 4g^4 \int \frac{d^4k}{(2\pi)^4} \frac{k\cdot(p+k)F_-(g^2k^2, m'^2/k^2) - mm'F'_-(g^2k^2, m'^2/k^2)}{(k+p)^2+m^2}, \tag{A6}$$

for spacelike p^2 and Euclidean integration $\int d^4k$. Introducing four-dimensional polar coordinates for the integration variable, $k_\mu = (k \sin\nu \sin\theta \sin\phi, k \sin\nu \sin\theta \cos\phi, k\cos\nu, k \sin\nu \cos\theta)$ for

$$0 \leq \phi \leq 2\pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \nu \leq \pi, \quad 0 \leq k \leq \infty \tag{A7}$$

and letting $p_1 = p_2 = p_4 = 0$, we find that the integrands in Eqs. (A4)–(A6) depend on the variables p, k, ν. If the ν integration is performed first, the resulting integrands have singularities for fixed p at

$$k = 0 \quad \text{and} \quad k = \pm p \pm im. \tag{A8}$$

The integrand in Eq. (A6) also is singular when

$$k = \pm im'. \tag{A9}$$

Thus the singularities in k lie in a bounded region of the complex k plane. Continuation in g^2 away from real values can be achieved by rotating the k contour outside this region of singularities in such a way that convergence of the k integration at infinity is always maintained. Consequently, the functions F_-, F'_-, and A_- are singular in g^2 only at the point $g^2 = 0$. It is not difficult to see from Eqs. (A4) and (A5) that continuation of F_- and F'_- in g^2 clockwise through 2π about the origin yields new functions, corresponding to the deformed k integration contour shown in Fig. 10, which differ from the first by added functions of the form

$$-4\pi(i\ln g^2 + \pi)\mathfrak{F}_2(g^2) - 2\pi i \mathfrak{F}_1(g^2), \tag{A10}$$

where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are analytic functions of g^2. Therefore, the functions F_- and F'_- have the g^2 dependence

$$\mathfrak{F}_2(g^2)\ln g^2 + \mathfrak{F}_1(g^2)\ln g^2 + \mathfrak{F}_0(g^2) - \frac{1}{g^2(p^2+m^2)}, \tag{A11}$$

where $\mathfrak{F}_0$ is an analytic function of g^2. Using the same methods, it can be easily seen that the function A_- has a form similar to (A11) but contains an additional term proportional to $\ln^3 g^2$ and lacks the pole at $g^2 = 0$.

APPENDIX B

We discuss below the construction of fourth-order amplitudes from products of lower-order amplitudes for either real or imaginary g. Three graphs are shown in Fig. 11, each belonging to one of three classes of fourth-order graphs with no internal fermion lines. Those fourth-order graphs possessing internal fermion lines will not be explicitly discussed since such a graph can be treated in essentially[11] the same manner as the graph obtained from it by cutting all internal fermion lines. Clearly the amplitude represented in Fig. 11(a) can be obtained for $g^2>0$ as a product of six second-order amplitudes (B_-), and therefore falls into the category (a) defined in Sec. IV. Figure 11(b), on the other hand, represents an amplitude which can be constructed for negative g^2 as a convergent integral of a product of second-order amplitudes and thus belongs to category (b). However, if the amplitude represented by Fig. 11(c) is written directly as an integral of products of second-order amplitudes, divergent integrals result for g^2 either positive or negative. Instead, as will be demonstrated, this amplitude can be constructed for negative g^2 from the product of one third-order amplitude and three second-order amplitudes.

The appropriate third-order amplitude corresponds to the graph obtained from that shown in Fig. 11(c) by deleting vertex number 4 and all the lines connected to it. This amplitude can be constructed directly for g^2 positive and is given by

$$M_{3,1}(-p,p'',p')^{\beta_1,\beta_2,\beta_3}_{\alpha_1,\alpha_2,\alpha_3}=[(\tau_3)_{\alpha_1,\beta_1}(\tau_3)_{\alpha_2,\beta_2}(\tau_3)_{\alpha_3,\beta_3}-(\tau_3)_{\alpha_1,\beta_1}(\tau_2)_{\alpha_2,\beta_2}(\tau_2)_{\alpha_3,\beta_3}-(\tau_2)_{\alpha_1,\beta_1}(\tau_2)_{\alpha_2,\beta_2}(\tau_3)_{\alpha_3,\beta_3}$$
$$-(\tau_2)_{\alpha_1,\beta_1}(\tau_3)_{\alpha_2,\beta_2}(\tau_2)_{\alpha_3,\beta_3}]\,\frac{(-i\Delta m_0)^3}{4}\left(\frac{ig^4}{\pi^2}\right)^3\frac{1}{(2\pi)^{12}}$$
$$\times i\int d^4k\,B_-\left(\frac{g^2k^2}{4\pi^2}\right)B_-\left(\frac{g^2(p+k)^2}{4\pi^2}\right)B_-\left(\frac{g^2(p'+k)^2}{4\pi^2}\right)\delta^4(p''+p'-p), \tag{B1}$$

where this amplitude has been labeled as a member of the class $i=1$. For simplicity let us define

$$M_{3,1}(-p,p'',p')^{1,1,1}_{1,1,1}=\left(\frac{\Delta m_0 g^4}{\pi^2}\right)^3\frac{\delta^4(p''+p'-p)}{4(2\pi)^{12}}E(p^2,p'^2,(p-p')^2) \tag{B2}$$

for g^2 positive and

$$\overline{M}_{3,1}(-p,p'',p')^{1,1,1}_{1,1,1}=\left(\frac{\Delta m_0 g^4}{\pi^2}\right)^3\frac{\delta^4(p''+p'-p)}{4(2\pi)^{12}}\overline{E}(p^2,p'^2,(p-p')^2) \tag{B3}$$

for g^2 negative. We will now show that the amplitude represented by Fig. 11(c) can be constructed for g^2 negative by integrating a product of $\overline{E}(p^2,p'^2,(p-p')^2)$ with three second-order $\overline{B}_+$ functions over Euclidean[8] p and p':

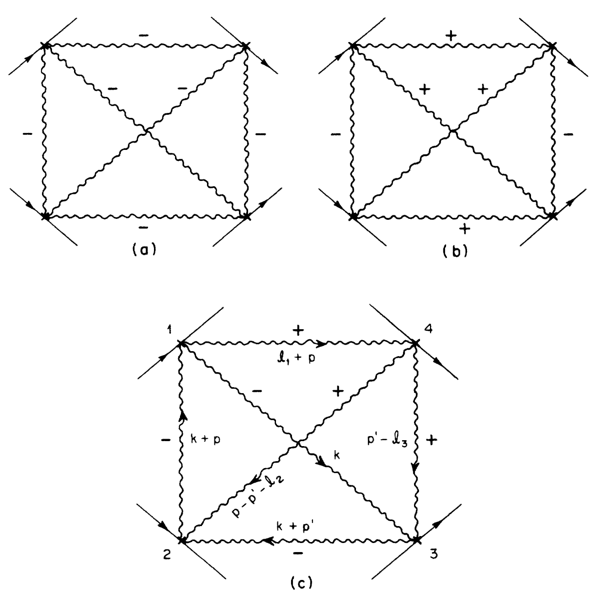

FIG. 11. Three graphs representing amplitudes of fourth order in Δm_0 which contain no internal fermion lines.

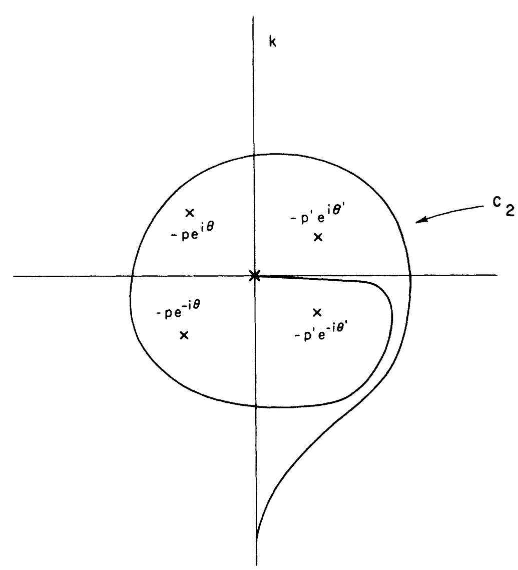

FIG. 12. The contour followed by the k integration in Eq. (B5) if g^2 is continued from positive to negative values along a path circling the origin twice counterclockwise. The small crosses in the figure specify the location of the singularities of the integrand in k.

$$\int d^4p \int d^4p' \bar{E}(p^2, p'^2, (p-p')^2)\, \bar{B}_+\left(\frac{g^2(p+l_1)^2}{4\pi^2}\right) \bar{B}_+\left(\frac{g^2(p'-l_3)^2}{4\pi^2}\right) \bar{B}_+\left(\frac{g^2(p-p'-l_2)^2}{4\pi^2}\right). \tag{B4}$$

In order to study the convergence of the integral (B4) it is necessary to determine the asymptotic behavior for large p and p' of the function $\bar{E}$ obtained, for negative g^2, by analytic continuation from the function E. We begin by introducing angular variables in the integral (B1). For $k=(k^2)^{1/2}$, $p=(p^2)^{1/2}$, $p'=(p'^2)^{1/2}$, $k\cdot p = kp\cos\theta$, $k\cdot p' = kp'\cos\theta'$, $p\cdot p' = pp'\cos\nu$ $(0\le\nu\le\pi)$, and g^2 positive, E is given by

$$E(p^2, p'^2, (p-p')^2) = \frac{2\pi}{\sin\nu}\int_{-1}^{1} d(\cos\theta)\int_{\cos(\theta+\nu)}^{\cos(\theta-\nu)} d(\cos\theta')\int_0^\infty k^3dk\, B_-\left(\frac{g^2k^2}{4\pi^2}\right) B_-\left(\frac{g^2(p+k)^2}{4\pi^2}\right) B_-\left(\frac{g^2(p'+k)^2}{4\pi^2}\right). \tag{B5}$$

Written in this form we can easily continue E in g^2 by deforming the k contour for large k in such a way that the k integral is always convergent. For example, the function $[E]_2$ for negative g^2 obtained by continuing E in g^2 along a path circling the origin twice counterclockwise is given by an expression identical to (B5) except that the k contour is changed from the positive real axis to the contour c_2 shown in Fig. 12. Thus the continued function $\bar{E}$ for negative g^2 is given by

$$\begin{aligned}\bar{E}(p^2, p'^2, (p-p')^2) = {} & \frac{2\pi}{\sin\nu}\int_{-1}^{1} d(\cos\theta)\int_{\cos(\theta+\nu)}^{\cos(\theta-\nu)} d(\cos\theta') \\ & \times \sum_{m=-\infty}^{\infty}\Bigg\{ d_m(3,1)\left(\sum_{m'=1}^{(|m|+m)/2} - \sum_{m'=(m-|m|)/2}^{-2}\right) \\ & \qquad \times \int_{c_{\rm I}} k^3dk \left[B_-\left(\frac{g^2k^2}{4\pi^2}\right)\right]_{m'} \left[B_-\left(\frac{g^2(p+k)^2}{4\pi^2}\right)\right]_{m'} \left[B_-\left(\frac{g^2(p'+k)^2}{4\pi^2}\right)\right]_{m'} \\ & + \left[d_{2m+1}(3,1)\int_{c_{\rm II}} k^3dk + d_{2m}(3,1)\int_{c_{\rm III}} k^3dk\right] B_-\left(\frac{g^2k^2}{4\pi^2}\right) B_-\left(\frac{g^2(p+k)^2}{4\pi^2}\right) B_-\left(\frac{g^2(p'+k)^2}{4\pi^2}\right)\Bigg\},\end{aligned} \tag{B6}$$

where the k contours $c_{\rm I}$, $c_{\rm II}$, and $c_{\rm III}$ are shown in Fig. 13 and Eqs. (B5) and (42) have been used.[15] We can now bound the asymptotic behavior of $\bar{E}$ by bounding each term in the sum found in Eq. (B6), assuming for each factor $[B_-]_{m'}$, $m'\neq 0$, the largest asymptotic behavior consistent with Eq. (A2).

Let us first consider a term in Eq. (B6) with k contour $c_{\rm I}$. For $z=e^{i\theta}$ and $z'=e^{i\theta'}$ such a term can be written as

$$\frac{\pi}{2\sin\nu}\int_c \left(1-\frac{1}{z^2}\right)dz \int_{c'}\left(1-\frac{1}{z'^2}\right)dz' \int_{c_{\rm I}} k^3dk \left[B_-\left(\frac{g^2k^2}{4\pi^2}\right)\right]_{m'} \left[B_-\left(\frac{g^2(k+pz)(k+p/z)}{4\pi^2}\right)\right]_{m'} \left[B_-\left(\frac{g^2(k+p'z')(k+p'/z')}{4\pi^2}\right)\right]_{m'}, \tag{B7}$$

where the contours c and c' are shown in Fig. 14. Since the singularities of $B_-(x)$ are of the type $1/x$ or $\ln x$ we can split the integrand in the expression (B7) into four parts each containing only three singularities in k. If we later symmetrize in p and p' we need only consider two terms: the first, A, singular when $k=pz, p'z'$, or 0 and the second, B, singular when $k=pz$, p'/z', or 0. The z' and z contours for the second term, B, are deformed so that $|z'|=|z|=(p'/p)^{1/2}$ where possible. Then, for each of these two terms the k contour is shrunk around the singularities and split into two parts so that the integral (B7) is made up of four terms having the singularities and contours shown in Fig. 15. Since we will later symmetrize in p and p', only the terms A_1 and B_1 whose contours are identified in Fig. 15 need be investigated. Further deformation of these contours to those shown in Fig. 16 yields for the two terms

$$\begin{aligned}|A_1(p,p',\nu)| \le {} & \frac{8}{\sin\nu}\int_e^1 d\xi\int_0^\nu d\theta'\int_0^{p\xi-\epsilon}\kappa^3d\kappa\left(\frac{2}{\sqrt{3}}\right)^3\left(\frac{4\pi^2}{g^2}\right)^4\left(\frac{1}{\xi^2}-1\right)\sin\theta' \\ & \times \frac{|\ln(\kappa-p'\xi e^{i\theta'})| + |\ln(\kappa-p'e^{-i\theta'}/\xi)|}{\kappa^{8/3}|(\kappa-p\xi)(\kappa-p/\xi)|^{4/3}|(\kappa-p'\xi e^{+i\theta'})(\kappa-p'e^{-i\theta'}/\xi)|^{4/3}} \\ & \times\left|\exp\left[3\left(\frac{-g^2}{4\pi^2}\right)^{1/3}\{(-\kappa^2)^{1/3}+[-(\kappa-p\xi)(\kappa-p/\xi)]^{1/3}+[-(\kappa-p'\xi e^{i\theta'})(\kappa-p'e^{-i\theta'}/\xi)]^{1/3}\}\right]\right|\end{aligned} \tag{B8}$$

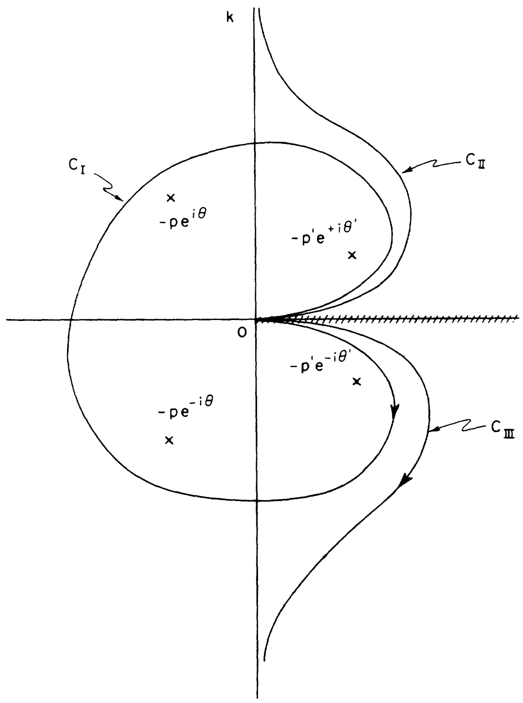

FIG. 13. The k integration contours $c_{\rm I}$, $c_{\rm II}$, and $c_{\rm III}$ which occur in Eq. (B6). The hatched line locates a branch cut; note however that not all of the branch cuts present have been so specified.

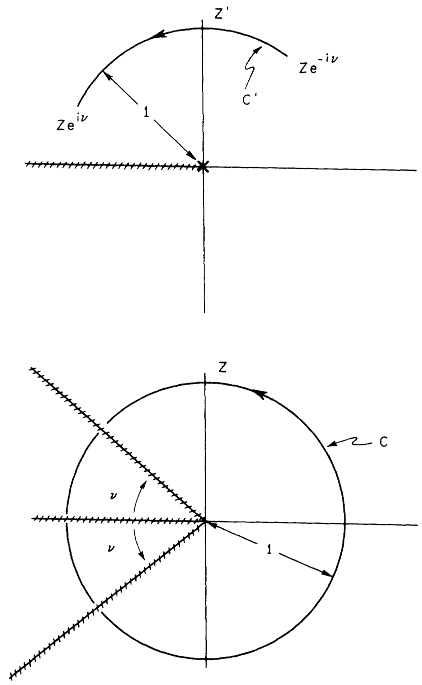

FIG. 14. The contours c' and c followed by the integration variables z' and z in the integral (B7). Breaks in the integration contours signal a change from one integrand to another.

and

$$|B_1(p,p',\nu)| \leqslant |A_1(p,p',\nu)| + \frac{16\pi}{\sin\nu}\int_{\epsilon}^{(p'/p)^{1/2}} d\xi \int_{\xi+\epsilon}^{(p'/p)(1/\xi)-e} d\xi' \int_0^{p\xi-\epsilon} \kappa^3 d\kappa \left(\frac{2}{\sqrt{3}}\right)^3 \left(\frac{4\pi^2}{g^2}\right)^4 \left|\left(1-\frac{e^{-i\nu}}{\xi^2}\right)\left(1-\frac{e^{i\nu}}{\xi'^2}\right)\right| \kappa^{-8/3}$$
$$\times |(\kappa-p\xi)(\kappa-pe^{-i\nu}/\xi)|^{-4/3} |(\kappa-p'\xi' e^{-i\nu})(\kappa-p'/\xi')|^{-4/3}$$
$$\times \left|\exp\left[3\left(\frac{-g^2}{4\pi^2}\right)^{1/3}\{[-\kappa^2 e^{+i\nu}]^{1/3}+[-\kappa^2 e^{i\nu}-p'^2+\kappa p'(\xi'+e^{i\nu}/\xi')]^{1/3}+[-\kappa^2 e^{i\nu}-p^2+\kappa p(\xi e^{i\nu}+1/\xi)]^{1/3}\}\right]\right| \tag{B9}$$

for g^2 real and negative, ϵ very small and positive.[16] The bounds (B8) and (B9) can now be used to investigate the convergence of the integral (B4). For this purpose we neglect the external momenta l_1, l_2, and l_3 and define $r=p'/p$ so that the integral (B4) becomes

$$16\pi^3\int_0^1 r^3 dr \int_0^{\pi} \sin^2\nu\, d\nu \int_0^{\infty} p^7 dp\, \overline{E}(p^2,p'^2,(p-p')^2)\, \overline{B}_+\!\left(\frac{g^2p^2}{4\pi^2}\right)\overline{B}_+\!\left(\frac{g^2p'^2}{4\pi^2}\right)\overline{B}_+\!\left(\frac{g^2(p-p')^2}{4\pi^2}\right). \tag{B10}$$

A straightforward computer search using the bounds (B8) and (B9) reveals that if the term (B7) is substituted for $\overline{E}$, the integral over p in the expression (B10) is convergent if $r>0.04$ and either $r<0.8$ or $\nu>\frac{1}{6}\pi$.[17] A similar analysis can be performed for the terms involving the k contours $c_{\rm II}$ and $c_{\rm III}$ yielding identical conclusions.

Thus, we must examine the regions $r<0.04$ and $r>0.8$, $\nu<\frac{1}{6}\pi$ in more detail. Since these two regions correspond to small p'^2 or small $(p-p')^2$ and are consequently related by a change of variables, only the region $p'<0.04p$ need be considered. Direct evaluation indicates that even for $p'<0.04p$ the p integral in the expression (B10) converges if the representation (B7) is used for $\overline{E}$, contours of the type shown in Fig. 16 chosen and the region $-p\leqslant k<-0.6p$ omitted. Fortunately, the large-p asymptotic behavior of the integrand in Eq. (B7) for this choice of contour, $p'<0.04p$ and $|k|>0.6p$ can be determined quite simply if

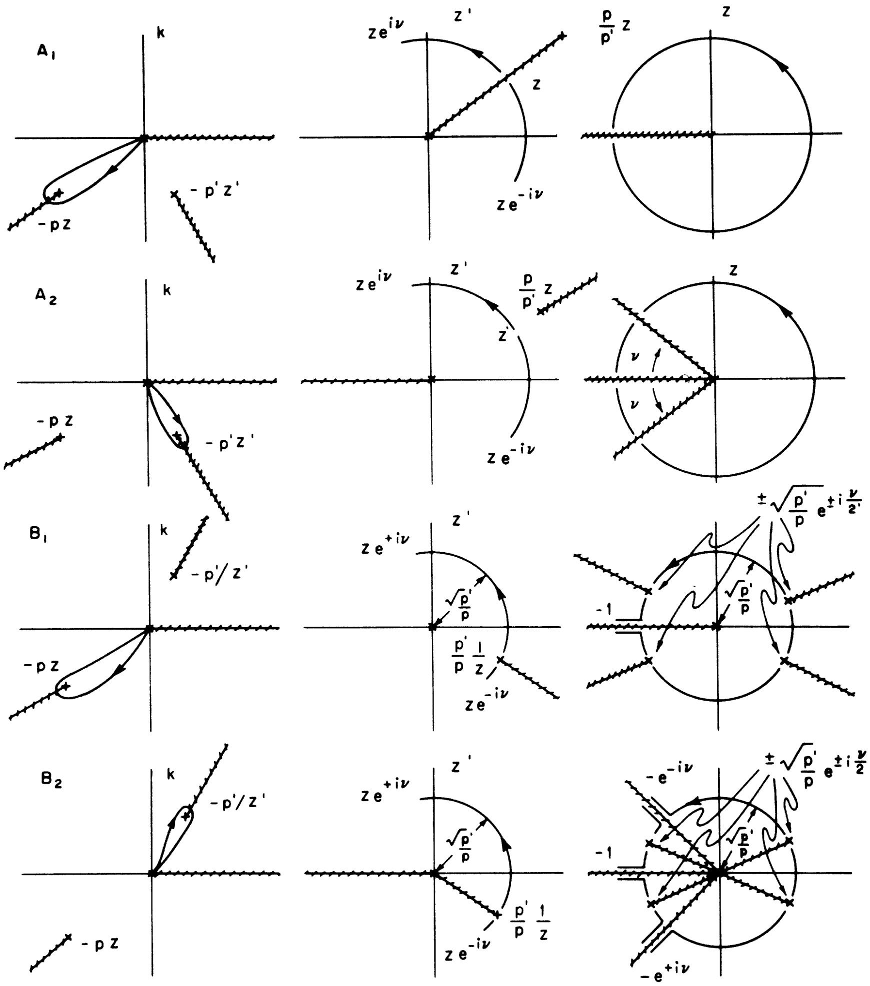

FIG. 15. The k, z', and z contours which result if the integrand in the expression (B7) is subdivided and the c_{I} contour deformed and split in the manner described in the text.

we expand in the small quantities p'/p and $(p+k)/p$. Each term in this integrand grows exponentially with an exponent whose real part is given by

$$\frac{3}{2}\left(\frac{-g^2}{4\pi^2}\right)^{1/3}\left(2p^{2/3}+[(p+k)^2-p^2(1+z)^2]^{1/3}-\frac{2p'\cos\theta'}{3p^{1/3}}-\frac{4}{3}\frac{p+k}{p^{1/3}}+\text{higher-order terms}\right) \tag{B11}$$

for $-p\leq k\leq -0.6p$, $-1\leq z\leq k/p$, $-\nu\leq\theta'\leq\nu$, and g^2 negative. The exponent (B11) is a maximum for $z=1$, $k=-\frac{7}{8}p$ and is readily seen to grow more rapidly with increasing p than the damping exponent arising from the three factors of $\overline{B}_+$ in the integral (B10), whose real part is

$$-\frac{3}{2}\left(\frac{-g^2}{4\pi^2}\right)^{1/3}\left(2p^{2/3}+p'^{2/3}-\frac{2}{3}\frac{p'\cos\nu}{p^{1/3}}\right). \tag{B12}$$

This discussion suggests that for the ratio p'/p fixed and small, the integrand in the expression (B4) may grow with increasing p. It is possible that this asymptotic behavior in p for small p' may be radically altered if we *integrate* over p' in the region $p'<0.04p$ and include the amplitude corresponding to the graph obtained from Fig. 11(c) by deleting the line connecting vertices 3 and 4.[18] Therefore let us first integrate over p' in the expression (B4), including this second amplitude, and substituting for $\overline{E}$ only the contribution to the integral in Eq. (B6) coming from the region $-p\leq k\leq -0.6p$ discussed above. Labeling this truncated function $\overline{E}_T$, we wish to determine the large p behavior of

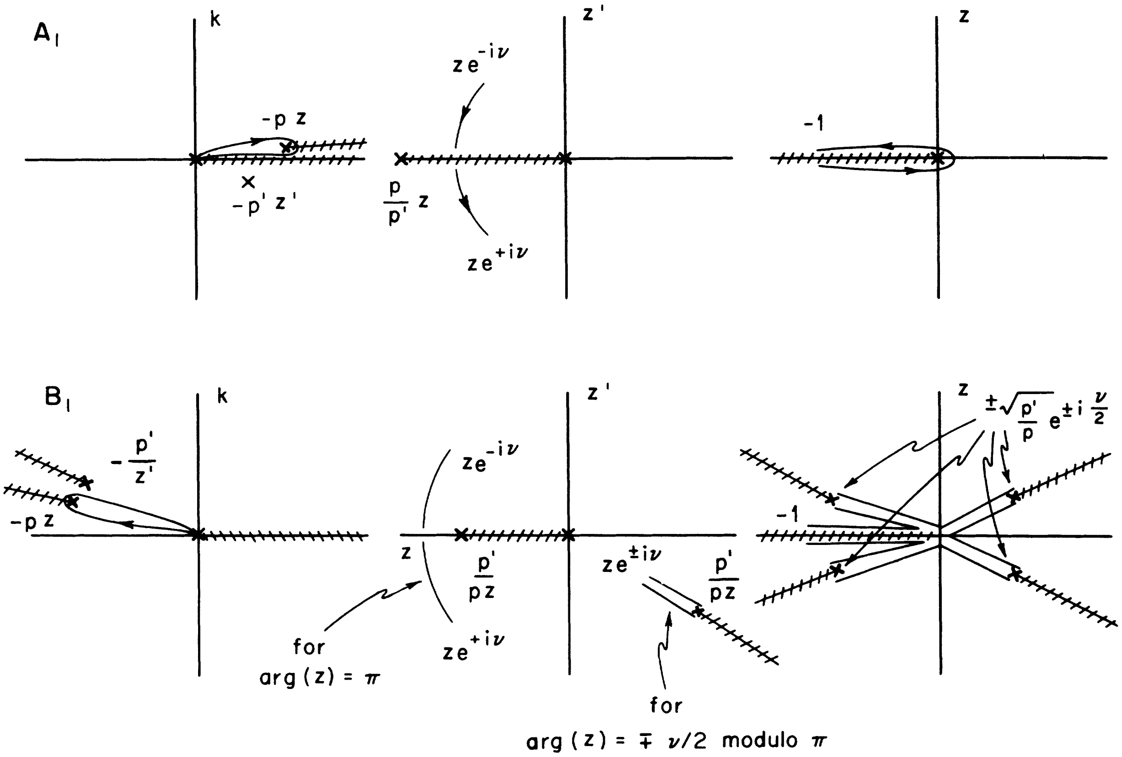

FIG. 16. The k, z', and z contours used to obtain the bounds (B8) and (B9).

$$\bar{B}_+\left(\frac{g^2p^2}{4\pi^2}\right)\prod_{i=1}^{4}\left(\int_{-0.02p}^{+0.02p} dp_i'\right)\left[\bar{B}_+\left(\frac{g^2p'^2}{4\pi^2}\right)+\frac{\pi^2}{ig^4}(2\pi)^4\delta^4(p')\right]\bar{B}_+\left(\frac{g^2(p-p')^2}{4\pi^2}\right)\bar{E}_T(p^2,p'^2,(p-p')^2)\,. \tag{B13}$$

If the quantity in large square brackets is written in terms of its Fourier transform using Eqs. (22) and (20), the resulting expression can be divided into 16 terms in such a way that within a single term each component of x has only one sign:

$$\bar{B}_+\left(\frac{g^2p'^2}{4\pi^2}\right)+\frac{\pi^2}{ig^4}(2\pi)^4\delta^4(p')=\frac{1}{16}\frac{\pi^2}{ig^4}\sum_{\sigma_i=\pm1}\int d^4x\,\exp\left(-i\sum_{j=1}^{4}\sigma_j p_j'|x_j|\right)e^{+4ig^2\Delta_F(x^2)}\,. \tag{B14}$$

If any of these terms is substituted into Eq. (B13), the path of each p_i' integration can be deformed into a semicircular contour in the upper ($\sigma_i=-1$) or lower ($\sigma_i=+1$) half of the complex plane. The asymptotic behavior for large p of the contribution of any of these terms to the integrand in expression (B13) for values of p_i' lying on these contours is such as to insure convergence of the final p integration.

APPENDIX C

The analytic structure in g^2 specified by Eq. (43) for the amplitudes $M_{n;i}(g^2)$ and $\bar{M}_{n,i}(g^2)$ can be deduced from the following theorem.

Theorem: Let $f_n(g, k_1, \ldots, k_n)$ be a function of $n+1$ complex variables which satisfies the following.

(i) $f_n(g, \omega_1/g, \ldots, \omega_n/g)$ is an analytic function of the variables $g, \omega_1, \ldots, \omega_n$ except possibly for the point $g=0$, or points satisfying one of the equations

$$Q_n^{(j)}(g,\omega_1,\ldots,\omega_n)=0,\quad 1\le j\le M_n\,, \tag{C1}$$

where $Q_n^{(j)}$ is a homogeneous polynomial of degree ν_j in the $n+1$ variables $g, \omega_1, \ldots, \omega_n$ which cannot be factored into the product of two similar, nonconstant polynomials. Each polynomial $Q_n^{(j)}$ possesses the following property: If the n real numbers $\bar{\omega}_1, \ldots, \bar{\omega}_n$ satisfy

$$Q_n^{(j)}(0,\bar{\omega}_1,\ldots,\bar{\omega}_n)=0\,, \tag{C2}$$

then

$$Q_n^{(j)}(g,\omega_1,\ldots,\omega_n)=Q_n^{(j)}(g,\omega_1+\bar{\omega}_1,\ldots,\omega_n+\bar{\omega}_n) \tag{C3}$$

for arbitrary complex values of $g, \omega_1, \ldots, \omega_n$.

(ii) $f_n(g, k_1, \ldots, k_n) = \sum_{l=0}^{N_n} f_n^{(l)}(g, k_1, \ldots, k_n) \ln^l g$, (C4)

where $f_n^{(l)}(g, k_1, \ldots, k_n)$, $0 \le l \le N_n$ are single-valued functions of g, analytic in the entire g plane except for the point $g=0$. The function

$$f_n(g, a_1 + b_1/g, \ldots, a_n + b_n/g)$$

for real values of $b_1, \ldots, b_n$ has a singularity at $g=0$ of a similar form,

$$f_n(g, a_1 + b_1/g, \ldots, a_n + b_n/g) \sum_{l=0}^{N_n'} h_n^{(l)}(g, a_1, \ldots, a_n, b_1, \ldots, b_n) \ln^l g, \tag{C5}$$

where, given $\delta > 0$, there exists $\epsilon > 0$ such that the functions $h_n^{(l)}(g, a_1, \ldots, a_n, b_1, \ldots, b_n)$ depend analytically on g in the region $0 < |g| < \epsilon \left(\sum_{i=1}^n |b_i|^2\right)^{1/2}$, provided

$$|Q_n^{(j)}(0, b_1, \ldots, b_n)| > \delta\left(\sum_{i=1}^n |b_i|^2\right)^{\nu_j/2}, \quad 1 \le j \le M_n. \tag{C6}$$

(iii) If the n variables $k_1, \ldots, k_n$ are split into two groups, $k_{\sigma_1}, \ldots, k_{\sigma_m}$ and $k_{\sigma_{m+1}}, \ldots, k_{\sigma_n}$, then we have the following:

(a) For fixed, real $k_{\sigma_{m+1}}, \ldots, k_{\sigma_n}$, g and real values of $\omega_{\sigma_i} = g k_{\sigma_i}$, $1 \le i \le m$, there exist positive real numbers A, B, and R such that

$$|f_n(g, k_1, \ldots, k_n)| < A \exp\left(-B \sum_{j=1}^m |\omega_{\sigma_j}|^{2/3}\right) \tag{C7}$$

if $|\omega_{\sigma_i}| > R$ for all $1 \le i \le m$.

(b) Equation (C7) remains valid if the variables $g, \omega_1, \ldots, \omega_n$ are continued from the real values $g^{(0)}, \omega_1^{(0)}, \ldots, \omega_n^{(0)}$ into the region

$$\big||g| - |g^{(0)}|\big| < \delta_1 |g^{(0)}| \left(\sum_{i=1}^n |\omega_i - \omega_i^{(0)}|^2\right)^{1/2} < \gamma W_m(\sigma), \tag{C8}$$

$$|Q_n^{(j)}(g, \omega_1, \ldots, \omega_n)| > \delta_2 |W_m(\sigma)|^\nu \tag{C9}$$

for

$$W_m(\sigma) = \left(\sum_{i=1}^m |\omega_{\sigma_i}^{(0)}|^2\right)^{1/2},$$

some real preassigned $\nu < \nu_j$, $1 > \delta_1 > 0$, $\delta_2 > 0$, sufficiently small γ and all values of $\omega_{\sigma_i}^{(0)}$, $1 \le i \le m$, obeying $|\omega_{\sigma_i}^{(0)}| > R$.

If these conditions are satisfied, then

$$\prod_{i=0}^n \left(\int_{-\infty}^{\infty} dk_i\right) f_n(g, k_1, \ldots, k_n) = \sum_{l=0}^N f^{(l)}(g) \ln^l g, \tag{C10}$$

where $f^{(l)}(g)$ is a single-valued function of g analytic in the entire g plane except at the point $g=0$.

Proof: We will establish this theorem by induction, showing that if $f_n(g, k_1, \ldots, k_n)$ satisfies the conditions (i), (ii), and (iii) then the function

$$f_{n-1}(g, k_1, \ldots, k_{n-1}) = \int_{-\infty}^{\infty} d\omega_n f_n(g, k_1, \ldots, k_{n-1}, \omega_n/g) \tag{C11}$$

also does. It is convenient to choose[19] the variables $k_1, \ldots, k_n$ so that each of the polynomials $Q_n^{(j)}(g, gk_1, \ldots, gk_n)$ appearing in Eq. (C1) contains a term of the form $(k_n)^{\nu_j}$.

Consider first condition (i). Label the r_j distinct singularities of the integrand in Eq. (C11) by $\omega_n^{j,i}(g, \omega_1, \ldots, \omega_{n-1})$, where

$$Q_n^{(j)}(g, \omega_1, \ldots, \omega_{n-1}, \omega_n^{j,i}(g, \omega_1, \ldots, \omega_{n-1})) = 0 \tag{C12}$$

for $1 \le i \le r_j \le \nu_j$ and $1 \le j \le M_n$. Since $Q_n^{(j)}$ contains a nonzero term of the form $(\omega_n)^{\nu_j}$, $\omega_n^{j,i}$ is finite for all finite values of $\omega_1, \ldots, \omega_{n-1}$. Therefore, $f_{n-1}(g^2, k_1, \ldots, k_{n-1})$ is an analytic function of the variables g and $\omega_i = gk_i$, $1 \le i \le n-1$ at all points except possibly those satisfying

$$\omega_n^{j,i}(g, \omega_1, \ldots, \omega_{n-1}) = \omega_n^{j',i'}(g, \omega_1, \ldots, \omega_{n-1}) \tag{C13}$$

for all $i \ne i'$ or $j \ne j'$. To see that the conditions (C13) can be written in the form of Eq. (C1) we need only apply a basic theorem[20] obeyed by polynomials in many variables: Given two polynomials $P_1(x_1, \ldots, x_n)$ and $P_2(x_1, \ldots, x_n)$ in n variables, the equations

$$P_1(x_1, \ldots, x_n) = 0 \quad \text{and} \quad P_2(x_1, \ldots, x_n) = 0 \tag{C14}$$

have a common root when viewed as equations in x_n if and only if

$$R(x_1, \ldots, x_{n-1}) = 0, \tag{C15}$$

where the resultant $R(x_1, \ldots, x_{n-1})$ is a polynomial in $x_1, \ldots, x_{n-1}$ which can be written

$$R(x_1, \ldots, x_{n-1}) = f_1(x_1, \ldots, x_n) P_1(x_1, \ldots, x_n) + f_2(x_1, \ldots, x_n) P_2(x_1, \ldots, x_n), \tag{C16}$$

f_1 and f_2 being polynomials in $x_1, \ldots, x_n$. The resultant $R(x_1, \ldots, x_{n-1})$ cannot be identically zero unless P_1 and P_2 have a common factor. Thus in the case $j \ne j'$, Eq. (C13) is equivalent to

$$R_{j,j'}(g, \omega_1, \ldots, \omega_{n-1}) = 0, \tag{C17}$$

where $R_{j,j'}$ is the nontrivial resultant of the polynomials $Q_n^{(j)}$ and $Q_n^{(j')}$. Likewise, for the case $j = j'$ it is not difficult to see that Eq. (C13) is equivalent to the requirement that the equations

$$Q_n^{(j)}(g, \omega_1, \ldots, \omega_n) = 0$$

and (C18)

$$\frac{\partial}{\partial \omega_n} Q_n^{(j)}(g, \omega_1, \ldots, \omega_n) = 0$$

be simultaneously valid, which in turn implies Eq. (C17) if we identify $R_{j,j}(g, \omega_1, \ldots, \omega_{n-1})$ as the resultant of the polynomials $Q_n^{(j)}$ and $(\partial/\partial\omega_n)Q_n^{(j)}$. Thus $f_{n-1}(g, \omega_1/g, \ldots, \omega_{n-1}/g)$ is singular for values of $g, \omega_1, \ldots, \omega_{n-1}$ satisfying an equation of the form (C1). Finally, if

$$R_{j,j'}(0, \bar{\omega}_1, \ldots, \bar{\omega}_{n-1}) = 0 \quad \text{(C19)}$$

for real $\bar{\omega}_1, \ldots, \bar{\omega}_{n-1}$ and if the point $g = 0$, $\omega_l = \bar{\omega}_l$, $1 \leq l \leq n-1$ is an actual singularity of

$$f_{n-1}(g, \omega_1/g, \ldots, \omega_{n-1}/g),$$

then, as will be seen below, either

$$Q_n^{(j)}(0, \bar{\omega}_1, \ldots, \bar{\omega}_n) = Q_n^{(j')}(0, \bar{\omega}_1, \ldots, \bar{\omega}_n) = 0 \quad \text{for } j \neq j'$$

or (C20)

$$Q_n^{(j)}(0, \bar{\omega}_1, \ldots, \bar{\omega}_n) = \frac{\partial}{\partial \omega_n} Q_n^{(j)}(0, \bar{\omega}_1, \ldots, \bar{\omega}_n) = 0 \quad \text{for } j = j'$$

for some real value of $\bar{\omega}_n$. It then follows from the standard Euclidean construction[21] of $R_{j,j'}$ and Eq. (C3), that

$$R_{j,j'}(g, \omega_1, \ldots, \omega_{n-1}) = R_{j,j'}(g, \omega_1 + \bar{\omega}_1, \ldots, \omega_{n-1} + \bar{\omega}_{n-1}) \quad \text{(C21)}$$

for all $g, \omega_1, \ldots, \omega_{n-1}$. Thus f_{n-1} satisfies condition (i) if n is replaced by $n-1$.

Let us turn to condition (ii). If the function $f_{n-1}(g, k_1, \ldots, k_{n-1})$, for fixed values of $k_1, \ldots, k_{n-1}$, is continued in g along a path circling the origin once counterclockwise, the function $[f_{n-1}]_1$ results. This continuation can be performed by simultaneously deforming the path of the ω_n integration in Eq. (C11) in order to avoid the singularities (in ω_n) of the integrand, all of which are linearly proportional to g. An ω_n contour of the sort shown in Fig. 17(a) results. Condition (iii b) implies that our integral is convergent throughout this continuation. From Eq. (C4) and Eq. (C5) for $b_1, \ldots, b_{n-1}$, a_n all zero we can deduce that for real values of ω_n lying on this contour which are either near zero or very large, the integrand differs from that in Eq. (C11) by continuation in g through 2π about a "logarithmic" branch point. Thus, for sufficiently

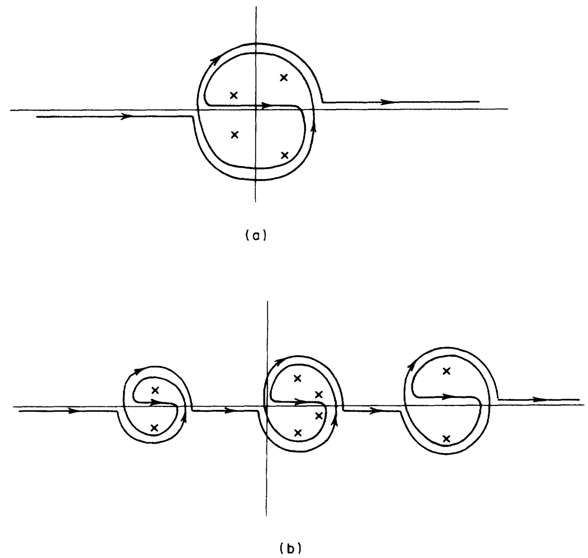

FIG. 17. An example of the ω_n contour which results if: (a) the function $f_{n-1}(g, k_1, \ldots, k_{n-1})$ is continued in g through 2π about the origin, (b) the function $f_{n-1}(g, a_1+b_1/g, \ldots, a_{n-1}+b_{n-1}/g)$ is continued counterclockwise in g along a very small circle about the origin. The crosses locate singularities of the integrand, f_n, in ω_n.

large N,

$$\sum_{m=0}^{N} (-1)^m \frac{N!}{m!(N-m)!}[f_{n-1}(g, k_1, \ldots, k_{n-1})]_m \quad \text{(C22)}$$

can be written as an integral over a contour lying wholly on a circle enclosing all the singularities in g. Therefore, Eq. (C4) is satisfied when n is replaced by $n-1$. Likewise, if

$$f_{n-1}(g, a_1 + b_1/g, \ldots, a_{n-1} + b_{n-1}/g)$$

is continued in g along a sufficiently small[22] circle about the origin, the resulting ω_n contour has the form shown in Fig. 17(b). Using an argument like that above, we can then conclude that Eq. (C5) is also obeyed when n is replaced by $n-1$. The form of the contour shown in Fig. 17(b) depends critically on the linearity in g of those singularities in ω_n which approach the real axis as g tends to zero, i.e., the linearity in g of

$$\Delta\omega^{j,i}(g) = \omega_n^{j,i}(g, ga_1 + b_1, \ldots, ga_{n-1} + b_{n-1}) - \omega_n^{j,i}(0, b_1, \ldots, b_{n-1}) \quad \text{(C23)}$$

for real $\omega_n^{j,i}(0, b_1, \ldots, b_{n-1})$. Using Eq. (C3) we can rewrite Eq. (C12) as

$$Q_n^{(j)}(g, ga_1, \ldots, ga_{n-1}, \Delta\omega_n^{j,i}(g)) = 0 \quad \text{(C24)}$$

which implies this linearity.

Let us now show that f_{n-1} also obeys condition (iii). Divide the $n-1$ variables into two groups,

$k_{\sigma_1}, \ldots, k_{\sigma_m}$ and $k_{\sigma_{m+2}}, \ldots, k_{\sigma_n}$, and let $\sigma_{m+1} = n$. For real values of $k_1, \ldots, k_{n-1}, g$, the condition (iii a) obeyed by f_n implies that

$$|f_{n-1}(g, k_1, \ldots, k_{n-1})| \leq A' \exp\left(-B' \sum_{j=1}^{m} |\omega_{\sigma_j}|^{2/3}\right) \quad \text{(C25)}$$

if $|\omega_{\sigma_i}| \geq R' = \max(R_1, R_2)$ for all $1 \leq i \leq m$. Here R_1 is chosen so that if $|\omega_{\sigma_i}| > R_1$ for all $1 \leq i \leq m+1$, then

$$|f_n(g, k_1, \ldots, k_n)| < A_1 \left(-B_1 \sum_{j=1}^{m+1} |\omega_j|^{2/3}\right) \quad \text{(C26)}$$

while R_2 is the largest value of R required for the validity of Eq. (C7) as $|\omega_n|$ is varied from zero to R_1. The constants A' and B' are given by

$$A' = \frac{3}{2}\sqrt{\pi}\,\frac{A_1}{B_1^{3/2}} + 2R_1 A, \quad B = \min(B, B_1). \quad \text{(C27)}$$

Thus $f_{n-1}(g, k_1, \ldots, k_{n-1})$ satisfies condition (iii a). Now continue f_{n-1} analytically from the point $g^{(0)}, \omega_1^{(0)}, \ldots, \omega_{n-1}^{(0)}$ into the region

$$||g| - |g^{(0)}|| < \delta_1 |g^{(0)}| \left(\sum_{i=1}^{n-1} |\omega_i - \omega_i^{(0)\,2}\right)^{1/2} < \gamma' W_m(\sigma), \quad \text{(C28)}$$

$$|Q_{n-1}^{(j)}(g, \omega_1, \ldots, \omega_{n-1})| > \delta_2' [W_m(\sigma)]^{\nu'} \quad \text{(C29)}$$

for real $g^{(0)}, \omega_1^{(0)}, \ldots, \omega_{n-1}^{(0)}$, and $|\omega_{\sigma_i}^{(0)}| > R, 1 \leq i \leq m$. Define

$$\overline{Q}_n^{(j)}(\omega_{\sigma_1}, \ldots, \omega_{\sigma_m}, \omega_n) = Q_n^{(j)}(0, \omega_1, \ldots, \omega_n)|_{\omega_{\sigma i}=0}, \quad m+1 < i \leq n. \quad \text{(C30)}$$

By assumption $\overline{Q}_n^{(j)}$ is a homogeneous polynomial in $\omega_{\sigma_1}, \ldots, \omega_{\sigma_m}, \omega_n$ of degree ν_j. Given $\gamma > 0$ there exist values of R_1 and γ' such that

$$|\omega_n^{j,i}(g, \omega_1, \ldots, \omega_n) - \overline{\omega}_n^{j,i}(\omega_{\sigma_1}^{(0)}, \ldots, \omega_{\sigma_m}^{(0)})| < \tfrac{1}{2}\gamma W_m \quad \text{(C31)}$$

uniformly in $\omega_{\sigma_1}^{(0)}, \ldots, \omega_{\sigma_m}^{(0)}$, where $\overline{\omega}_n^{j,i}$ is a root of $\overline{Q}_n^{(j)}$,

$$\overline{Q}_n^{(j)}(\omega_{\sigma_1}, \ldots, \omega_{\sigma_m}^{(0)}, \overline{\omega}_n^{j,i}) = 0. \quad \text{(C32)}$$

Thus as we continue $g, \omega_1, \ldots, \omega_{n-1}$ into the region specified by Eqs. (C28) and (C29), the singularities in ω_n stay sufficiently close to a discrete set of fixed points to insure the validity of Eq. (C8).

Therefore, the inequality (25) can be maintained during the continuation described above if we are able to choose the ω_n contour in a manner consistent with Eq. (C9). This is assured if we require

$$|\omega_n - \omega_n^{j,i}| > \delta'' W_m^{\nu/\nu_j} \quad \text{(C33)}$$

for all $1 \leq i \leq r_j$. The ω_n contour can be chosen to satisfy (C33) provided

$$|\omega_n^{j,i} - \omega_n^{j'i'}| > 2\delta'' W_m^{\nu/\nu_j} \quad \text{(C34)}$$

for $i \neq i'$ or $j \neq j'$. For $j \neq j'$ use Eq. (C16) to write

$$R_{j,j'}(g, \omega_1, \ldots, \omega_{n-1}) = f_j(g, \omega_1, \ldots, \omega_n) Q_n^{(j)}(g, \omega_1, \ldots, \omega_n) + f_{j'}(g, \omega_1, \ldots, \omega_n) Q_n^{(j')}(g, \omega_1, \ldots, \omega_n). \quad \text{(C35)}$$

If we set $\omega_n = \omega_n^{j,i}$ in this equation and require

$$|R_{j,j'}(g, \omega_1, \ldots, \omega_{n-1})| > \delta_2' W_m^{\nu'}, \quad \text{(C36)}$$

we obtain

$$\delta_2' W_m^{\nu'} < f_j(g, \omega_1, \ldots, \omega_{n-1}, \omega_n^{j,i'}) \prod_{i=1}^{\nu_j} (\omega_n^{j',i'} - \omega_n^{j,i}), \quad \text{(C37)}$$

which implies Eq. (34) for some value of ν' less than the degree of $R_{j,j'}$. Similarly for $j = j'$, write

$$R_{j,j}(g, \omega_1, \ldots, \omega_{n-1}) = f_j(g, \omega_1, \ldots, \omega_n) Q_n^j(g, \omega_1, \ldots, \omega_n) + f_j'(g, \omega_1, \ldots, \omega_n) \frac{\partial}{\partial \omega_n} Q_n^j(g, \omega_1, \ldots, \omega_n), \quad \text{(C38)}$$

let $\omega_n = \omega_n^{j,i}$ and assume $R_{j,j}$ bounded from below by $\delta_2' W_m^{\nu'}$; one obtains

$$\delta_2' W_m^{\nu'} \leq f_j'(g, \omega_1, \ldots, \omega_n^{i,j}) \prod_{i' \neq i} (\omega_n^{j,i'} - \omega_n^{j,i}), \quad \text{(C39)}$$

which is equivalent to Eq. (C34). Thus for values of $g, \omega_1, \ldots, \omega_{n-1}$ consistent with Eqs. (C28) and (C29), the ω_n contour can be chosen so that the bound (C25) is satisfied.

APPENDIX D

We will now demonstrate that the terms $S_{n,i}(g)$ in the S-matrix expansion (38), (39) which can be constructed for real g directly from lower-order unitary amplitudes automatically satisfy the unitarity condition

$$S_{n,i}(g) + S_{n,i}(g)^\dagger = -\sum_{n'=1}^{n-1} \sum_{j,k} [S_{n',j}(g) S_{n-n',k}(g)^\dagger]_i. \quad \text{(D1)}$$

Such terms correspond to case (a) in the discussion of Sec. IV. Likewise it will be shown that those terms $U_{n,i}(g)$ falling into case (b) must obey

$$U_{n,i}(g) + U_{n,i}(g^*)^\dagger = -\sum_{n'=1}^{n-1} \sum_{j,k} [U_{n',j}(g) U_{n-n',k}(g^*)^\dagger]_i \quad \text{(D2)}$$

if they are constructed out of lower-order amplitudes which obey a similar pseudounitarity condition. The method of proof developed below was suggested by the solution to a similar problem obtained by Veltman.[23] For simplicity, we will limit this discussion to amplitudes which correspond to graphs containing no internal fermion lines.

Let us begin by considering the nth-order amplitude

$$M_{n,1}(g^2, p_1, \ldots, p_n)^{\beta_1,\ldots,\beta_n}_{\alpha_1,\ldots,\alpha_n}$$

corresponding to the class $i=1$ of connected graphs containing only minus lines. This amplitude can be expressed as a series of terms, each term corresponding to graphs containing a particular arrangement of L internal lines. We will assign the integers $k=1,\ldots,n$ to the vertices and $j=1,\ldots,L$ to the internal lines of such a graph. Let $P_k = (\vec{P}_k, iE_k)$ be the total four-momentum entering the kth vertex and $l_j = (\vec{l}_j, ie_j)$ the four-momentum carried by the jth internal line from vertex a_j to vertex b_j. An example of such a graph is shown in Fig. 18. The term in the amplitude $M_{n,i}$ represented by graphs with such an internal structure can be written

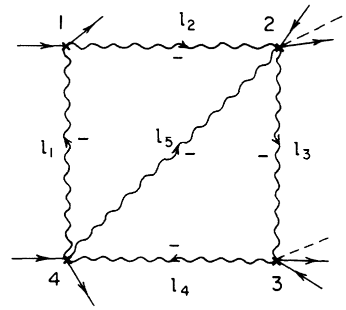

FIG. 18. A fourth-order graph constructed of minus lines only. For this example $n=4$, $L=5$, $a_5=4$, $b_5=2$, etc.

$$\frac{(-i\Delta m_0)^n}{(2\pi)^{4(L)}}\left(\frac{ig^4}{\pi^2}\right)^L E_{n,L}(p_1,\ldots,p_n)\prod_{j=1}^{L}\left(\sum_{\gamma_j=0}^{1}\right)\prod_{k=1}^{n}\left[\exp\left(-\tfrac{1}{4}i\pi\sum_{r=1}^{L}\tau_1|\epsilon_{r,k}|\gamma_r\right)\tau_3\exp\left(+\tfrac{1}{4}i\pi\sum_{r'=1}^{L}\tau_1|\epsilon_{r',k}|\gamma_{r'}\right)\right]_{\alpha_k\beta_k}, \tag{D3}$$

where p_k is the total external four-momentum entering the kth vertex and

$$\epsilon_{j,k} = \begin{cases} 1 & \text{if } k=b_j \\ -1 & \text{if } k=a_j \\ 0 & \text{otherwise}. \end{cases} \tag{D4}$$

The function $E_{n,L}$ is then given by

$$E_{n,L}(p_1,\ldots,p_n) = \prod_{r=1}^{L}\left[\prod_{i=1}^{3}\left(\int_{-\infty}^{\infty} d(l_r)_i\right)\int_{-i\infty}^{i\infty} de_r B_-\left(\frac{g^2 l_r^2}{4\pi^2}\right)\right]\prod_{k=1}^{n}\delta^4(P_k) \tag{D5}$$

and is initially defined for imaginary external energies $(p_k)_0$ and internal integration over Euclidean four-momenta as indicated by the imaginary e_r contour in Eq. (D5).

Using the techniques developed in Appendix C, it is not difficult to show that $E_{n,L}(p_1,\ldots,p_n)$ can be continued to real values of $(p_1)_0,\ldots,(p_n)_0$. Consider

$$F_{n,L}(z, e'_1,\ldots,e'_{n'}) = \prod_{k=1}^{n'}\left(\int d^3l'_k\right)\prod_{j=1}^{L} B_-\left(\frac{g^2}{4\pi^2}\left[\left(\sum_{i=1}^{n'}\eta_{j,i}\vec{l}'_i+\vec{p}'_j\right)^2-\left(\sum_{i=1}^{n'}z\eta_{j,i}e'_i+z(p'_j)_0\right)^2\right]+\lambda_j^{\,2}\right), \tag{D6}$$

where the δ functions in Eq. (D5) have been eliminated, the resulting integration variables labeled $l'_i = (\vec{l}'_i, ie'_i)$, $1\le i\le n'$, and the total external four-momentum carried by the jth line labeled p'_j, $1\le j\le L$. The small "masses" λ_j have been introduced to regulate the familiar infrared divergences present in such a theory of interacting massless scalar and massive spinor particles. The function $F_{n,L}(z, e'_1,\ldots,e'_{n'})$ depends analytically on the variables $z, e'_1,\ldots,e'_{n'}$ at all points except possibly those satisfying

$$P_n^{(j)}(z, e'_1,\ldots,e'_{n'}) = 0, \quad 1\le j\le N_n \tag{D7}$$

where $P_n^{(j)}$ is a polynomial in the variables $z, e'_1,\ldots,e'_{n'}$. It follows directly from Eq. (D6) that for imaginary values of e'_i, $1\le i\le n'$ and $(p'_j)_0$, $1\le j\le L$, $F_{n,L}(z, e'_1,\ldots,e'_n)$ is analytic in the half-plane $\mathrm{Re}\,z>0$ and can be continued to all but a finite number of points on the imaginary z axis. If the variables $e'_1,\ldots,e'_{n'}$ are divided into two subsets, $e'_{\sigma_1},\ldots,e'_{\sigma_m}$ and $e'_{\sigma_{m+1}},\ldots,e'_{\sigma_{n'}}$, then there exist positive real numbers A, B, and R such that $|e_{\sigma_i}|>R$ for all $1\le i\le m$ implies

$$|F_{n,L}(z, e'_1,\ldots,e'_{n'})| \le A\exp\left(-B\sum_{i=1}^{m}|ze'_{\sigma_i}|^{2/3}\right) \tag{D8}$$

for z real and e_i', $1 \leqslant i \leqslant n'$, imaginary. For sufficiently large R and A, Eq. (D8) remains valid if the variables $z, e_1', \ldots, e_{n'}'$ are continued from the real values $z^{(0)}, e_1^{(0)}, \ldots, e_{n'}^{(0)}$ into the region

$$\left||z| - |z^{(0)}|\right| < \delta_1 |z^{(0)}|\,, \quad \sum_{i=1}^{n'} |z e_i' - z^{(0)} e_i^{(0)}|^2 < \gamma^2 \sum_{i=1}^{m} |z e_{\sigma_i}^{(0)}|^2\,, \quad |P_n^{(j)}(z, e_1', \ldots, e_{n'}')| > \delta_2 \sum_{i=1}^{m} |z e_{\sigma_i}^{(0)}|^{\nu} \tag{D9}$$

for some real, predetermined ν, $1 > \delta_1 > 0$, $\delta_2 > 0$, sufficiently small γ, and all values of $e_{\sigma_i}^{(0)}$, $1 \leqslant i \leqslant m$ obeying $|e_{\sigma_i}^{(0)}| > R$, $1 \leqslant i \leqslant m$. Using the method of Appendix C, we can sequentially integrate over $e_{n'}', e_{n'-1}', \ldots, e_1'$ maintaining the analytic structure in z. Note that since the location of the singularities in z does not depend on g, the singularity structure in g, established for Euclidean momenta, is not altered by the continuation in z from real to imaginary values.

In order to show that the amplitude $M_{n,i}$ is unitary after being continued to real values of the external energies, it is convenient to use a somewhat different representation for the term $E_{n,L}$:

$$E_{n,L}(p_1, \ldots, p_n) = \prod_{r=1}^{L} \left(\int d^3 l_r \right) \prod_{k=1}^{n} \delta^3(\vec{P}_k) \prod_{r'=1}^{L} \left[\int_{c_{r'}} d e_{r'} B_-\!\left(\frac{g^q l_{r'}{}^2}{4\pi^2} \right) \right] \left(\frac{1}{2\pi i} \right)^n \prod_{k'=1}^{n} \left(\frac{1}{E_{k'} + \epsilon} - \frac{1}{E_{k'} - \epsilon} \right), \tag{D10}$$

where for imaginary $(p_1)_0, \ldots, (p_n)_0$, the contours c_r coincide with the imaginary axis and the small real quantity ϵ indicates in what way each c_r avoids the singularities $1/E_k$, $1 \leqslant k \leqslant n$. As we continue the external energies from imaginary to real values, we will distort each contour c_r to avoid a discrete set of singularities. We will choose each contour c_r for real external energies so that (i) c_r lies along the real axis for $|e_r| < E$, where E is a real number much greater than the magnitude of any external energy or momentum,[24] and (ii) the quantity $\mathrm{Re}(e_r)$ never decreases as e_r follows the contour c_r. For example, the contours c_1 and c_2 entering the amplitude corresponding to the graph found in Fig. 18 are shown in Fig. 19. In addition to avoiding the singularities in e_r of the function

$$\mathcal{E}_{n,r-1}(e_r, \ldots, e_L) = \prod_{r'=1}^{r-1} \left[\int_{c_{r'}} d e_{r'} B_-\!\left(\frac{g^2 l_{r'}{}^2}{4\pi^2} \right) \right] \prod_{k'=1}^{n} \left(\frac{1}{E_{k'} + \epsilon} - \frac{1}{E_{k'} - \epsilon} \right), \tag{D11}$$

we will also route the contour c_r around left and right distinguished values of e_r defined inductively as follows: A distinguished value of the function $\mathcal{E}_{n,r-1}(e_r, \ldots, e_L)$ is either the location of a singularity in e_r or a value of e_r for which a left and a right distinguished value of e_{r-1} for $\mathcal{E}_{n,r-2}(e_{r-1}, e_r, \ldots, e_n)$ coincide. A distinguished point is called left if it moves to the left when ϵ is increased and called right if it moves to the right. The singularities of $E_{n,r}$ in e_r naturally fall into pairs of points whose difference is real and depends only on ϵ and $|\vec{l}_1|, \ldots, |\vec{l}_r|$. Each distinguished point lies on a straight line connecting two members of such a pair. We will choose c_r to pass above right distinguished points and below left distinguished points. The contour c_2 in Fig. 19 is so chosen.

Given a permutation $\sigma_1, \ldots, \sigma_n$ of the first n integers, define a new function $E_n(p_{\sigma_1}, \ldots, p_{\sigma_\rho} | p_{\sigma_{\rho+1}}, \ldots, p_{\sigma_n})$ from Eq. (D10) by altering the contours c_r, $1 \leqslant r \leqslant L$. Let $a(r) = \sigma_{\alpha(r)}$, $b(r) = \sigma_{\beta(r)}$, $1 \leqslant r \leqslant L$ and distinguish four cases:

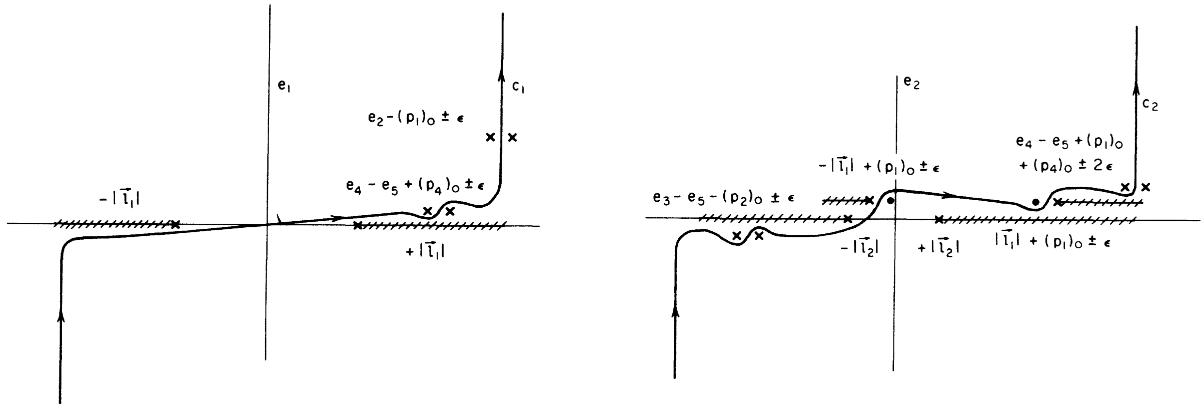

FIG. 19. The contours c_1 and c_2 followed by the variables of integration e_1 and e_2 appearing in the amplitude $E_{4,5}(p_1, p_2, p_3, p_4)$, derived from the graph in Fig. 18. These contours result after continuation from imaginary to real external energies.

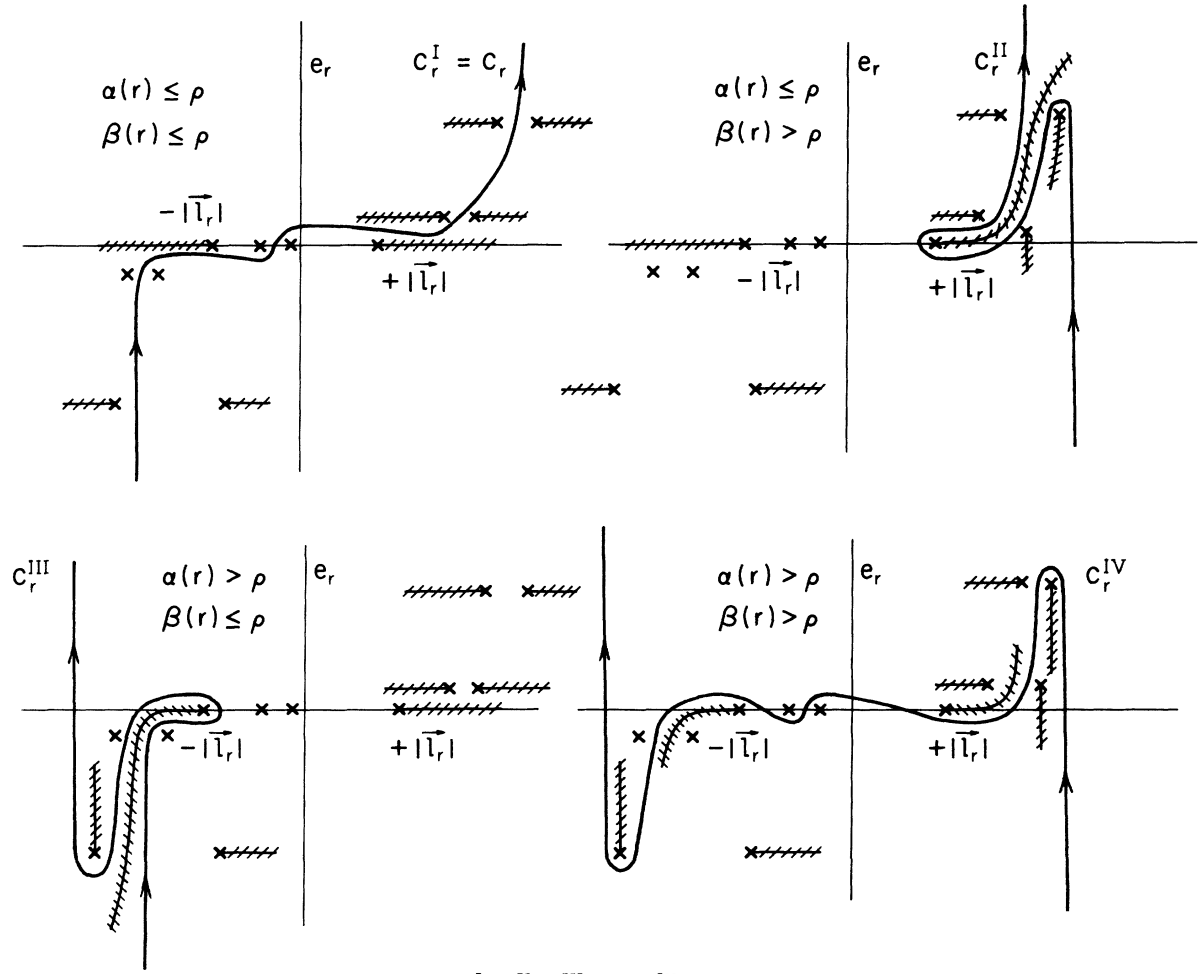

FIG. 20. The integration contours c_r^{I}, c_r^{II}, c_r^{III}, and c_r^{IV} required by the definition of the function $E_{n,L}(p_{\sigma_1},\ldots,p_{\sigma_\rho}|p_{\sigma_{\rho+1}},\ldots,p_{\sigma_n})$.

$$
\begin{array}{ll}
\mathrm{I.} & \alpha(r)\leqslant\rho\,,\quad \beta(r)\leqslant\rho\,,\\
\mathrm{II.} & \alpha(r)\leqslant\rho\,,\quad \beta(r)>\rho\,,\\
\mathrm{III.} & \alpha(r)>\rho\,,\quad \beta(r)\leqslant\rho\,,\\
\mathrm{IV.} & \alpha(r)>\rho\,,\quad \beta(r)>\rho\,.
\end{array}
\tag{D12}
$$

In case I the contour $c_r^{\mathrm{I}}=c_r$ is used. For case II the contour c_r is followed down from $+\infty$ until it passes above the singularity $e_r=+|\vec{1}_r|$; this singularity is circled counterclockwise and the contour c_r followed back toward $+\infty$ until the last distinguished point on c_r is reached; the contour c_r^{II} then circles this point clockwise and is directed downward, parallel to the imaginary axis, to $-\infty$. The contour c_r^{III} is defined so that if e_r follows c_r^{III}, $-e_r$ will follow a contour similar to c_r^{II}. Finally, c_r^{IV} is a combination of those parts of c_r^{II} and c_r^{III} which differ from c_r. Examples of these four contours are shown in Fig. 20. It should be noted that

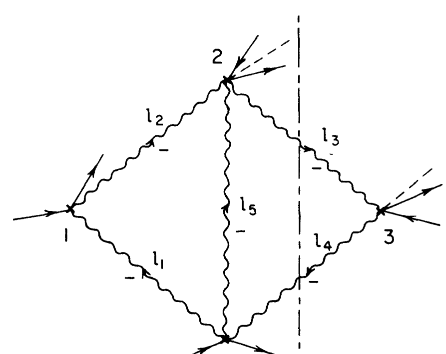

FIG. 21. The graph corresponding to the function $E_{4,5}(p_1,p_2,p_4|p_3)$ formed by appropriate modification of the amplitude represented in Fig. 18.

the location of the distinguished points in e_r of the function $\mathcal{E}_{n,r-1}(\sigma,\rho,e_r,\ldots,e_n)$, defined from Eq. (D11) by making the above replacement of contours, is the same as for the original function $\mathcal{E}_{n,r-1}(e_r,\ldots,e_n)$. However, the locations of actual singularities in e_r are different for the two functions. Our new function $E_{n,L}(p_{\sigma_1},\ldots,p_{\sigma_\rho}|p_{\sigma_{\rho+1}},\ldots,p_{\sigma_n})$ can be represented graphically by (i) moving the vertices of a graph corresponding to the original amplitude $E_{n,L}$ so that the vertices $\sigma_1,\ldots,\sigma_\rho$ all lie to the left of the vertices $\sigma_{\rho+1},\ldots,\sigma_n$ and (ii) drawing a vertical dashed line separating these two groups of vertices. Thus if we consider the amplitude $E_{4,5}(p_1,p_2,p_3,p_4)$ corresponding to the graph in Fig. 18, the graph representing $E_{4,5}(p_1,p_2,p_4|p_3)$ is shown in Fig. 21.

With this choice of contours, the integration variables $e_1,\ldots,e_L$ entering the definition of $E_{n,L}(p_{\sigma_1},\ldots,p_{\sigma_\rho}|p_{\sigma_{\rho+1}},\ldots,p_{\sigma_n})$ obey

$$\begin{aligned}&\mathrm{Re}(e_{r_i})\geq 0, \quad 1\leq i\leq d\\ &\mathrm{Re}(-e_{r_i})\geq 0, \quad d< i\leq D\end{aligned} \tag{D13}$$

where $\{r_i\}_{1\leq i\leq d}$ is the set of lines of case II, $\alpha(r_i)\leq\rho$, $\beta(r_i)>\rho$, $1\leq i\leq d$, while for $d<i\leq D$, $\alpha(r_i)>\rho$, $\beta(r_i)\leq\rho$. Since energy conservation requires

$$-\sum_{i=\rho+1}^{n}(p_{\sigma_i})_0=\sum_{i=1}^{\rho}(p_{\sigma_i})_0=\sum_{i=1}^{d}e_{r_i}-\sum_{i=d+1}^{D}e_{r_i}, \tag{D14}$$

only those parts of c_r^{II} and c_r^{III} lying on the real axis can contribute to $E_{n,L}(p_{\sigma_i},\ldots,p_{\sigma_\rho}|p_{\sigma_{\rho+1}},\ldots,p_{\sigma_n})$. Thus we can write

$$\begin{aligned}E_{n,L}(p_{\sigma_1},\ldots,p_{\sigma_\rho}|p_{\sigma_{\rho+1}},\ldots,p_{\sigma_n})=&\prod_{i=1}^{D}\left[\int d^4l_{r_i}2i\,\mathrm{Im}B_-\left(\frac{g^2l_{r_i}^{\,2}}{4\pi^2}-i\epsilon\right)\right]\prod_{i=1}^{d}[\theta(e_{r_i}-|\vec{1}_{r_i}|)]\prod_{i=d+1}^{D}[\theta(-e_{r_i}-|\vec{1}_{r_i}|)]\\ &\times E_{\rho,L'}\left(p_{\sigma_1}+\sum_{j=1}^{D}\epsilon_{r_j,\sigma_1}l_{r_j},\ldots,p_{\sigma_\rho}+\sum_{j=1}^{D}\epsilon_{r_j,\sigma_\rho}l_{r_j}\right)\\ &\times E_{n-\rho,L-L'-D}\left(\,\Big|\,p_{\sigma_{\rho+1}}+\sum_{j=1}^{D}\epsilon_{r_j,\sigma_{\rho+1}}l_{r_j},\ldots,p_{\sigma_n}+\sum_{j=1}^{D}\epsilon_{r_j,\sigma_n}l_{r_j}\right).\end{aligned} \tag{D15}$$

The amplitudes $E_{\rho,L'}$ and $E_{n-\rho,L-L'-D}$ correspond to the left- and right-hand graphs obtained by dropping all of the internal lines cut by the dashed line in the graphical representation of $E_{n,L}(p_{\sigma_1},\ldots,p_{\sigma_\rho}|p_{\sigma_{\rho+1}},\ldots,p_{\sigma_n})$. The second amplitude $E_{n-\rho,L-L'-D}$ can be related to a conventional amplitude containing only contours c_r^{I} by applying the following equation:

$$\mathcal{E}_{n_1,r}(\sigma,0,e_{r+1},\ldots,e_{L_1})=(-1)^{n_1+r}\,\mathcal{E}_{n_1,r}(e_{r+1},\ldots,e_{L_1})^* \tag{D16}$$

valid for real $e_{r+1},\ldots,e_{L_1}$. Equation (D15) can be derived by using induction on r. First, for $r=0$ the validity of Eq. (D16) follows directly from Eq. (D11). Next, assume that Eq. (D16) holds for $r\leq r_0$ and consider

$$\int_{c^{\mathrm{IV}}}de_{r_0+1}B_-\left(\frac{g^2l_{r_0+1}^{\,2}}{4\pi^2}\right)\mathcal{E}_{n_1,r_0}(\sigma,0,e_{r_0+1},\ldots,e_{L_1}) \tag{D17}$$

and

$$(-1)^{n_1+r_0+1}\int_{c^{\mathrm{I}}}de_{r_0+1}B_-\left(\frac{g^2l_{r_0+1}^{\,2}}{4\pi^2}\right)\mathcal{E}_{n_1,r_0}(e_{r_0+1},\ldots,e_{L_1}) \tag{D18}$$

for real values of $e_{r_0+2},\ldots,e_{L_1}$. If the internal lines $1\leq r\leq r_0$ make up a disconnected graph, then our induction hypothesis easily implies that the expression (D17) is the complex conjugate of (D18). Otherwise, the singularities in e_{r_0+1} of $\mathcal{E}_{n_1,r_0}(\sigma,0,e_{r_0+1},\ldots,e_{L_1})$, an amplitude defined entirely with the contours c_r^{IV}, can be naturally grouped into pairs whose difference is real just as in the case of $\mathcal{E}_{n_1,r_0}(e_{r_0+1},\ldots,e_{L_1})$. However, for each such pair, there are distinguished values of e_{r_0+1} lying on a straight line connecting the pair, a right distinguished point lying to the right of the right-hand member of the pair and a left distinguished point lying to the left of the left-hand member. Typical contours c^{IV} and c^{I} occurring in the expressions (D17) and (D18) are shown in Fig. 22. Continuing Eq. (D16) for $r=r_0$ in e_{r_0+1} and referring to Fig. 22, we can conclude that the quantity (D17) is the complex conjugate of (D18). Thus Eq. (D16) is established by induction. If Eq. (D16) for $n_1=n-\rho$, $L_1=L-L'-D$, and $r=L_1$ is integrated over the internal spatial momenta $\vec{1}_i$, $1\leq i\leq L_1$, we obtain

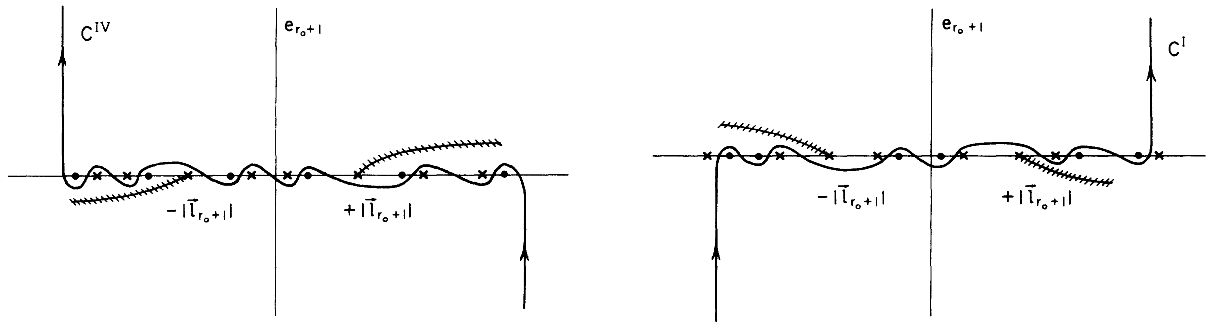

FIG. 22. Typical contours $c^{\rm IV}$ and $c^{\rm I}$ occurring in the expressions (D17) and (D18).

$$E_{n-\rho,L-L'-D}(|k_1,\ldots,k_{n-\rho})=(-1)^{L-L'-D}E_{n-\rho,L-L'-D}(k_1,\ldots,k_{n-\rho})^* \tag{D19}$$

for any real four momenta $k_1,\ldots,k_{n-\rho}$.

Finally let us observe that

$$\sum_{(\sigma,\rho)}(-1)^\rho E_{n,L}(p_{\sigma_1},\ldots,p_{\sigma_\rho}|p_{\sigma_{\rho+1}},\ldots,p_{\sigma_n})=0\,, \tag{D20}$$

where $\sum_{(\sigma,\rho)}$ indicates a sum over all distinct divisions of the first n integers into two groups $\sigma_1,\ldots,\sigma_\rho$ and $\sigma_{\rho+1},\ldots,\sigma_n$. Equation (D20) can be derived from Eq. (D10) if we first write

$$\prod_{k'=1}^{n}\left(\frac{1}{E_{k'}+\epsilon}-\frac{1}{E_{k'}-\epsilon}\right)=\sum_{m=0}^{n}\sum_{\delta}\left[\prod_{n'=1}^{m}\left(\frac{1}{\sum_{j=1}^{n'}E_{\delta_j}+\epsilon}\right)\prod_{n''=m+1}^{n}\left(\frac{1}{-\sum_{j'=n''}^{n}E_{\delta_{j'}}+\epsilon}\right)\right], \tag{D21}$$

where $\sum_\delta$ represents a sum over all permutations $\delta_1,\ldots,\delta_n$ of the first n integers. If Eqs. (D10) and (D21) are substituted into the left-hand side of Eq. (D20), we obtain

$$\frac{1}{(2\pi i)^n}\sum_{(\sigma,\rho)}\prod_{r=1}^{L}\left(\int d^3l_r\right)\prod_{k=1}^{n}[\delta^3(P_k)](-1)^\rho\prod_{r'=1}^{L}\left[\int_{c_{r'}(\sigma,\rho)}de_{r'}\,B_-\left(\frac{g^2l_{r'}^2}{4\pi^2}\right)\right]\sum_{m=0}^{n}\sum_{\delta}\prod_{n'=1}^{m}\left(\frac{1}{\sum_{j=1}^{n'}E_{\delta_j}+\epsilon}\right)\prod_{n''=m+1}^{n}\left(\frac{1}{-\sum_{j'=n''}^{n}E_{\delta_{j'}}+\epsilon}\right), \tag{D22}$$

where the contour $c_{r'}(\sigma,\rho)$, for a particular grouping (σ,ρ) and variable $e_{r'}$, is $c_{r'}^{\rm I}$, $c_{r'}^{\rm II}$, $c_{r'}^{\rm III}$, or $c_{r'}^{\rm IV}$, following Eq. (D12). Consider a particular term in this sum with a specific division of the vertices into left and right groups, $\sigma_1,\ldots,\sigma_\rho$; $\sigma_{\rho+1},\ldots,\sigma_n$ and a definite permutation δ. Assume that $\delta_1=\sigma_k$ and that $k\le\rho$. A second term in this sum with the same permutation δ and a grouping $(\sigma',\rho-1)$ of the form $\sigma_1,\ldots,\sigma_{k-1}$, $\sigma_{k+1},\ldots,\sigma_\rho$; $\sigma_k,\sigma_{\rho+1},\ldots,\sigma_n$ differs from the first term only in the choice of contours c_r for r such that $\alpha(r)=k$ or $\beta(r)=k$. However, if $\beta(r)=k$ then the form of the energy denominators in Eq. (D22) implies that the variable e_r has no singularities to the right of the contour $c_r(\sigma,\rho)$ except for the point $+|\vec{l}_r|$. Inspection of Fig. 20 reveals that for $\alpha(r)\le\rho$ the appropriate contours $c_r^{\rm I}$ and $c_r^{\rm II}$ are equivalent; or in the case $\alpha(r)>\rho$ the contours $c_r^{\rm III}$ and $c_r^{\rm IV}$ are equivalent. Similarly, if $\alpha(r)=k$, then e_r has no singularities to the left of c_r except the point $-|\vec{l}_r|$. Then if $\beta(r)\le\rho$ the relevant contours $c_r^{\rm I}$ and $c_r^{\rm III}$ are equivalent while for $\beta(r)>\rho$ the contours $c_r^{\rm II}$ and $c_r^{\rm IV}$ can be deformed into each other. The case $k>\rho$ behaves in a similar manner. Thus the sum in Eq. (D22) can be separated into pairs of canceling terms and Eq. (D20) is established.

If Eq. (D20) is combined with Eqs. (D15) and (D19) and summed over all possible, connected, arrangements of minus lines, then Eq. (D1) for $i=1$ follows.

Clearly this method of proof applies directly to all amplitudes formed as a multiple integral of a product of the functions B_- and B_+. However, our method can also be applied to the general case with only slight modification. If, for example, a subgraph (not necessarily connected) of the graph of interest must be computed for imaginary g and then continued to real g, we will apply the above method integrating over that subgraph's internal momenta first. Equations (D15) and (D19) are then satisfied, for imaginary g, by the amplitude obtained after integrating over only those variables appearing in this subgraph provided the other variables have values lying on the appropriate contours. If we continue in g according to Eqs. (45) and (63), Eqs. (D15) and (D19) will continue to hold. Equation (D15) and Eq. (D13), valid for the remaining integration variables, will then imply that the full amplitude obeys Eq. (D15), while Eq. (D19) can be established inductively for the full amplitude as before. Finally Eq. (D20) can be derived using the same arguments as above, so that unitarity is established.

APPENDIX E

Let us briefly consider two other nonrenormalizable field theories to which our techniques can be applied:

(i) A theory containing two scalar fields $\phi(x)$ and $\theta(x)$ coupled by the interaction Lagrangian density

$$\mathcal{L}_I(x) = -\lambda : \theta^2(x) e^{g\phi(x)} : , \tag{E1}$$

where the interacting field $\phi(x)$ is massless while $\theta(x)$ has a nonzero mass. In this theory only the function B_+ occurs and, therefore, the operator $U(g)$ can be computed to *arbitrary* order in λ. The results of Appendix C, when combined with our continuation procedure, then yield a finite, unitary scattering matrix depending on λ, g, the θ particle's mass, and an additional real parameter b.[25]

(ii) A neutral-vector-boson theory determined by the Lagrangian

$$\mathcal{L}(x) = -\frac{1}{4}\left(\frac{\partial W_\mu}{\partial x_\nu} - \frac{\partial W_\nu}{\partial x_\mu}\right)\left(\frac{\partial W_\mu}{\partial x_\nu} - \frac{\partial W_\nu}{\partial x_\mu}\right) - \tfrac{1}{2} m_{W,0}{}^2 W_\mu W_\mu - \bar{l}'(x)\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_0 + \Delta m_0' \tau_3\right) l'(x) - igW_\mu \bar{l}'(x)\gamma_\mu \tau_1 l'(x) , \tag{E2}$$

where $W_\mu(x)$ is a vector field and all other notation is the same as that appearing in Eq. (1). The scattering matrix predicted by the Lagrangian (E2) can be developed as a standard perturbation-series expansion in $\Delta m_0'$ and g. In such an expansion we can divide the vector-boson propagator $\Delta_F(x)_{\mu\nu}$ into two terms:

$$\begin{aligned}\Delta_F(x)_{\mu\nu} &= \Delta_F^{(1)}(x)_{\mu\nu} + \Delta_F^{(0)}(x)_{\mu\nu} \\ &= -\int \frac{d^4q}{(2\pi)^4} e^{iq\cdot x} \frac{\delta^{\mu\nu} - q^\mu q^\nu/q^2}{q^2 + m_W{}^2 - i\epsilon} - \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot x} \frac{q^\mu q^\nu}{m_W{}^2(q^2 - i\epsilon)} .\end{aligned} \tag{E3}$$

If taken by itself, the first term, $\Delta_F^{(1)}$, yields amplitudes no more divergent than those of a renormalizable field theory. On the other hand, the second term has exactly the same effect as an exchange of the derivative-coupled scalar boson analyzed before. Thus, if all Feynman graphs of a given order in $\Delta m_0'$ are grouped according to the number of powers of $\Delta_F^{(1)}$ and $\Delta_F^{(0)}$ which they contain, our methods can be used to sum the contribution of all powers of $\Delta_F^{(0)}$ for a fixed power of $\Delta m_0'$ and $\Delta_F^{(1)}$. The result is a series expansion for the scattering matrix of the form

$$S = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \sum_{n''=0}^{r(n')} g^n (\Delta m_0)^{n'} \ln^{n''}(g') B_{n,n',n''} . \tag{E4}$$

For $n' = 0$, the coefficients $B_{n,0,0}$ are simply the predictions of a renormalizable field theory. For $n' \neq 0$, the results presented previously imply that the $B_{n,n',n''}$ are unique functions of the physical masses m, m_W and a new real parameter b, and are explicitly finite for $n' \leq 3$.

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†Alfred P. Sloan Foundation Fellow.

[1]This theory, or one similar to it, has been discussed previously by a number of authors: M. K. Volkov, Commun. Math. Phys. 7, 289 (1968); S. Okubo, Progr. Theoret. Phys. (Kyoto) 11, 80 (1954); R. Arnowitt and S. Deser, Phys. Rev. 100, 349 (1955); B. A. Arbuzov, N. M. Atakishiev, and A. T. Filippov, Yadern. Fiz. 8, 385 (1968) [Soviet J. Nucl. Phys. 8, 222 (1969)]; A. T. Filippov, *Topical Conference on Weak Interactions* (CERN, Geneva, 1969), p. 395; F. J. Dyson, Phys. Rev. 73, 929 (1948); B. Klaiber, Nuovo Cimento 36, 165 (1965); T. D. Lee, *ibid.* 59A, 579 (1969).

[2]Throughout this paper we specify a four-vector p by three spatial components p_1, p_2, p_3 and an imaginary time component $p_4 = ip_0$; $p^2 = p_1{}^2 + p_2{}^2 + p_3{}^2 - p_0{}^2$. We use $\not{p} = -i\gamma_\mu p_\mu$, $\Box = \partial^2/\partial t^2 - \nabla^2$ and for a Dirac spinor U, $\bar{U} = U^\dagger\gamma^4$. On occasion, the same symbol p will be used to represent $\sqrt{p^2}$.

[3]Similar techniques have been used previously by many authors: G. V. Efimov, Zh. Eksperim. i Teor. Fiz. 44, 2107 (1963) [Soviet Phys. JETP 17, 1417 (1963)]; Phys. Letters 4, 314 (1963); Nuovo Cimento 32, 1046 (1964); Nucl. Phys. 74, 657 (1965); E. S. Fradkin, *ibid.* 49, 624 (1963); 76, 588 (1966); G. Feinberg and A. Pais, Phys. Rev. 131, 2724 (1963); 133, B477 (1964); M. B. Halpern, *ibid.* 140, B1570 (1965); B. A. Arbuzov and A. T. Filippov, Nuovo Cimento 38, 796 (1965); B. A. Arbuzov and A. T. Filippov, Zh. Eksperim. i Teor. Fiz. 49, 990 (1965) [Soviet Phys. JETP 22, 688 (1966)]; H. M. Fried, Nuovo Cimento 52A, 1333 (1967); Phys. Rev. 174, 1725 (1968);

M. K. Volkov, Ann. Phys. (N.Y.) 49, 202 (1968); R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. 187, 1999 (1969); in addition to the first three references listed in Ref. 1.

[4]F. J. Dyson, Phys. Rev. 73, 929 (1948).

[5]The Feynman functions $\triangle_F(x)$ and $S_F(x;m)$ are given by

$$\triangle_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{ip\cdot x}\frac{1}{p^2 - i\epsilon},$$

$$S_F(x;m) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{ip\cdot x}\frac{\not{p} + m}{p^2 + m^2 - i\epsilon}.$$

[6]This analytic-continuation procedure, or one similar to it, has been suggested previously: Okubo, Ref. 1; Arnowitt and Deser, Ref. 1; L. Cooper, Phys. Rev. 100, 362 (1955); Volkov, Ref. 1, and Ann. Phys. (N.Y.) 49, 202 (1968); Arbuzov, Atakishiev, and Filippov, Ref. 1.

[7]Throughout this section we will ignore the mass counterterm $\bar{l}\delta M l$ found in Eq. (12) and omit any reference to wave-function renormalization constants since both quantities enter here in essentially the usual way (except for the situation discussed in Ref. 11). For a complete discussion of the usual situation found for example in ordinary quantum electrodynamics see J. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965). We will also omit the factors $e^{\pm ip\cdot x_i}/(2p_0)^{1/2}$ accompanying the absorption or emission of a particle of four-momentum p at the ith vertex.

[8]In both Secs. III and IV we will attempt to define an amplitude corresponding to a particular graph by first performing a phase rotation of $-\frac{1}{2}\pi$ for all time coordinates or $+\frac{1}{2}\pi$ for all energies so that a Euclidean region is reached. We will then return to the physical region by reversing this procedure and continuing all energies back to real values in a clockwise sense. Our ability to perform this second continuation is established in Appendix D.

[9]Our scattering amplitudes are invariantly normalized matrix elements of $T = i(S-1)$.

[10]The functions $S_F(x;m)\exp[\pm 4ig^2\triangle_F(x)]$ and $\mathrm{tr}[S_F(x;m)S_F(-x;m')]\exp[\pm 4ig^2\triangle_F(x)]$ each contain a term with no boson propagator and consequently correspond to the sum of two graphs, one with and one without a wavy line connecting the two ends of the fermion line(s). Note that the integral in Eq. (35) defining A_- is convergent only when both terms are included.

[11]Note that the center diagram in Fig. 6(c), containing the mass counterterm δM is proportional to the mass difference $\triangle m - \triangle m_0$. Only the $\triangle m_0$ term enters the amplitude A_+. The term containing the factor $\triangle m$ is to be combined with the quadratically divergent first-order term represented by the left-hand graph in Fig. 6(c). The amplitude resulting from this combination still contains a logarithmic divergence, proportional to $\triangle m_0(\triangle m)^3$ which is canceled in a similar way by terms of third and fourth order in our expansion.

[12]These operators are of course independent only for values of $n, n', l, l', l'', i, j, k$ for which none of them vanish.

[13]Although the parameters $f_{n,i}^{(l)}$ and $h_{n,i}^{(l)}$ are all that is needed to determine $S_n(g)$ it is interesting to ask what series of constants $c_m(n,i)$ and $d_m(n,i)$ are required by Eqs. (45), (46), (65), and (66). These equations can be reduced to

$$\sum_{m=-\infty}^{\infty} c_m(n,i)m^l = \left(\frac{b-\pi i}{2\pi i}\right)^l$$

and

$$\sum_{m=-\infty}^{\infty} d_m(n,i)m^l = \left(\frac{-b-\pi i}{2\pi i}\right)^l$$

for $l \le \triangle(n,i)$. The above equations do not uniquely determine $c_m(n,i)$ and $d_m(n,i)$; however, there is no solution for real b in which $c_m(n,i)$ and $d_m(n,i)$ are independent of n and i.

[14]S. Okubo, Progr. Theoret. Phys. (Kyoto) 11, 80 (1954); M. K. Volkov, Commun. Math. Phys. 7, 289 (1969); A. T. Filippov, *Topical Conference on Weak Interactions* (Ref. 1), p. 395.

[15]The function $[B_-(z)]_m$ is obtained by continuing $B_-(z)$ from real values of z along a path circling the origin through a total angle ϕ, where $(2m-1)\pi < \phi \le (2m+1)\pi$. It is intended, for example, that in the k integration over the contour c_I in Eq. (B6) the function $[B_-(g^2k^2/4\pi^2)]_{m'}$ be used for k near the starting point 0 of the c_I contour, but as the argument of k decreases through $-\pi$ the function $[B_-(g^2k^2/4\pi^2)]_{m'}$ should be replaced by its analytic continuation $[B_-(g^2k^2/4\pi^2)]_{m''-1'}$.

[16]The small positive number ϵ is introduced because of the divergence of the integrals in Eqs. (B8) and (B9). at the end points. The extra terms, corresponding to those parts of the contours circling these singular points, have been omitted. It can be shown that the inclusion of these extra terms will not alter our estimate of the asymptotic behavior of $\bar{E}$.

[17]There is a small region around $\nu = \frac{1}{2}\pi$ and $0.7 \le r \le 1$ for which the bounds (B8) and (B9) do not imply convergence. However, if for values of r and ν in this range the k contour is deformed slightly from a straight line, the resulting bounds are sufficiently improved that they require the convergence of the p integration in the integral (B10).

[18]A similar phenomenon can be seen by comparing Eqs. (A3) and (A4). The asymptotic growth for large p and fixed k of the integrand in Eq. (A4) is considerably greater than that obtained after the integral over k is performed as is indicated by Eqs. (A3) and (24).

[19]Maxime Bocher, *Introduction to Higher Algebra* (Dover, New York, 1964), p. 185.

[20]Ref. 19, pp. 212–216.

[21]This construction is presented for the case of two variables on p. 206 of Ref. 19.

[22]Using an argument similar to that found in the beginning of the final paragraph of Appendix C, one can show that if $|Q_{n-1}^{(j)}(0,b_1,\ldots,b_{n-1})| \ge \delta'(\sum_{i=1}^{n-1}|b_i|^2)^{\nu j/2}$ for all j and positive δ' and if $0<|g|<\epsilon'(\sum_{i=1}^{n-1}|b_i|^2)^{1/2}$ for sufficiently small ϵ', then the singularities of $f_n(g, a_1+gb_1, \ldots, a_{n-1}+gb_{n-1}, \omega_n)$ in ω_n are clustered in small groups about widely separated roots of $Q_n^{(j)}(0,b_1,\ldots,b_{n-1},\omega_n)$, $1 \le j \le M_n$, as is shown in Fig. (17b).

[23]M. Veltman, Physica 29, 186 (1963). The author is indebted to Professor H. Lehmann for bringing this paper to his attention.

[24]This choice can be accomplished by first considering the final integration over e_L, the integrals over $e_{L-1},\ldots,e_1$ having been performed. The resulting integrand will be singular in e_L at the points $z_{1,L},\ldots,z_{s,L}$

and c_L can be chosen so that $|e_L - z_{i,L}| < 3^L E$ implies that $e_L - z_{i,L}$ is real. This procedure can then be carried out inductively, routing c_r in such a way that $|e_r - z_{i,r}| < 3^r E$ implies that $e_r - z_{i,r}$ is real. Such a choice of c_r might be impeded by two singularities $z_{r,i}$ and $z_{r,j}$ with $\text{Im}(z_{r,i} - z_{r,j}) \neq 0$ and $|z_{r,i} - z_{r,j}| \leq 2 \times 3^r E$. However, the choice of the contour c_{r+1} rules out this possibility: $|z_{r,i} - z_{r,j}| < 3^{r+1} E$ implies that $z_{r,i} - z_{r,j}$ is real.

[25]For a slightly more complete discussion of this model, see N. Christ, in *Nonpolynomial Lagrangians, Renormalisation and Gravity*, Proceedings of the 1971 Coral Gables Conference on Fundamental Interactions at High Energy, Vol. 1 (Gordon and Breach, New York, 1971), p. 69.

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Application to the Scattering Problem of a Higher-Order Modified WKB Approximation Due to Miller and Good

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The modified WKB method due to Miller and Good is used to derive the scattering phase-shift formula including $\hbar^4$ terms for the radial equation of the three-dimensional scattering problem.

I. INTRODUCTION

Previous papers have developed a derivation, in the spirit of Miller and Good,[1] which yields the scattering phase shifts in terms of an expansion in powers of $\hbar$. A previous paper[2] presents the terms to the order $\hbar^2$.

The present paper is an extension of the previous result to terms of the order of $\hbar^4$. Section II of the present paper recapitulates the rigorous derivation of Ref. 2. A rigorous derivation of the scattering phase shifts to order $\hbar^4$ by the method of Ref. 2 would, however, require a very great expenditure of labor.

Therefore, in Sec. III we present a less rigorous derivation which can serve the purpose nicely. This same procedure can be easily followed for obtaining terms of higher order than $\hbar^4$ for the method developed is a straightforward expansion. This is, however, not so if we follow the rigorous derivation and try to obtain some higher-order terms by that process.

The divergence of the perturbation terms to $\hbar^2$ was successfully avoided in the scattering cases.[3,4] The main idea there is to replace the divergent integrals by contour integrals. As a result, we were able to obtain some meaningful results. In the spirit of the modified WKB method of Miller and Good, we can express the phase shifts which we want in terms of the known phase shifts of a known potential. In the specific example chosen in Ref. 4, the terms of second order in $\hbar$, in general, contribute to the phase shifts to the second decimal place while the zeroth-order terms in some places cannot give the result even to the first decimal place. An important question to be answered in this note is the extent to which improvement is possible. When the potential of the known part is sufficiently different from the potential of the unknown part, we may need to go to terms of higher order than $\hbar^2$ in order to get results of higher precision. We, therefore, investigate the contribution due to $\hbar^4$ terms.

Throughout this paper we avoid Langer's substitution, which replaces the three-dimensional distance r by e^x with x being the one-dimensional distance. Breit[5] raised the point that it is difficult to give a physical justification for the lower limit used in the integration if Langer's substitution is introduced. Here we simply consider the case where the angular momentum quantum number $L \neq 0$ for expansion in $1/L$ will be made later.

In the derivation of the formula in Secs. II and III the basic assumption is that $\psi(r)$ has the form $\psi(r) = T(r)\phi(S(r))$, where $\psi(r)$ is the unknown wave function and $\phi(S)$ is the known wave function that we want to make use of. Care should be exercised in the choice of the known part, $\phi(S)$. For example, in solving to order $\hbar^4$, we find that it is inadequate to represent the Coulomb scattering problem by the fractional-order Bessel-function formula as was possible when solving to order $\hbar^2$. This situation may be changed if we set, for example, $\psi(r) = T_1\phi(S(r)) + T_2\phi'(S(r))$ where T_1 and T_2 are functions dependent on r and $\phi'(S(r)) = d\phi/dS$. However, if short-range forces are the main con-