alous" diagrams makes no difference to our general argument.

[19]For any time-ordered single-particle electroproduction diagram, the leading power of $q_\perp^{-1}$ in the $q_\perp \to \infty$ limit can be checked following the general power-counting method described in Sec. IV in connection with elastic *ep* scattering. The only subtlety in the present case is the proper identification of each of the intermediate states that contributes a factor $\propto q_\perp^{-1}$ to the diagram from its energy denominator. Any intermediate state that appears after the interaction of the virtual photon and before the absorption of the last wee parton has to be included independent of whether it contains a wee particle or not. In the energy denominator of such a state the terms proportional to $q_\perp^2$ cancel out because of over-all energy conservation, but the terms proportional to $q_\perp$ persist. See the particular example in Appendix C.

[20]This buttresses our claim made towards the beginning of Sec. V that the external spin factors do not affect the leading $q_\perp$ dependence.

[21]These examples are not expected to give the correct $q_\perp$ dependences for the corresponding physical processes. Only their general feature is being used to propose a relation between W_2 and $[F]^2$.

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SU(3)×SU(3) Symmetry and K_{l3} Decays*

V. S. Mathur and T. C. Yang†
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

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The problem of the quadratic dependence on momentum transfer of the form factors in K_{l3} decay is analyzed. Assuming that the $(3, 3^*) + (3^*, 3)$ model provides a reasonable description of approximate SU(3)×SU(3) symmetry, several new sum rules have been written down which involve form factors defined at different momentum transfers. A solution of the various K_{l3} parameters is obtained and discussed.

I. INTRODUCTION

There has been a great deal of discussion recently[1] on the validity of the soft-pion theorem[2] in K_{l3} decays. The difficulty arises because the soft-pion theorem relates form factors evaluated at the momentum transfer $t = m_K^2$, which lies far beyond the physical region $m_l^2 \leq t \leq (m_K - m_\pi)^2$. Thus in order to test such a relation, one necessarily runs into the problem of momentum dependence of the form factors in the unphysical region. It has been customary to assume a linear t dependence of the form factors and indeed such a linear dependence seems to be consistent with the present experimental data.[1] However, if this linear relationship is extrapolated to the unphysical region, the experimental results seem to contradict the soft-pion theorem. A possible way out is that the form factors as a function of t may have nonlinear terms, which may be relatively unimportant in the physical region, but may become sizable in the extrapolated region near $t \sim m_K^2$. This paper deals with an investigation of this possibility.

Theoretically, one can study the momentum dependence of the form factors through the use of dispersion theory. In this case, inevitably, two problems arise. The first problem has to do with the number of subtractions to be used in the dispersion relations, or the knowledge of the asymptotic behavior of the form factors, about which very little is known. The second problem arises in evaluating the absorptive part of the form factors, for which one usually assumes pole dominance. Since the K^* and κ poles are rather far removed from the K_{l3} decay region of interest, it is not quite clear how good this assumption is.[3]

In this paper, we avoid dispersion theory and instead parametrize the t dependence of the form factors, retaining, however, the quadratic terms in t^2, which are generally ignored. The K_{l3} problem is then studied solely within the framework of chiral SU(3)×SU(3) symmetry. Such an investigation has many advantages, and in particular it serves to sharpen the experimental confrontation of our ideas regarding chiral SU(3)×SU(3) symmetry and its breaking. Our attitude in this paper will be to start with the idea that SU(3)×SU(3) symmetry realized through an octet of pseudoscalar Goldstone bosons, is indeed a reasonable approximate description of nature. Based on this description, two sum rules relevant to K_{l3} decays are well known: the usual soft-pion SU(2)×SU(2) theorem[2] and the Dashen-Weinstein relation.[4] A third soft-pion SU(3)×SU(3) sum rule has also been obtained by us[5] recently. To investigate the quadratic dependence on momentum transfer of the form factors, clearly more information is required. In this paper we derive what we believe is a maximal set of SU(3)×SU(3) sum rules relevant to K_{l3} decays. Specifically, these sum rules are derived on the basis of the $(3, 3^*)+(3^*, 3)$ model of SU(3)×SU(3) symmetry breaking, using a technique discussed recently in the literature,[6] and summarized in Sec. II. These sum rules are not obtained by using perturbation theory around the SU(3)×SU(3) limit, but are expected to be good if the $(3, 3^*)+(3^*, 3)$ model provides a reasonable description of an approximately SU(3)×SU(3)-invariant world. The precise sense in which we define approximate SU(3)×SU(3) symmetry is also discussed in Sec. II.

After some preliminaries in Sec. III, we obtain in Secs. IV and V various soft-pion SU(2)×SU(2) and soft-kaon chimeral[7] SU(3) sum rules. Aside from the usual soft-pion and -kaon sum rules, we obtain two extra relations using the extrapolation technique outlined in Sec. II. Sections VI and VII are devoted to obtaining what we call the "physical" sum rules. Numerical results of our investigation are presented in Sec. VIII, which are discussed in Sec. IX.

II. DESCRIPTION OF APPROXIMATE SU(3)×SU(3)

The $(3, 3^*)+(3^*, 3)$ model of Gell-Mann, Oakes, and Renner[8] and of Glashow and Weinberg[9] can be described by the Hamiltonian density

$$H = H_0 + \epsilon_0 S_0 + \epsilon_8 S_8 , \tag{2.1}$$

where S_α ($\alpha = 0, 1, \ldots, 8$) is a nonet of scalar densities which, together with a suitable nonet of pseudoscalar densities, transforms as the $(3, 3^*)+(3^*, 3)$ representation of SU(3)×SU(3). Also in Eq. (2.1), ϵ_0 and ϵ_8 are the symmetry-breaking parameters. It will be profitable if instead we choose these parameters to be ϵ_0 and a, where[6,7]

$$a = \frac{\epsilon_8}{\sqrt{2}\,\epsilon_0} . \tag{2.2}$$

H_0 describes an SU(3)×SU(3)-invariant world realized through the zero-mass octet (π, K, η) of pseudoscalar mesons.

It is well known[8] and simple to check that the Hamiltonian density (2.1) would be SU(2)×SU(2)-invariant for $a = -1$. In this limit the pions are massless but K and η acquire masses. Also at $a = 0$, one realizes the usual SU(3) symmetry and at $a = 2$, H is invariant under chimeral[7] SU(3) symmetry realized by massless K but massive π and η. Note that these subgroups of SU(3)×SU(3) are realized for arbitrary ϵ_0. Also, in the SU(3)×SU(3) limit, $\epsilon_0 \to 0$, $\epsilon_8 \to 0$, and since the way in which ϵ_0 and ϵ_8 go to zero is not important, a can take any arbitrary value. Thus if the matrix elements are well defined in the SU(3)×SU(3) limit, they cannot depend on a in this limit. If now SU(3)×SU(3) symmetry is broken, dependence on a would develop, but if the symmetry provides a reasonably approximate description, we do not expect any drastic dependence on a. To be precise, we shall take approximate SU(3)×SU(3) symmetry to mean that the matrix elements are as smooth functions of a as possible consistent with various subgroups constraints. Thus, if a matrix element M is known in the SU(2)×SU(2), SU(3) and chimeral SU(3) limits, and if SU(3)×SU(3) is an approximate symmetry, the matrix element may be written by at most a quadratic function of a:

$$M = M_0(\epsilon_0) + a M_1(\epsilon_0) + a^2 M_2(\epsilon_0). \tag{2.3}$$

Note, in the SU(3)×SU(3)-symmetric limit, we must have $M_1(\epsilon_0 = 0) = M_2(\epsilon_0 = 0) = 0$. Since, we shall not discuss in the following the exact SU(3)×SU(3) limit, the ϵ_0 dependence can be ignored. Stated otherwise we may fix ϵ_0 to be equal to its "physical" value. Since the various subgroup symmetries are realized at $a = -1$, 0, and 2 irrespective of the value of ϵ_0, there is no loss of generality in fixing ϵ_0 as far as considerations of these subgroups symmetries are concerned. The physical value of a, where the pion and kaon acquire the experimentally known masses, is given by the well-known formula[8]

$$a = \frac{m_\pi^2 - m_K^2}{m_K^2 + \frac{1}{2} m_\pi^2} \simeq -0.89. \tag{2.4}$$

Thus knowning M in the various subgroup limits, the smooth or gentle a dependence in Eq. (2.3), allows us to obtain the physical value of M for a given by Eq. (2.4).

An alternative but related way to look at the smoothness assumption is to argue as follows: The usual hypothesis of partially conserved axial-vector current (PCAC) for pions implies gentleness in the extrapolation from the physical point to $a = -1$, while the kaon PCAC hypothesis implies a similar gentle behavior when extrapolated to $a = 2$. Furthermore, near $a = 0$, the gentle behavior such as (2.3) can be viewed as a perturbation expansion about the SU(3)-symmetric limit. The combined success of the pion PCAC hypothesis, SU(3) perturbation theory, as well as the fact that the kaon PCAC hypothesis is not inconsistent with experiments suggests strongly that the assumption of gentleness in the whole region $-1 \leq a \leq 2$ may indeed be quite reasonable. It is worth emphasizing that statements of smoothness such as (2.3), which are valid for all a in the region $-1 \leq a \leq 2$, are *not* consequences of a perturbation theory in the variable a.

III. THE K_{l3} PROBLEM: DEFINITIONS

To be specific we shall consider K_{l3}^+ decay. The form factors are defined through the matrix element

$$\langle\pi^0(q)|V_\mu^{4-i5}(0)|K^+(p)\rangle$$
$$= -\frac{1}{\sqrt{2}}\frac{1}{(4p_0q_0V^2)^{1/2}}[(p+q)_\mu f_+(t)+(p-q)_\mu f_-(t)], \quad (3.1)$$

where $t=-(p-q)^2$. Note that in the SU(3) limit $f_+(0)=1$ and $f_-(t)=0$. For the physical decay $m_l^2 < t < (m_K-m_\pi)^2$. We shall use the metric such that the scalar product of two four-vectors is given by $a\cdot b=\vec{a}\cdot\vec{b}+a_4b_4=\vec{a}\cdot\vec{b}-a_0b_0$. For later convenience, we shall define the divergence of the matrix element (1) as

$$\langle\pi^0(q)|\partial_\mu V_\mu^{4-i5}(0)|K^+(p)\rangle=\frac{i}{\sqrt{2}}\frac{1}{(4p_0q_0V^2)^{1/2}}d(t) \quad (3.2)$$

so that

$$d(t)=(m_K^2-m_\pi^2)f_+(t)+tf_-(t). \quad (3.3)$$

We shall parametrize $f_\pm(t)$ as follows:

$$f_\pm(t)=f_\pm(0)\left(1+\lambda_\pm\frac{t}{m_\pi^2}+\lambda'_\pm\frac{t^2}{m_\pi^4}\right),\quad 0\le t<m_K^2 \quad (3.4)$$

and define, as usual,

$$\xi=f_-(0)/f_+(0). \quad (3.5)$$

From Eqs. (3.3)–(3.5), we may express

$$d(t)=d(0)+d'(0)t+d''(0)\frac{t^2}{2!}+d'''(0)\frac{t^3}{3!}, \quad (3.6)$$

where

$$d(0)=(m_K^2-m_\pi^2)f_+(0), \quad (3.7)$$

$$d'(0)=f_+(0)\left(\xi+\frac{m_K^2-m_\pi^2}{m_\pi^2}\lambda_+\right), \quad (3.8)$$

$$d''(0)=\frac{2f_+(0)}{m_\pi^2}\left(\lambda_-\xi+\frac{m_K^2-m_\pi^2}{m_\pi^2}\lambda'_+\right), \quad (3.9)$$

$$d'''(0)=\frac{6f_+(0)}{m_\pi^4}\lambda'_-\xi. \quad (3.10)$$

For later use, we shall also define the π, K, and κ leptonic decay constants f_π, f_K, and f_κ through the matrix elements

$$\langle 0|A_\mu^3(0)|\pi^0(q)\rangle=\frac{i}{(2q_0V)^{1/2}}\frac{f_\pi}{\sqrt{2}}q_\mu, \quad (3.11)$$

$$\langle 0|A_\mu^{4-i5}(0)|K^+(p)\rangle=\frac{i}{(2p_0V)^{1/2}}f_Kp_\mu, \quad (3.12)$$

$$\langle 0|V_\mu^{4-i5}(0)|\kappa^+(p)\rangle=\frac{i}{(2p_0V)^{1/2}}f_\kappa p_\mu. \quad (3.13)$$

In the SU(3)-symmetry limit, note that $f_K=f_\pi$ and $f_\kappa=0$.

IV. SOFT-PION RELATIONS

We have shown elsewhere[5] that in the soft-pion limit, besides the usual soft-pion theorem[2]

$$f_+(m_K^2)+f_-(m_K^2)=f_K/f_\pi, \quad (4.1)$$

one can obtain another result,

$$f_+(m_K^2)-f_-(m_K^2)=\frac{1}{2f_Kf_\pi}(3f_K^2-f_\pi^2+f_\kappa^2). \quad (4.2)$$

Note that the relation (4.1) is strictly valid in the SU(2)×SU(2)-symmetric world, irrespective of how good chiral SU(3)×SU(3) symmetry is. The validity of the relation (4.2), however, requires the stronger assumption that SU(3)×SU(3) symmetry is also a good approximation. To see how this arises and for completeness, we shall briefly sketch the derivation of (4.2) here.

Consider the matrix element

$$M_{\mu\nu}=i\int d^4x\,e^{-iq\cdot x}\langle 0|T^*(A_\nu^3(x)V_\mu^{4-i5}(0))|K^+(p)\rangle, \quad (4.3)$$

which from general invariance arguments can be written as

$$M_{\mu\nu}=-\frac{i}{(2p_0V)^{1/2}}(Ap_\mu q_\nu+Bp_\nu q_\mu+Cp_\mu p_\nu+Dq_\mu q_\nu+E\delta_{\mu\nu}), \quad (4.4)$$

where the invariant amplitudes, A, B, ..., E are functions of q^2 and $t=-(p-q)^2$. In order to derive the soft-pion results, we start with exact SU(2)×SU(2) symmetry, realized through a zero-mass pion. From Eq. (4.3), and using the definition (3.12), we then obtain the Ward identity,

$$q_\nu M_{\mu\nu}=-\frac{1}{2}\frac{i}{(2p_0V)^{1/2}}f_Kp_\mu, \quad (4.5)$$

so that from Eq. (4.4), we get

$$q^2A+p\cdot qC=\tfrac{1}{2}f_K, \quad (4.6)$$

$$p\cdot qB+q^2D+E=0. \quad (4.7)$$

The relations (4.6) and (4.7) are valid in an SU(2)×SU(2)-invariant world for all q^2 and t. Letting now $q^2\to 0$, only the zero-mass-pion poles in A and D will contribute, the residues being related to the K_{l3} form factors defined in Eq. (3.1). Thus for the massless pion, we get

$$f_+(t)+f_-(t)=\frac{1}{f_\pi}[f_K-(t-m_K^2)C(q^2=0,t)], \quad (4.8)$$

$$f_+(t)-f_-(t)=-\frac{1}{f_\pi}[2E(q^2=0,\,t)+(t-m_K{}^2)B(q^2=0,\,t)]. \tag{4.9}$$

In the soft-pion limit ($q\to 0$, $t\to m_K{}^2$), since the invariant amplitudes have no poles at $t=m_K{}^2$, Eq. (4.8) reduces to the well-known soft-pion theorem (4.1). In this limit Eq. (4.9) becomes

$$f_+(m_K{}^2)-f_-(m_K{}^2)=-\frac{2}{f_\pi}E(q^2=0,\,t=m_K{}^2). \tag{4.10}$$

In order to obtain an expression for $E(q^2, t)$ in the soft-pion limit, we follow the procedure discussed in Sec. II. Accordingly, we first obtain expressions for E in the exact-SU(3) and chimeral-SU(3) limits. If SU(3)×SU(3) is an approximate symmetry in the sense of Sec. II, we would then extrapolate our results to obtain E in the SU(2) ×SU(2) limit. First note that in the limit of exact SU(3) symmetry, $\partial_\mu V_\mu^{4-i5}=0$, so that from Eqs. (4.3) and (3.12) we obtain the Ward identiy

$$k_\mu M_{\mu\nu}=\frac{i}{2}\,\frac{1}{(2p_0V)^{1/2}}f_K p_\nu, \tag{4.11}$$

where

$$k=p-q \quad\text{and}\quad t=-k^2. \tag{4.12}$$

Letting $k\to 0$ and using (4.4), we obtain from Eq. (4.11)

$$E(k^2=0,\,p^2=q^2)=-\tfrac{1}{2}f_K \quad [\mathrm{SU(3)}]. \tag{4.13}$$

In the soft-kaon limit, the matrix element (4.3) can be expressed in terms of the two-point functions for vector and axial-vector currents. Using the Lehmann-Källén spectral representation ($\alpha, \beta=1, 2, \ldots, 8$)

$$i\int d^4x\, e^{-iq\cdot x}\langle 0|T(V_\mu^\alpha(x)V_\nu^\beta(0))|0\rangle=\delta_{\mu\nu}\int\frac{\rho_1^{\alpha\beta}(m^2,V)}{q^2+m^2}dm^2+q_\mu q_\nu\int\frac{\rho_2^{\alpha\beta}(m^2,V)}{(q^2+m^2)m^2}dm^2-\delta_{\mu4}\delta_{\nu4}\int\frac{\rho_2^{\alpha\beta}(m^2,V)}{m^2}dm^2 \tag{4.14}$$

and a similar one for axial-vector currents, we get

$$E(k^2=q^2,\,p^2=0)=\frac{1}{f_K}\int_0^\infty\frac{\rho_1^{33}(m^2,A)-\rho_1^{44}(m^2,V)}{q^2+m^2}dm^2 \quad [\text{chimeral SU(3)}], \tag{4.15}$$

where ρ_1 is the spin-one spectral function, and $\rho_2\equiv\rho_1+\rho_0$, ρ_0 being the spin-zero spectral function.

We now obtain E in the soft-pion limit from the information (4.13) and (4.15) which are valid in the limits of exact SU(3) and chimeral SU(3) symmetry, respectively. Following the ideas outlined in Sec. II, it is now profitable to consider the following function of a for fixed ϵ_0:

$$\chi(a)=E\left(k^2=\frac{-(m_K{}^2-m_\pi{}^2)(2m_K{}^2-m_\pi{}^2)}{2m_K{}^2+m_\pi{}^2},\,p^2=-m_K{}^2,\,q^2=-m_\pi{}^2\right)-\tfrac{1}{4}(a-2)f_K-\frac{a}{2}\,\frac{1}{f_K}\int\frac{\rho_1^{33}(m^2,A)-\rho_1^{44}(m^2,V)}{m^2-m_\pi{}^2}dm^2. \tag{4.16}$$

From Eqs. (4.13) and (4.15), we observe that

$$\chi(a=0)=\chi(a=2)=0. \tag{4.17}$$

Note that the various masses and couplings in Eq. (4.16) are also in general functions of a. If now SU(3)×SU(3) is a good symmetry in nature, the discussion in Sec. II leads us to expect that $\chi(a)$ is a reasonably smooth continuous function of a in the region $-1\leqslant a\leqslant 2$ so that we may express $\chi(a)$ by a linear function of a. Equation (4.17) then implies $\chi(a)=0$ for all a in this region, and in particular

$$\chi(a=-1)=0. \tag{4.18}$$

Evaluating (4.16) at $a=-1$, we then get

$$E(k^2=-m_K{}^2,p^2=-m_K{}^2,\,q^2=0)=-\tfrac{3}{4}f_K-\frac{1}{2f_K}\int\frac{\rho_1^{33}(m^2,A)-\rho_1^{44}(m^2,V)}{m^2}dm^2. \tag{4.19}$$

Now using Weinberg's first sum rule,[10]

$$\int\frac{\rho_1^{33}(m^2,A)-\rho_1^{33}(m^2,V)}{m^2}dm^2=-\int\frac{\rho_0^{33}(m^2,A)}{m^2}dm^2, \tag{4.20}$$

and the asymptotic SU(3) sum rule,[11]

$$\int\frac{\rho_1^{44}(m^2,V)-\rho_1^{33}(m^2,V)}{m^2}dm^2=-\int\frac{\rho_0^{44}(m^2,V)}{m^2}dm^2, \tag{4.21}$$

we get

$$\int\frac{\rho_1^{33}(m^2,A)-\rho_1^{44}(m^2,V)}{m^2}dm^2=-\int\frac{\rho_0^{33}(m^2,A)-\rho_0^{44}(m^2,V)}{m^2}dm^2. \tag{4.22}$$

Using π and κ dominance[12] to evaluate the integrals over spin-zero spectral functions in Eq. (4.22), we obtain on substitution in Eq. (4.19),

$$E(k^2=-m_K{}^2, p^2=-m_K{}^2, q^2=0)$$
$$=\frac{1}{2}\left[-\tfrac{3}{2}f_K+\frac{1}{2f_K}(f_\pi{}^2-f_\kappa{}^2)\right] \quad [\mathrm{SU}(2)\times\mathrm{SU}(2)]. \tag{4.23}$$

Substituting Eq. (4.23) in Eq. (4.10), one obtains the soft-pion sum rule (4.2).

V. SOFT-KAON RELATIONS

Proceeding in a fashion analogous to Sec. IV, we now obtain the following two soft-kaon results:

$$f_+(m_\pi{}^2)-f_-(m_\pi{}^2)=\frac{f_\pi}{f_K}, \tag{5.1}$$

$$f_+(m_\pi{}^2)+f_-(m_\pi{}^2)=\frac{1}{f_\pi f_K}(3f_\pi{}^2-2f_K{}^2+2f_\kappa{}^2). \tag{5.2}$$

The relation (5.1) is the analog of Eq. (4.1), and as before the soft-kaon result (5.2) is valid only if SU(3)×SU(3) symmetry is a good symmetry. In order to derive these results, we consider now the following matrix element:

$$T_{\mu\nu}=i\int d^4x\, e^{ip\cdot x}\langle\pi^0(q)|T^*(A_\nu^{4+i5}(x)V_\mu^{4-i5}(0))|0\rangle, \tag{5.3}$$

which may be written in terms of invariant amplitudes

$$T_{\mu\nu}=\frac{-i}{(2q_0V)^{1/2}}(A'p_\mu q_\nu+B'p_\nu q_\mu+C'p_\mu p_\nu+D'q_\mu q_\nu+E'\delta_{\mu\nu}). \tag{5.4}$$

Working in the exact chimeral-SU(3) limit, realized through massless kaons, one obtains the Ward identity

$$p_\nu T_{\mu\nu}=\frac{i}{(2q_0V)^{1/2}}\frac{f_\pi}{\sqrt{2}}q_\mu \tag{5.5}$$

using the definition (3.11). From Eqs. (5.4) and (5.5), we get

$$p^2B'+p\cdot qD'=-\frac{f_\pi}{\sqrt{2}}, \tag{5.6}$$

$$p\cdot qA'+p^2C'+E'=0. \tag{5.7}$$

In the limit $p^2\to 0$, only the massless-kaon pole contributes to B' and C', and we obtain, using the definition (3.1),

$$f_+(t)-f_-(t)=\frac{f_\pi}{f_K}-\frac{1}{\sqrt{2}}(t-m_\pi{}^2)D'(p^2=0,t), \tag{5.8}$$

$$f_+(t)+f_-(t)=\frac{\sqrt{2}}{f_K}[E'(p^2=0,t)+\tfrac{1}{2}(t-m_\pi{}^2)A'(p^2=0,t)]. \tag{5.9}$$

Since D' and A' do not have poles at $t=m_\pi{}^2$, in the soft-kaon limit ($p\to 0$, $t\to m_\pi{}^2$), Eq. (5.8) reduces to the result (5.1), and Eq. (5.9) becomes

$$f_+(m_\pi{}^2)+f_-(m_\pi{}^2)=\frac{\sqrt{2}}{f_K}E'(p^2=0, t=m_\pi{}^2). \tag{5.10}$$

As before, in order to evaluate $E'(p^2, t)$ in the soft-kaon limit, we first discuss its value in the SU(3) and SU(2)×SU(2) limits. In the SU(3) limit, we get from Eq. (5.3) the Ward identity

$$k_\mu T_{\mu\nu}=\frac{i}{(2q_0V)^{1/2}}\frac{f_\pi}{\sqrt{2}}q_\nu, \quad k=p-q. \tag{5.11}$$

On letting $k\to 0$, one obtains

$$E'(k^2=0, p^2=q^2)=\frac{f_\pi}{\sqrt{2}} \quad [\mathrm{SU}(3)]. \tag{5.12}$$

In the soft-pion limit the matrix element (5.3) can be expressed in terms of the two-point functions as before. Using Eq. (4.14) and a similar definition for the case of axial-vector currents, we get

$$E'(k^2=p^2, q^2=0)=\frac{\sqrt{2}}{f_\pi}\int\frac{\rho_1^{44}(m^2,V)-\rho_1^{44}(m^2,A)}{p^2+m^2}dm^2 \quad [\mathrm{SU}(2)\times\mathrm{SU}(2)]. \tag{5.13}$$

We now define

$$\chi'(a)=E'\left(k^2=-\frac{(m_\pi{}^2-m_K{}^2)(2m_\pi{}^2-m_K{}^2)}{2m_\pi{}^2+m_K{}^2}, p^2=-m_K{}^2, q^2=-m_\pi{}^2\right)$$
$$-(a+1)\frac{f_\pi}{\sqrt{2}}+\frac{\sqrt{2}\,a}{f_\pi}\int\frac{\rho_1^{44}(m^2,V)-\rho_1^{44}(m^2,A)}{m^2-m_K{}^2}dm^2. \tag{5.14}$$

Equations (5.12) and (5.13) imply $\chi'(a=0)=\chi'(a=-1)=0$. Assuming, as before, that $\chi(a)$ is a linear function of a, we see that $\chi(a)$ must vanish also in the chimeral-SU(3) limit, so that

$$E'(k^2=-m_\pi{}^2, p^2=0, q^2=-m_\pi{}^2)=\frac{3}{\sqrt{2}}f_\pi-\frac{2\sqrt{2}}{f_\pi}\int\frac{\rho_1^{44}(m^2,V)-\rho_1^{44}(m^2,A)}{m^2}dm^2. \tag{5.15}$$

Using Weinberg's first sum rule in the form[10]

$$\int \frac{\rho_1^{44}(m^2, V) - \rho_1^{44}(m^2, A)}{m^2} dm^2 = \int \frac{\rho_0^{44}(m^2, A) - \rho_0^{44}(m^2, V)}{m^2} dm^2 \simeq \frac{1}{2}(f_K{}^2 - f_\kappa{}^2), \tag{5.16}$$

we obtain

$$E'(k^2 = -m_\pi{}^2, p^2 = 0, q^2 = -m_\pi{}^2) = \frac{3}{\sqrt{2}} f_\pi - \frac{\sqrt{2}}{f_\pi}(f_K{}^2 - f_\kappa{}^2) \quad \text{[chimeral SU(3)]}. \tag{5.17}$$

From Eqs. (5.17) and (5.10) we obtain Eq. (5.2).

VI. "PHYSICAL" SUM RULES

Introducing the notation

$$\frac{f_K}{f_\pi} = x, \quad \frac{f_\kappa{}^2}{f_K f_\pi} = y, \tag{6.1}$$

we have shown that in the SU(2)×SU(2) limit, i.e., at $a = -1$,

$$f_+(m_K{}^2) + f_-(m_K{}^2) = x, \tag{6.2}$$

$$f_+(m_K{}^2) - f_-(m_K{}^2) = \tfrac{1}{2}(3x - 1/x + y), \tag{6.3}$$

and in the chimeral-SU(3) limit, i.e., at $a = 2$,

$$f_+(m_\pi{}^2) - f_-(m_\pi{}^2) = 1/x, \tag{6.4}$$

$$f_+(m_\pi{}^2) + f_-(m_\pi{}^2) = 3/x - 2x + 2y. \tag{6.5}$$

Our purpose now is to use Eqs. (6.2)–(6.5) and the information in the usual SU(3) limit as boundary conditions to extrapolate and obtain information for all a $(-1 \leq a \leq 2)$, and thence at the "physical" value of a given by Eq. (2.4). Obviously the extrapolation is not unique, but as discussed in Sec. II we shall be guided by requirements of minimal a dependence, in accordance with the expectation that SU(3)×SU(3) is a reasonably good symmetry.

At first sight it appears that the smoothest and the most obvious extrapolation is

$$\begin{aligned} &f_+(\Delta) + f_-(\Delta) = x, \\ &f_+(\Delta) - f_-(\Delta) = \tfrac{1}{2}(3x - 1/x + y), \\ &f_+(\Delta') - f_-(\Delta') = 1/x, \\ &f_+(\Delta') + f_-(\Delta') = 3/x - 2x + 2y, \end{aligned} \tag{6.6}$$

where

$$\begin{aligned} &\Delta = m_K{}^2 \text{ at } a = -1, \quad \Delta = 0 \text{ at } a = 0, \\ &\Delta' = m_\pi{}^2 \text{ at } a = 2, \quad \Delta' = 0 \text{ at } a = 0. \end{aligned} \tag{6.7}$$

Note in the SU(3) limit $x = 1$ and $y = O(\epsilon_8{}^2)$. However, from Eqs. (6.6) one readily obtains

$$f_-(\Delta) - f_-(\Delta') = \tfrac{3}{4}(x - 1/x) - \tfrac{5}{4}y. \tag{6.8}$$

Note that the right-hand side of Eq. (6.8) has first-order SU(3)-breaking terms. This can be seen explicitly if we write

$$x \equiv \frac{f_K}{f_\pi} = 1 + \epsilon, \tag{6.9}$$

where ϵ is a first-order SU(3)-breaking term, then correct to this order $x - 1/x = 2\epsilon$, so that the right-hand side of Eq. (6.8) has $O(\epsilon_8)$ terms. The left-hand side of Eq. (6.8), on the other hand, is at least $O(\epsilon_8{}^2)$. This is because $f_-(\Delta) - f_-(\Delta') = (\Delta - \Delta') f_-'(0) + \cdots$, and the leading terms itself is at least $O(\epsilon_8{}^2)$ if $\Delta - \Delta' = O(\epsilon_8)$, since $f_-'(0) = O(\epsilon_8)$ in the SU(3) limit. Furthermore, if we make the obvious choice $\Delta = -\Delta' = m_K{}^2 - m_\pi{}^2$, which is consistent with Eq. (6.7), then it is easy to check by adding all the equations in (6.6) that one violates the Ademollo-Gatto theorem.[13] Clearly then the set (6.6) is unacceptable.

The next step is to try the following extrapolation:

$$\begin{aligned} &f_+(\Delta) + f_-(\Delta) = -ax + (1 + a)F(x), \\ &f_+(\Delta) - f_-(\Delta) = -\tfrac{1}{2}a(3x - 1/x + y) + (1 + a)G(x), \\ &f_+(\Delta') - f_-(\Delta') = \frac{a}{2x} + \left(1 - \frac{a}{2}\right)H(x), \\ &f_+(\Delta') + f_-(\Delta') = \frac{a}{2}\left(\frac{3}{x} - 2x + 2y\right) + \left(1 - \frac{a}{2}\right)I(x), \end{aligned} \tag{6.10}$$

with the condition

$$F(0) = G(0) = H(0) = I(0) = 1. \tag{6.11}$$

At this stage, it is worthwhile to point out that x itself depends implicitly on a. However, it is evident that x would be a rather insensitive function of a. Note, for instance, that at $a = 0$, $x = 1$, whereas at the physical value $a \simeq -0.89$, one knows that f_K/f_π is not too different from unity. Indeed in the model (2.1) x is given by[6]

$$x \equiv \frac{f_K}{f_\pi} = \left(\frac{1 - \frac{1}{2}b}{1 + b}\right)^{1/2}, \tag{6.12}$$

where

$$b = \frac{1}{\sqrt{2}} \frac{\langle 0|S_8|0\rangle}{\langle 0|S_0|0\rangle}. \tag{6.13}$$

It has been argued[6] that $|b| \ll 1$ for all a, $-1 \leq a \leq 2$, so that the vacuum state is nearly invariant under SU(3).[8] In terms of the parameter ϵ introduced in Eq. (6.9), we then have $|\epsilon| \ll 1$ for a in the range $-1 \leq a \leq 2$, so that we may to a good approximation express the functions $F(x)$, $G(x)$, $H(x)$, and $I(x)$ in Eq. (6.10) by expanding them in powers of ϵ, and neglecting for the present quadratic and higher terms. We shall return to a discussion of the quadratic terms later. Now the κ parameter y defined in Eq. (6.1) also represents a second-order SU(3)-breaking constant. Also at the physical value (2.4) of a, most estimates show that y is small compared to unity. Thus it seems plausible that $y \equiv f_\kappa^2/f_\pi f_K = O(\epsilon^2)$. Indeed in the model (2.1), it has been shown that[14]

$$y = \frac{f_\kappa^2}{f_\pi f_K} = \frac{9}{4}\,\frac{b^2}{(1-\frac{1}{2}b)^{3/2}(1+b)^{1/2}} = O(\epsilon^2). \tag{6.14}$$

Dropping all $O(\epsilon^2)$ and higher terms, we obtain from Eqs. (6.10) and (6.11)

$$f_+(\Delta)+f_-(\Delta) = -ax+(1+a)(1+\alpha\epsilon) = 1+\alpha\epsilon+(\alpha-1)a\epsilon, \tag{6.15}$$

$$f_+(\Delta)-f_-(\Delta) = -\tfrac{1}{2}a(3x-1/x)+(1+a)(1+\beta\epsilon) = 1+\beta\epsilon+(\beta-2)a\epsilon, \tag{6.16}$$

$$f_+(\Delta')-f_-(\Delta') = \frac{a}{2x}+\left(1-\frac{a}{2}\right)(1+\gamma\epsilon) = 1+\gamma\epsilon-\tfrac{1}{2}(\gamma+1)a\epsilon, \tag{6.17}$$

$$f_+(\Delta')+f_-(\Delta') = \frac{a}{2}\left(\frac{3}{x}-2x\right)+\left(1-\frac{a}{2}\right)(1+\delta\epsilon) = 1+\delta\epsilon-\tfrac{1}{2}(\delta+5)a\epsilon, \tag{6.18}$$

where α, β, γ, and δ are constants.

It is important to note that although we have neglected $O(\epsilon^2)$ terms, we do not neglect terms proportional to $a\epsilon$, since we do not consider a to be a perturbation parameter. In the SU(3) limit, i.e., near $a \simeq 0$, of course the $a\epsilon$ term is $O(\epsilon_8^2)$; however, for large values of a, and in particular near the physical value of a, the $a\epsilon$ term is only $O(\epsilon)$. Note, as also emphasized in Sec. II, the linear structure in a on the right-hand side of Eqs. (6.15)–(6.18) is a consequence of our smoothness assumption. Perturbation expansion is made only for the parameter ϵ defined in Eq. (6.9) or equivalently for the parameter b in Eq. (6.13).

We now turn our attention to the constants α, β, γ, and δ appearing in the sum rules (6.15)–(6.18). Note, hitherto Δ and Δ' satisfy the conditions (6.7). How Δ and Δ' go to zero in the SU(3) limit is, however, not specified in Eq. (6.7). Also unspecified is the information on what Δ is at $a=2$, or Δ' at $a=-1$. For the moment, leaving unspecified how Δ or Δ' go to zero in the SU(3) limit, we will first show that in the special case when $\Delta=\Delta'$ at $a=-1$, 0, and 2, the constants α, β, γ, and δ can be determined uniquely. In this case, comparing Eqs. (6.15) and (6.18) at $a=-1$ and at $a=2$, we get

$$\alpha=\delta=-1. \tag{6.19}$$

Also comparing Eqs. (6.16) and (6.17) at these values of a, we obtain

$$\beta=\gamma=1. \tag{6.20}$$

Note that, adding Eqs. (6.15) and (6.16), we now obtain $f_+(\Delta)=1-\frac{3}{2}a\epsilon$. Around the SU(3)-symmetry limit, since the $a\epsilon$ term is $O(\epsilon_8^2)$, expansion around the SU(3) value $\Delta=0$, and the use of the Ademollo-Gatto theorem $f_+(0)=1+O(\epsilon_8^2)$, then implies that Δ must go to zero in the SU(3) limit at least as $O(\epsilon_8^2)$. Thus, if we take $\Delta=\Delta'=\Delta''$, where

$$\Delta''=m_K^2 \text{ at } a=-1,\quad \Delta''=m_\pi^2 \text{ at } a=2,\quad \Delta''=O(\epsilon_8^2) \text{ at } a=0, \tag{6.21}$$

the various symmetry constraints on the set of Eqs. (6.15)–(6.18) are satisfied if we choose α, β, γ, and δ given by Eqs. (6.19) and (6.20). At $a=0$, we have chosen Δ'' as $O(\epsilon_8^2)$, although in principle a faster approach to zero is also consistent. There is no ambiguity, however, if we confine ourselves to the philosophy of minimal a dependence. Indeed a minimal variation with respect to a, consistent with the conditions (6.21), yields[15]

$$\Delta''=\frac{(m_K^2-m_\pi^2)^2(4m_K^2+m_\pi^2)}{(2m_K^2+m_\pi^2)^2}. \tag{6.22}$$

For the momentum transfer (6.22), Eqs. (6.15)–(6.20) then yield the following sum rules[16]:

$$f_+(\Delta'')+f_-(\Delta'')=1-\epsilon-2a\epsilon,\qquad f_+(\Delta'')-f_-(\Delta'')=1+\epsilon-a\epsilon. \tag{6.23}$$

For the physical masses and the coupling ratio f_K/f_π, and with a given by Eq. (2.4), Eqs. (6.23) constitute the "physical" sum rules valid for the squared momentum transfer Δ'' given by Eq. (6.22). Numerically, $\Delta''\simeq 0.8m_K^2$. This is still in the unphysical K_{l3}-decay region, but is somewhat closer to the physical region than the corresponding value of the momentum transfer that appears in the soft-pion result Eq. (4.1).

If we make the usual assumption that $f_+(t)$ is given by a linear function of t in the region $m_l^2<t<\Delta''$, the sum rules (6.23) can be solved to obtain

$$\lambda_+=\frac{m_\pi^2}{\Delta'' f_+(0)}\left[1-f_+(0)-\tfrac{3}{2}a\epsilon\right]. \tag{6.24}$$

For $f_K/f_\pi=1.28$, $f_+(0)=1$, as an illustration, we obtain the result

$$\lambda_+=0.036. \tag{6.25}$$

If we further assume that $d(t)$ defined in Eq. (3.3) is a linear function of t for $m_l^2 < t < \Delta''$, which for a linear $f_+(t)$, implies a constant f_-, we obtain for the parameter ξ defined in Eq. (3.5) the corresponding numerical result

$$\xi = -0.16. \tag{6.26}$$

The results (6.25) and (6.26) are numerically close to the corresponding solutions[5] of the soft-pion results (4.1) and (4.2) obtained under similar assumptions of linearity. In order to go beyond the linearity assumption, clearly more information is required. In Sec. VII we show that more relations at different momentum transfer can indeed be derived starting from the results (6.15)–(6.18).

Before we conclude this section, we would like to point out that the sum rules (6.23) have been derived by neglecting terms of order ϵ^2. We can, however, go to the next approximation and retain terms of order ϵ^2. It would be interesting to see how the results (6.25) and (6.26) change. For the squared momentum transfer Δ'' defined in Eq. (6.21), and using the technique that leads to the sum rules (6.23), one can easily show that correct to $O(\epsilon^2)$, the sum rules are

$$\begin{aligned} f_+(\Delta'') + f_-(\Delta'') &= -ax + (1+a)/x + \tfrac{2}{3}(1+a)y, \\ f_+(\Delta'') - f_-(\Delta'') &= (1-a/2)x + a/2x + \tfrac{1}{3}(1-\tfrac{1}{2}a)y, \end{aligned} \tag{6.27}$$

where x and y are defined in Eq. (6.1). We may now estimate λ_+ and ξ from Eq. (6.27) assuming linear expansions for $f_+(t)$ and $d(t)$, as before. There are various estimates for y in the literature. Using Eqs. (6.12) and (6.14) for small b, $y \approx 4(1-x)^2$. Then for $x = 1.28$, $f_+(0) = 1$, we get

$$\lambda_+ = 0.044, \quad \xi = -0.20. \tag{6.28}$$

Alternatively, we may use the formula of Glashow and Weinberg[9]

$$2f_+(0) = x + 1/x - y, \tag{6.29}$$

so that for $x = 1.28$, $f_+(0) = 1$, we obtain

$$\lambda_+ = 0.037, \quad \xi = -0.15. \tag{6.30}$$

VII. MORE "PHYSICAL" SUM RULES

Recall that the results (6.23) were derived for a special value of the squared momentum transfer Δ'' [see Eq. (6.21)] which in the SU(3) limit is of order ϵ_8^2. What about the case when Δ and Δ' are nonvanishing up to first order in SU(3) breaking. In this case, the arguments of Sec. VI show that we cannot have $\Delta = \Delta'$. This section is devoted to a discussion when Δ and Δ' have the following properties:

$$\begin{aligned} &\Delta = m_K^2 \ \text{at} \ a = -1, \quad \Delta = O(\epsilon_8) \ \text{at} \ a = 0, \\ &\Delta' = m_\pi^2 \ \text{at} \ a = 2, \quad \Delta' = O(\epsilon_8) \ \text{at} \ a = 0. \end{aligned} \tag{7.1}$$

It is simple to see now that the considerations used in Sec. VI do not determine all the constants α, β, γ, and δ uniquely. Note first that the requirement that Δ and Δ' have minimal a dependence now leads to the following values of the squared momentum transfer:

$$\Delta = -\Delta' = m_K^2 - m_\pi^2. \tag{7.2}$$

Now, adding Eqs. (6.15) and (6.17) and similarly Eqs. (6.16) and (6.18), and using Eq. (7.2) and the SU(3) constraints, it is easy to check that

$$\alpha + \gamma = \beta + \delta = 0. \tag{7.3}$$

It is worth emphasizing that the constants α, β, γ, and δ as they appear in Eqs. (6.15)–(6.18) are a completely different set in the two cases defined by the properties of Δ and Δ' given by Eqs. (6.21) in the previous case and by Eq. (7.1) in the present case. For economy, we shall, however, use the same symbols. It is easy to check that Eqs. (7.3) are the only conditions we can derive in the present case, so that two out of the four constants α, β, γ, and δ cannot be fixed from the present considerations.

We shall now show that one of the two unknown constants can be fixed if we use the Dashen-Weinstein[4] result. In our notation this result is

$$(m_K^2 - m_\pi^2)f_+'(0) + f_-(0) = \tfrac{1}{2}(f_K/f_\pi - f_\pi/f_K) + \text{second-order terms in SU(3)}\times\text{SU(3) breaking}. \tag{7.4}$$

We would like to mention that the Dashen-Weinstein relation is a perturbative result and may not be correct if the perturbation theory around SU(3) ×SU(3) limit breaks down. In contrast, we emphasize again that the relations (6.15)–(6.18) are correct up to second-order perturbation only in ϵ, defined by Eq. (6.9). We shall show later that our numerical results for λ_+ and ξ do not depend or do not depend sensitively on the use of Eq. (7.4). Subtracting Eq. (6.17) from Eq. (6.15) and using Eqs. (7.2) and (7.3) together with the definition (6.9), one readily sees that Eq. (7.4) implies

$$\alpha = 1. \tag{7.5}$$

Using the results (7.2) and (7.3) in Eqs. (6.15)–(6.18), we obtain the following relations:

$$f_+(\Delta) = 1 + (\alpha+\beta)\tfrac{1}{2}\epsilon + (\alpha+\beta-3)\tfrac{1}{2}a\epsilon, \tag{7.6}$$

$$f_+(-\Delta) = 1 - (\alpha+\beta)\tfrac{1}{2}\epsilon + (\alpha+\beta-6)\tfrac{1}{4}a\epsilon, \tag{7.7}$$

$$f_-(\Delta) = (\alpha-\beta)\tfrac{1}{2}\epsilon + (\alpha-\beta+1)\tfrac{1}{2}a\epsilon, \tag{7.8}$$

$$f_-(-\Delta)=(\alpha-\beta)\tfrac{1}{2}\epsilon-(\alpha-\beta+4)\tfrac{1}{4}a\epsilon, \qquad (7.9)$$

where $\alpha=1$ if we accept Eq. (7.4). Furthermore, from Eq. (6.23), we have the extra relations[17]

$$f_+(\Delta'')=1-\tfrac{3}{2}a\epsilon, \qquad (7.10)$$

$$f_-(\Delta'')=-(1+\tfrac{1}{2}a)\epsilon. \qquad (7.11)$$

Now using the quadratic expansion (3.4), we may solve Eqs. (7.6), (7.7), and (7.10) to obtain $\alpha+\beta$, λ_+, and λ_+'. Note that since we are dropping terms of order ϵ^2, we should for consistency put $f_+(0)=1$. This is because we expect the SU(3)-breaking term in $f_+(0)$ to be proportional to ϵ^2 rather than $a\epsilon$ or a^2. Indeed this follows if we use Eqs. (6.29), (6.14), and (6.12). From Eqs. (7.6), (7.7), and (7.10), we then get[18]

$$\alpha+\beta=-\frac{3a}{1+\frac{1}{4}a(1+3\Delta''/\Delta)}\frac{\Delta}{\Delta''}\left(1-\frac{\Delta''^2}{\Delta^2}\right), \qquad (7.12)$$

$$\lambda_+=\frac{m_\pi^2}{\Delta}\left(1+\frac{a}{4}\right)\frac{\epsilon}{2}(\alpha+\beta), \qquad (7.13)$$

$$\lambda_+'=\frac{m_\pi^4}{\Delta^2}\,\frac{3a\epsilon}{8}(\alpha+\beta-4). \qquad (7.14)$$

For the physical value of a given by Eq. (2.4), and the experimental value of f_K/f_π, one can then obtain λ_+ and λ_+' without requiring α and β separately The results for λ_+ and λ_+' thus do not require the special value of α obtained in Eq. (7.5). These numerical results will be discussed in Sec. VIII.

Using the quadratic expansion (3.4) for $f_-(t)$, we can further solve Eqs. (7.8), (7.9), and (7.11) to obtain ξ, λ_-, and λ_-'. We get

$$\xi=-\frac{\epsilon}{(1-\Delta''^2/\Delta^2)}\left[1+\frac{a}{4}\left(2+3\frac{\Delta''}{\Delta}-\frac{\Delta''^2}{\Delta^2}\right)+(\alpha-\beta)\frac{\Delta''}{8\Delta}\left(3a+\frac{\Delta''}{\Delta}(4+a)\right)\right], \qquad (7.15)$$

$$\lambda_-=\frac{3}{8}\,\frac{m_\pi^2}{\xi}\,\frac{a\epsilon}{\Delta}(\alpha-\beta+2), \qquad (7.16)$$

$$\lambda_-'=\frac{m_\pi^4}{\xi\Delta^2}\left[-\xi+(\alpha-\beta)\frac{\epsilon}{2}+\frac{a\epsilon}{8}(\alpha-\beta-2)\right]. \qquad (7.17)$$

For the numerical evaluation of ξ, λ_-, and λ_-' we now need the value of $\alpha-\beta$. This can be obtained if we use Eq. (7.12) and the result (7.5) based on the Dashen-Weinstein sum rule. Note, however, that in the evaluation of ξ from Eq. (7.15), the multiplicative factor of $\alpha-\beta$ for physical a is so small that unless $\alpha-\beta$ is very large, ξ is sensitively independent of the value of $\alpha-\beta$. In contrast, λ_- and λ_-' do depend quite sensitively on $\alpha-\beta$.

VIII. NUMERICAL RESULTS

From Eqs. (7.12)–(7.14), we evaluate λ_+ and λ_+' using Eq. (2.4) for the physical value of a, and Eqs. (6.23) and (7.2) for expressions of Δ'' and Δ. Eq. (7.12) gives

$$\alpha+\beta\simeq3.79. \qquad (8.1)$$

The numerical estimates of λ_-, λ_-', and ξ follow from Eqs. (7.15)–(7.17), if we use $\alpha-\beta=-1.79$ as determined from Eqs. (7.5) and (8.1). Note that ξ is essentially independent of $\alpha-\beta$, as mentioned before. For f_K/f_π we take the value

$$\frac{f_K}{f_\pi}\simeq1.28,$$

which follows from the experimentally determined value[19] $f_K/f_\pi f_+(0)=1.28\pm0.06$ if we use $f_+(0)=1$. We then obtain the results

$$\begin{aligned}&\lambda_+=0.037,\quad \lambda_+'=-0.0001,\\&\lambda_-=0.010,\quad \lambda_-'=-0.0014,\\&\xi=-0.16.\end{aligned} \qquad (8.2)$$

With the values of $\lambda_\pm$ and $\lambda_\pm'$ in Eq. (8.2), we observe that the quadratic terms in the t dependence of $f_\pm(t)$ and $d(t)$ are negligible in the physical K_{l3}-decay region and do not make any sizable contribution even up to the soft-pion point $t=m_K^2$.

We would also like to point out that from rather general considerations, Okubo and Shih[20] have recently derived the following bounds on the derivatives of $d(t)$ at $t=0$:

$$\begin{aligned}&-0.095\le d'(0)\le0.52,\\&-1.1\times10^{-2}\le m_\pi^2d''(0)\le3.7\times10^{-2},\\&-1.6\times10^{-3}\le m_\pi^4d'''(0)\le4.6\times10^{-3}.\end{aligned} \qquad (8.3)$$

Our results for the derivatives are consistent with these bounds. For comparison, we list our numerical values,

$$\begin{aligned}&d'(0)=0.27,\\&m_\pi^2d''(0)=-0.57\times10^{-2},\\&m_\pi^4d'''(0)=1.35\times10^{-3}.\end{aligned} \qquad (8.4)$$

IX. DISCUSSION

Using approximate SU(3)×SU(3) symmetry within the framework of the (3, 3*)+(3*, 3) model, we have obtained in this paper several new sum rules involving the K_{l3} form factors. Specifically we have assumed that the extrapolation of the results from one subgroup of SU(3)×SU(3) to another is as smooth as possible, consistent with the constraints available when the various subgroup symmetries are realized exactly. We have argued that if SU(3)

×SU(3) symmetry is a reasonably good symmetry of nature, such a smooth extrapolation is to be expected. Using the sum rules, we have been able to solve for the various K_{l3} parameters using quadratic expansions in t for the form factors $f_\pm(t)$. Our main results can be summarized as follows:

(1) We find that the quadratic terms make a negligible contribution in the region of interest. In particular, $f_+(t)$ and $d(t)$ are reasonably well described by linear functions of t up to the soft-pion point $t=m_K^2$.

(2) Although our value of λ_+ is consistent with experiments, most experiments yield a larger negative value for ξ. The experimental spread is however substantial, and for comparison, we quote results from a recent Rochester-Wisconsin collaboration,[21]

$$\lambda_+ = 0.04 \pm 0.015, \quad \xi = -0.36^{+0.29}_{-0.31}. \tag{9.1}$$

(3) A crucial quantity is the slope of $d(t)$ vs t. With the expansion

$$d(t) = d(0)\left(1 + \frac{\lambda_0}{m_\pi^2}t + \cdots\right), \tag{9.2}$$

we get

$$\lambda_0 = m_\pi^2 \frac{d'(0)}{d(0)} \simeq 0.022, \tag{9.3}$$

which is almost the same result as one gets from the soft-pion theorem (4.1), and is inconsistent with the experimental value[1] -0.024 ± 0.02. Should future experiments support this latter value, it would imply a drastic modification in the soft-pion result. Such a situation would imply a breakdown of our smoothness hypothesis or even of the $(3, 3^*) + (3^*, 3)$ model if indeed approximate SU(3)×SU(3) symmetry itself still makes sense. It has been suggested[1,22] that the experimental contradiction of the soft-pion value of λ_0 may be explained if nonlinear terms in $d(t)$ become important near the soft-pion value of the momentum transfer. Our calculations is this paper do not bear out this possibility. Clearly, a great deal of interest attaches to more experimental information on λ_0.

On the question of SU(3)×SU(3)-symmetry breaking, it is worthwhile to point out that a recent result of Cheng and Dashen[23] seems to suggest that the $(3, 3^*) + (3^*, 3)$ model is insufficient to explain the large σ term that the πN scattering analysis seems to require. It has, however, been pointed out[24] that if in the SU(3)×SU(3) limit, one also realizes scale invariance through a scalar Goldstone σ meson, this difficulty in the $(3, 3^*) + (3^*, 3)$ model disappears, with all the advantages of this model still preserved. A scalar σ, however, is of no interest in the K_{l3} problem.

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†Present address: Department of Physics, University of California, San Diego, La Jolla, Calif. 92037.

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[3]Note that some recent experiments suggest a value of λ_+ which is much larger than the prediction from the usual K^* dominance of the form factor $f_+(t)$. This would imply (1) if K^* dominance is correct, $f_+(t)$ would satisfy at least a twice-subtracted dispersion relation, or (2) if $f_+(t)$ satisfies an unsubtracted or once-subtracted dispersion relation, there must be other sizable contributions besides K^*.

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[16]If one insists on a perturbation description around the SU(3)×SU(3) limit, these sum rules are valid to fourth order in SU(3)×SU(3) breaking.

[17]Note that the sum rules (7.10) and (7.11) or equivalently Eqs. (6.23) as well as the set of relations (7.6)–(7.9) have been obtained from the same generating set (6.15)–(6.18). No circular argument is, however, involved in using all these sum rules together, since the momentum transfer involved is different in the two sets (7.6)–(7.9) and (7.10)–(7.11). We cannot derive any more useful relations starting with (6.15)–(6.18). Also note that the extension of the sum rules (7.6)–(7.9) retaining terms of order ϵ^2 involves many unknown constants, so that no useful information can be obtained. The situation here is unlike the case in the previous section where we could obtain the sum rules (6.27) correct to terms of order ϵ^2. However, in view of the fact that order-ϵ^2 terms do not make a sizable contribution, as evidenced by comparing the results (6.25) and (6.26) with (6.28) and (6.29), we believe the

relations (7.6)–(7.9) are quite reliable.

[18]It should be noted that $\alpha+\beta$ as given by Eq. (7.12) is independent of ϵ. This is as it should be if we recall the definitions of α and β. Furthermore, the a dependence is quite weak. At $a=-1$, $a=0$, and $a=2$, $\alpha+\beta$ is given by $\frac{56}{15}$, $\frac{54}{15}$, and $\frac{40}{15}$. This implies that the linear a dependence of the right-hand side in Eqs. (6.10) is quite resonable. Note that the right-hand side in Eqs. (6.10) cannot of course be exactly linear in a, due to the appearance of x, which has a weak implicit dependence on a as discussed before.

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