PHYSICAL REVIEW D VOLUME 5, NUMBER 10 15 MAY 1972

Weak Electromagnetic Fields Around a Rotating Black Hole*

Edward D. Fackerell† and James R. Ipser‡
California Institute of Technology, Pasadena, California 91109
(Received 12 April 1971)

Foundations are laid for studying weak electromagnetic fields around a rotating black hole. In particular, Maxwell's equations in the Kerr geometry of the black hole are reduced to a single second-order partial differential equation. Although this differential equation is probably not separable, it is sufficiently simple to yield theorems about electromagnetic properties of black holes.

The analysis of perturbations to the Kerr geometry is a problem of considerable importance for current research in relativistic astrophysics, since it seems quite likely that the gravitational collapse of a rotating star will result in a Kerr-metric black hole.[1-3] Despite the insight yielded by Carter's[3] recent theorem, it is clear that any detailed discussion of gravitational perturbations to the Kerr geometry will be rather complex. Consequently, it is desirable to deal first with the simpler problem of weak electromagnetic fields ("electromagnetic perturbations") in the Kerr geometry. An analysis of this problem may be expected to cast light upon the more difficult problem of gravitational perturbations. Moreover, electromagnetic perturbations are of interest in their own right; they deal with the evolution and fate of the magnetic field of a collapsing star, and with the form of the light and radio waves that a distant observer should receive in the late stages of collapse, if there is no obscuring matter.

In this paper we reduce Maxwell's equations to a single second-order, hyperbolic, partial differential equation, which governs the behavior of electromagnetic perturbations to the Kerr geometry. On separating out the time and the azimuthal dependence of the solution by using coordinates adapted to the two Killing-vector fields of the Kerr geometry, we obtain an elliptic partial differential equation in two independent variables. Unfortunately, it appears that this equation is not separable, so it is likely that a *complete* discussion of realistic perturbations to the Kerr geometry will prove to be very difficult.

The form of the Kerr metric which is most suitable for our purposes is that given by Newman and Janis,[4] namely,

$$ds^2 = \left(1 - \frac{2Mr}{R^2}\right) du^2 + 2du dr + \frac{4Mar}{R^2}\sin^2\theta\, du d\phi - 2a\sin^2\theta\, dr d\phi - R^2 d\theta^2 - \sin^2\theta\left(\frac{2Ma^2 r}{R^2}\sin^2\theta + r^2 + a^2\right) d\phi^2, \tag{1}$$

where

$$R^2 = r^2 + a^2\cos^2\theta, \tag{2}$$

M is the mass parameter, and a is the angular momentum per unit mass.

In order to derive the basic equation, we follow the technique used by Price[5] in his recent discussion of perturbations to the Schwarzschild geometry. This technique uses the Newman-Penrose[6] spin-coefficient formalism. In the case of the Kerr geometry we can choose the null tetrad $\underline{l}$, $\underline{n}$, $\underline{m}$, $\overline{\underline{m}}$ so that the spin coefficients ϵ, κ, λ, ν, and σ all vanish.[7] We shall follow Kinnersley and take the tetrad to be[7]

$$\underline{l} = \frac{\partial}{\partial r}, \tag{3}$$

$$\underline{n} = (r^2 + a^2)R^{-2}\frac{\partial}{\partial u} - \tfrac{1}{2}(r^2 - 2Mr + a^2)R^{-2}\frac{\partial}{\partial r} + aR^{-2}\frac{\partial}{\partial \phi}, \tag{4}$$

and

$$\underline{m} = \tfrac{1}{2}\sqrt{2}\,(r + ia\cos\theta)^{-1}\left(ia\sin\theta\frac{\partial}{\partial u} + \frac{\partial}{\partial\theta} + i\csc\theta\frac{\partial}{\partial\phi}\right). \tag{5}$$

Under these circumstances Maxwell's equations take the form

$$D\phi_1 - \bar{\delta}\phi_0 = 2\rho\phi_1 + (\pi - 2\alpha)\phi_0, \tag{6}$$

$$\Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1, \tag{7}$$

$$D\phi_2 - \bar{\delta}\phi_1 = 2\pi\phi_1 + \rho\phi_2, \tag{8}$$

and

$$\Delta\phi_1 - \delta\phi_2 = -2\mu\phi_1 + (2\beta - \tau)\phi_2. \tag{9}$$

Here

$$\phi_0 = F_{\mu\nu} l^\mu m^\nu, \quad \phi_1 = \tfrac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \overline{m}^\mu m^\nu), \quad \phi_2 = F_{\mu\nu}\overline{m}^\mu n^\nu, \tag{10}$$

and $F_{\mu\nu}$ is the electromagnetic field tensor. The nonvanishing spin coefficients are

$$\rho=-(r-ia\cos\theta)^{-1},$$
$$\beta=-\tfrac{1}{4}\sqrt{2}\,\cot\theta\,\bar\rho,$$
$$\pi=\tfrac{1}{2}\sqrt{2}\,ia\sin\theta\,\rho^2,$$
$$\alpha=\pi-\bar\beta, \tag{11}$$
$$\tau=-\tfrac{1}{2}\sqrt{2}\,ia\sin\theta\,\rho\bar\rho,$$
$$\gamma=\tfrac{1}{2}M\rho^2-\tfrac{1}{2}ia\cos\theta\,\rho\bar\rho+\tfrac{1}{2}a^2\sin^2\theta\,\rho^2\bar\rho,$$
$$\mu=\tfrac{1}{2}\rho+\tfrac{1}{2}M\rho(\rho+\bar\rho)+\tfrac{1}{2}a^2\sin^2\theta\,\rho^2\bar\rho.$$

It is convenient to introduce a new function

$$\chi=(r^2-2Mr+a^2)\rho^2\sin\theta \tag{12}$$

and new dependent variables Φ_0, Φ_1, and Φ_2 such that

$$\phi_0=\chi^{-1}\rho\Phi_0,\quad \phi_1=\rho^2\Phi_1,\ \text{and}\ \phi_2=\rho\Phi_2/\sin\theta. \tag{13}$$

We then find that Eqs. (6)–(9) become

$$D\Phi_1=\rho^{-1}\chi^{-1}\bar\delta\Phi_0, \tag{14}$$
$$\Delta\Phi_0=\rho\chi\delta\Phi_1, \tag{15}$$
$$D\Phi_2=\rho\sin\theta\,\bar\delta\Phi_1, \tag{16}$$

and

$$\rho\sin\theta\,\Delta\Phi_1=\delta\Phi_2. \tag{17}$$

Either on eliminating Φ_0 from Eqs. (14) and (15), or on eliminating Φ_2 from (16) and (17), we find that Φ_1 satisfies the equation

$$[\Delta D-(\bar\mu-\mu-\gamma-\bar\gamma)D-\bar\delta\delta+(\bar\tau+2\alpha)\delta]\Phi_1=0. \tag{18}$$

A useful alternate form of this equation is obtained if we introduce a new dependent variable Ω_1 by

$$\Omega_1=\rho\Phi_1, \tag{19}$$

since then we find that Ω_1 satisfies the equation

$$\Box\Omega_1+2M\rho^3\Omega_1=0, \tag{20}$$

where

$$\Box\Omega_1\equiv g^{\mu\nu}\Omega_{1;\mu;\nu}. \tag{21}$$

The equation $\Box\Omega_1=0$ is separable,[8] and the solutions have been investigated recently.[9] However, it does not appear that Eq. (20) can be dealt with so readily. When written out explicitly, Eq. (20) is equivalent to

$$\frac{\partial}{\partial r}\left[(r^2-2Mr+a^2)\frac{\partial\Omega_1}{\partial r}\right]+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Omega_1}{\partial\theta}\right)+a^2\sin^2\theta\frac{\partial^2\Omega_1}{\partial u^2}+\frac{1}{\sin^2\theta}\frac{\partial^2\Omega_1}{\partial\phi^2}$$
$$-2(r^2+a^2)\frac{\partial^2\Omega_1}{\partial r\partial u}-2a\frac{\partial^2\Omega_1}{\partial r\partial\phi}+2a\frac{\partial^2\Omega_1}{\partial u\partial\phi}-2r\frac{\partial\Omega_1}{\partial u}+\frac{2M(r+ia\cos\theta)}{(r-ia\cos\theta)^2}\Omega_1=0. \tag{22}$$

The local Lie group, G, generated by the solutions of Eq. (22) contains the subgroup S which is generated by all transformations of the form

$$(u',r',\theta',\phi')=(u,r,\theta,\phi),\qquad \Omega_1'=s\Omega_1+g(u,r,\theta,\phi),$$

where s is an arbitrary parameter and $g(u,r,\theta,\phi)$ is any solution of Eq. (22). Ovsjannikov[10] has shown that S is a normal divisor of G, and that the transformations of the factor group G/S are characterized by operators of the form

$$x=\xi^u\frac{\partial}{\partial u}+\xi^r\frac{\partial}{\partial r}+\xi^\theta\frac{\partial}{\partial\theta}+\xi^\phi\frac{\partial}{\partial\phi}+\zeta\Omega_1\frac{\partial}{\partial\Omega_1},$$

where the ξ's and ζ are functions of u, r, θ, ϕ only. In our case the determining equations for the ξ's are precisely the equations for conformal Killing motions in the Kerr geometry. Since there are only two of these conformal motions,[11] namely, the two Killing motions which correspond to ϕ and u being ignorable coordinates, we only obtain from the factor group G/S the obvious separation of variables

$$\Omega_1(u,r,\theta,\phi)=\exp(im\phi+i\omega u)f_1(r,\theta), \tag{23}$$

where m and ω are constants and $f_1(r,\theta)$ satisfies the partial differential equation

$$\frac{\partial}{\partial r}\left[(r^2-2Mr+a^2)\frac{\partial f_1}{\partial r}\right]+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f_1}{\partial\theta}\right)-2[\,i\omega(r^2+a^2)+ima]\,\frac{\partial f_1}{\partial r}$$
$$+\left[-\left(\omega a\sin\theta+\frac{m}{\sin\theta}\right)^2-2i\omega r+\frac{2M(r+ia\cos\theta)^3}{(r^2+a^2\cos^2\theta)^2}\right]f_1=0. \tag{24}$$

In the region outside the horizons the factor $r^2 - 2Mr + a^2$ is positive, so that Eq. (24) is an elliptic partial differential equation.

Equation (24) is the key to electromagnetic perturbations of the Kerr metric. Once it has been solved, everything else of interest can be calculated algebraically or by quadrature. In particular, Φ_1 is given by

$$\Phi_1 = \exp(im\phi + i\omega u)\rho^{-1} f_1(r, \theta). \tag{25}$$

Taking the same time and azimuthal dependence for Φ_0 and Φ_2, namely,

$$\Phi_0 = \exp(im\phi + i\omega u) f_0(r, \theta) \tag{26}$$

and

$$\Phi_2 = \exp(im\phi + i\omega u) f_2(r, \theta), \tag{27}$$

we find that Eqs. (14) and (15) are equivalent to the integrable Pfaffian differential equation[12]

$$df_0 = \left\{ 2(r^2 - 2Mr + a^2)^{-1}[i\omega(r^2+a^2) + iam] f_0 - ia\sqrt{2}\rho^2 \sin^2\theta f_1 + \sqrt{2}\rho \sin\theta \left[\frac{\partial f_1}{\partial\theta} - \left(a\omega\sin\theta + \frac{m}{\sin\theta}\right) f_1\right]\right\} dr + \left[-\left(a\omega\sin\theta + \frac{m}{\sin\theta}\right) f_0 - \sqrt{2}\rho(r^2 - 2Mr + a^2)\sin\theta\left(\frac{\partial f_1}{\partial r} - \rho f_1\right)\right] d\theta \tag{28}$$

and that Eqs. (16) and (17) are equivalent to the integrable Pfaffian differential equation

$$df_2 = -\frac{1}{\sqrt{2}}\rho^2 \sin\theta \left[\left(a\omega\sin\theta + \frac{m}{\sin\theta} - ia\rho\sin\theta\right) f_1 + \frac{\partial f_1}{\partial\theta}\right] dr + \left\{\left(\omega a\sin\theta + \frac{m}{\sin\theta}\right) f_2 - \sqrt{2}\rho\sin\theta\left[i\omega(r^2+a^2) f_1 + iam f_1 - \tfrac{1}{2}(r^2 - 2Mr + a^2)\left(\frac{\partial f_1}{\partial r} - \rho f_1\right)\right]\right\} d\theta. \tag{29}$$

In both cases the integrability condition is simply that Eq. (24) be satisfied. Once f_0 and f_2, and hence Φ_0 and Φ_2, have been found, ϕ_0, ϕ_1, and ϕ_2 are given by Eq. (13); and the components of the electromagnetic field tensor are

$$F_{\mu\nu} = 2(\phi_1 + \overline{\phi}_1) n_{[\mu} l_{\nu]} + 2\phi_2 l_{[\mu} m_{\nu]} + 2\overline{\phi}_2 l_{[\mu} \overline{m}_{\nu]} + 2\phi_0 \overline{m}_{[\mu} n_{\nu]} + 2\overline{\phi}_0 m_{[\mu} n_{\nu]} + 2(\phi_1 - \overline{\phi}_1) m_{[\mu} \overline{m}_{\nu]}, \tag{30}$$

where the square brackets denote antisymmetrization.

Equations (3)–(5), (11)–(13), and (24)–(30) are a complete set of equations for the electromagnetic field tensor in Newman-Janis coordinates.

It is worthwhile to note that the "longitudinal" solution of Maxwell's equations (6)–(9) which corresponds to the addition of charge to the source is obtained from Eq. (18) by setting Φ_1 = constant, that is, $\phi_1 = \text{const}\,\rho^2$, together with $\phi_0 = \phi_2 = 0$.[13]

In an analysis by one of the authors (JRI), which will be reported elsewhere, the fundamental equation (24) is used to prove two theorems: (i) The solution Φ_1 = const – which corresponds to passage from the Kerr metric to the charged Kerr metric – is the only physically acceptable, time-independent solution to Maxwell's equations in the Kerr geometry. (Hence a Kerr-metric black hole can have no electromagnetic "hair," except that obtained by adding charge in the standard "charged Kerr metric" manner.) (ii) All axisymmetric normal modes (physically acceptable perturbations with ω^2 real) of an electromagnetic field in the Kerr geometry have $\omega^2 > 0$ and hence are oscillatory. No modes grow exponentially. Assuming that these modes form a complete set, this guarantees that all axisymmetric, electromagnetic perturbations are stable.

---

*Work supported in part by the National Science Foundation (GP-27304 and GP-19887).

†Present address: Department of Mathematics, Monash University, Clayton Victoria 3168, Australia.

‡Present address: Department of Astronomy and Astrophysics, and Enrico Fermi Institute, University of Chicago, Chicago, Illinois.

[1]V. de la Cruz and W. Israel, Phys. Rev. 170, 1187 (1968).

[2]R. Penrose, Riv. Nuovo Cimento 1, 252 (1969).

[3]B. Carter, Phys. Rev. Letters 26, 331 (1971).

[4]E. T. Newman and A. I. Janis, J. Math. Phys. 6, 915 (1965).

[5]R. H. Price, Ph. D. thesis, California Institute of Technology, 1971 (unpublished); also this issue, Phys. Rev. D 5, 2419 (1972); 5, 2439 (1972).

[6]E. T. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).
[7]W. Kinnersley, J. Math. Phys. 10, 1195 (1969).
[8]B. Carter, Commun. Math. Phys. 10, 280 (1968).
[9]E. D. Fackerell and J. R. Ipser (unpublished report).
[10]L. V. Ovsjannikov, Soviet Math. 1, 481 (1960).
[11]This can be verified easily by integrating the equations given in C. D. Collinson and D. C. French, J. Math. Phys. 8, 701 (1967).
[12]I. N. Sneddon, *Elements of Partial Differential Equations* (McGraw-Hill, New York, 1957).
[13]E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. 6, 918 (1965).

PHYSICAL REVIEW D VOLUME 5, NUMBER 10 15 MAY 1972

Quantum Models for the Lowest-Order Velocity-Dominated Solutions of Irrotational Dust Cosmologies

E. P. T. Liang*

Center for Relativity Theory, University of Texas at Austin, Austin, Texas 78712
(Received 10 February 1972)

The lowest-order velocity-dominated solutions to the Einstein dust equations of Eardley, Liang, and Sachs are quantized using the canonical methods of DeWitt and of Arnowitt, Deser, and Misner. The quantum dynamics of these models is shown to be governed by the Einstein-Klein-Gordon (EKG) equation. Exact solutions of the decoupled EKG equations in the discrete limit are obtained, which have the striking feature that the state amplitude vanishes at the singularity for anisotropic models. The geometry of the manifold of the classical 3-metrics is studied and it turns out to be composed of conformally flat geodesic submanifolds. Other difficulties related to the quantum theory such as factor ordering, divergence, interpretation of the volume measure, etc. are also discussed.

I. INTRODUCTION

Recently, Eardley, Liang, and Sachs[1] introduced the concept of "velocity-dominated" singularities in irrotational dust cosmologies in general relativity and obtained the lowest-order solutions near these singularities by explicit integration. In this paper we are going to apply the methods of Dirac,[2] DeWitt,[3] and Arnowitt, Deser, and Misner[4] (ADM) to canonically quantize special models corresponding to these lowest-order velocity-dominated (LOVD) solutions. The purpose of this exercise is at least twofold: (a) to gain insight into the complicated formalism of canonical quantization in general relativity through the study of some simplified field models; (b) to obtain some meaningful physical results concerning the quantum structure of space-time singularities since the LOVD solutions may be a good approximation to the early universe. (It is at present still obscure whether or not the mixmaster[5] or mixmaster-like models[6] ultimately become velocity-dominated near the singularity.)

With irrotational dust as source, the dust flow lines provide a natural and unique $3+1$ decomposition of space-time, which is necessary for the canonical approach. The "velocity-dominated" assumption then simply says that the spatial curvature (${}^3R$ etc.) of the $t=\text{const}$ surfaces is small compared to time-derivative terms in the Einstein equations, and can be dropped near the singularities (where the matter-energy density becomes infinite). This is true for a large class of exact solutions. The reduced equations can then be explicitly integrated to give the lowest-order approximations near the singularity. Some of the integration functions are restricted because of the constraint equations and self-consistency requirements. These solutions may, of course, also be exact models whose spatial curvature is identically zero (i.e., exact models with flat 3-spaces). The crucial properties about these LOVD solutions are:

(a) They can be written in the form (8). (See Sec. III. This form is originally introduced by Lifshitz and Khalatnikov.[7]) In other words, they can be diagonalized by time-independent frame fields.

(b) They are spatially pointwise decoupled since all spatial derivatives are contained in the ${}^3R$ terms. The situation is a little similar to that of ordinary quantum field theory when one ignores the coupling between particles at $t=\pm\infty$ and quantizes the free fields, except that here the individual "particles" are not particles but field variables evaluated at different dust world lines. The LOVD solutions obviously contain less degrees of freedom than a generic exact solution. For more details, the reader is referred to Ref. 1.