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Relativistic Disks. I. Background Models*†

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Relativistic kinetic theory is used in conjunction with the theory of relativistic surface layers in order to study relativistic disks of matter. After a brief general discussion, attention is restricted to the case of counter-rotating disks. The general surface stress-energy tensors of such disks are exhibited and a distribution function which generates these stress-energy tensors is deduced. This is followed by a discussion of stability, and a criteria for the stability of particle orbits is derived. Finally, the question of central red shift is considered. It is shown that all counter-rotating disks without singularities at the rim will have a finite central red shift, but the question of the existence of a maximum central red shift remains open.

INTRODUCTION

At present, the only well-understood gravitational fields are those possessing spherical symmetry. This is unfortunate, for such a high degree of symmetry automatically excludes such interesting effects of relativistic gravitation as gravitational waves and magnetic-type fields, as well as placing strong restrictions on the kinds of sources which may be studied. The purpose of this paper is to provide a foundation for the study of those axially symmetric gravitational fields which may be regarded as having disks of matter as their source.

The interest in disks is motivated by the presence in nature of many disklike configurations and by recent suggestions that disks of stars may be used as models for supermassive objects in which relativistic effects would be of importance.

In the present paper, static disks of collisionless dust will be considered with the prime emphasis being on counter-rotating disks, i.e., those with zero net angular momentum and circular particle orbits. In subsequent papers, slowly rotating disks and nonstatic perturbations will be considered.

SURFACE LAYERS IN GENERAL RELATIVITY

The theory of surface layers in curved space-time has been developed by Israel[1] and by Papapetrou and Hemoui.[2]

Let M^+ and M^- be two manifolds having a common TL boundary Σ and let $K^{\pm}_{ab}$ be, respectively, the second fundamental form of Σ in M^+ and M^-. Denote the full space-time $M^+ \cup \Sigma \cup M^-$ by M. In a Gaussian normal coordinate system based on Σ such that $z=0$ defines Σ,

$$K^{\pm}_{ab} = \frac{\partial g^{\pm}_{ab}}{\partial z}\bigg|_{\Sigma}. \tag{1}$$

The hypersurface Σ is called a surface layer if $K^+_{ab} \neq K^-_{ab}$. The following argument suggested by Ehlers[3] gives a criterion for the existence of a surface layer:

Consider a one-parameter family of hypersurfaces Σ_ϵ such that $\Sigma = \Sigma_0$; and for every value ϵ construct Gaussian normal coordinates $x^a(\epsilon)$ based on Σ such that $g^+_{ab}|_\Sigma = g^-_{ab}|_\Sigma$. In such coordinates, the geodesic equations are[4]

$$\begin{aligned} &\ddot{z} + \Gamma^z_{bc}\dot{x}^b\dot{x}^c = 0\,, \\ &\ddot{x}^\alpha + \Gamma^\alpha_{bc}\dot{x}^b\dot{x}^c = 0\,. \end{aligned} \tag{2}$$

Assume initial conditions such that all particles remain between the hypersurfaces $\Sigma_{-\epsilon}$ and Σ_ϵ and that the limits as $\epsilon \to 0$ of $x^a(s;\epsilon)$, $\dot{x}^a(s;\epsilon)$, $\ddot{x}^a(s;\epsilon)$, and $g_{ab}(x^a(s;\epsilon);\epsilon)$ exist. Then in this limit, the geodesic equations become

$$\begin{aligned} &\Gamma^z_{\beta\sigma}\dot{x}^\beta\dot{x}^\sigma|_\Sigma = 0\,, \\ &\ddot{x}^\alpha + \Gamma^\alpha_{\beta\sigma}\dot{x}^\beta\dot{x}^\sigma = 0\,. \end{aligned} \tag{3}$$

In the chosen coordinates,

$$\lim_{\epsilon\to 0} \pm\Gamma^z_{\beta\sigma} = K^{\pm}_{\beta\sigma}\,, \tag{4}$$

so that the first equation of (3) means that $\hat{K}_{\beta\sigma}\dot{x}^\beta\dot{x}^\sigma = 0$, where $\hat{K}_{\beta\sigma}$ is some weighted average of $K^+_{\beta\sigma}$ and $K^-_{\beta\sigma}$. Since this must hold for all timelike $\dot{x}^\beta$, and $K_{\beta\sigma}$ is independent of $\dot{x}^\beta$, this means $\hat{K}_{\beta\sigma}=0$. Thus, if a surface layer is to exist, $K^+_{\beta\sigma}$ and $K^-_{\beta\sigma}$ must have opposite sign. In the case of a disk, the space-time must possess a discrete symmetry. If the disk is represented by Σ, then, upon reflection, $K^+_{\beta\sigma} = -K^-_{\beta\sigma}$.

The second equation of (3) is the geodesic equation with respect to the induced metric on Σ. Thus, free particles in a surface layer follow geodesics of that surface layer; and, without loss of generality, all calculations and results may be given in terms of the hypersurface Σ on which the matter resides.

If $K_{\alpha\beta}=\frac{1}{2}(K^{+}_{\alpha\beta}-K^{-}_{\alpha\beta})$, the three-dimensional surface stress-energy tensor is[1]

$$S_{\alpha\beta}=-\tfrac{1}{2}(K_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}K)\,, \tag{5}$$

obeying the conservation law $S_{\alpha}{}^{\beta}{}_{|\beta}=0$ in which represents covariant differentiation within Σ.

As expected, no normal stress is present.

THE EXTERIOR FIELD: WEYL SOLUTIONS

The obvious candidates for the external gravitational field of disks are the Weyl solutions, those solutions of the Einstein equations which are static and axially symmetric.

The canonical form for such metrics is

$$ds^2=e^{2(\nu-\gamma)}(d\eta^2+d\xi^2)+\rho^2e^{-2\gamma}d\varphi^2-e^{2\gamma}dt^2\,, \tag{6}$$

with ρ, ν, and γ functions only of η and ξ. Coordinates have been chosen so that the timelike and spacelike Killing vectors are, respectively, $\xi=\partial/\partial t$ and $\zeta=\partial/\partial\varphi$. If $D=\partial/\partial\eta+i\partial/\partial\xi$ and the surface layer Σ is defined by $\eta=0$, the field equations are[5]

$$\begin{aligned}&D\overline{D}\rho=e^{\nu-\gamma}S_{\xi}{}^{\xi}\sin^2\xi\,\delta(\eta)\,,\\&D\overline{D}\gamma+\frac{1}{2\rho}(D\rho\overline{D}+\overline{D}\rho D\gamma)=\frac{e^{\nu-\gamma}}{2\rho}(S_t{}^t-S_\alpha{}^\alpha)\delta(\eta)\,,\\&2D\nu D\rho=D^2\rho+2\rho(D\gamma)^2+e^{(\nu-\gamma)}S_{\xi}{}^{\xi}\sin^2\xi\,\delta(\eta)\,.\end{aligned} \tag{7}$$

For the purposes of this paper, a simpler set of equations will be used. Off Σ the functions ρ, ν, γ must satisfy the vacuum-field equations

$$\begin{aligned}&D\overline{D}\rho=0\,,\\&\Delta\gamma=0\,,\\&2D\rho D\nu=D^2\rho+2\rho(D\gamma)^2\,,\end{aligned} \tag{8}$$

in which $\Delta\gamma$ is the Laplacian in a flat space having coordinates η, ξ. Solutions of (8) are easy to obtain, and Eq. (5) then determines the surface stress-energy tensor. It will be shown that this procedure generates precisely the set of all counter-rotating disks.

The metric (6) and field equations (8) are form-invariant under any transformation

$$\begin{aligned}&\eta+i\xi=f(u+iv)\,,\\&\nu(u,v)=\nu(\eta,\xi)+\ln|f'|\,,\end{aligned} \tag{9}$$

in which f is an analytic function. In particular, if z is a conjugate function to ρ (which must be harmonic from the field equations), then $\eta+i\xi=\rho+iz$ leads to the canonical cylindrical coordinates of Weyl. With this definition of (ρ,z) choose (η,ξ) as the canonical spheroidal coordinates defined by

$$\rho+iz=R_0\cosh(\eta+i\xi)\,. \tag{10}$$

In these coordinates, the surface $\eta=0$ is a disk of coordinate radius R_0. Equations (1), (5), and (6) immediately give the nonzero components of the surface stress-energy tensor:

$$\begin{aligned}&S_{\xi\xi}=\frac{1}{\rho}e^{(\nu-\gamma)}\frac{\partial\rho}{\partial\eta}\bigg|_{\eta=0}\,,\\&S_{\varphi\varphi}=\rho^2e^{-(\nu+\gamma)}\frac{\partial\nu}{\partial\eta}\bigg|_{\eta=0}\,,\\&S_{tt}=-e^{-(\nu-3\gamma)}\left(\frac{\partial\nu}{\partial\eta}+\frac{1}{\rho}\frac{\partial\rho}{\partial\eta}-2\frac{\partial\gamma}{\partial\eta}\right)\bigg|_{\eta=0}\,.\end{aligned} \tag{11}$$

The field equations (8) and Eq. (10) give

$$\begin{aligned}&\frac{\partial\rho}{\partial\eta}\bigg|_{\eta=0}=0\,,\\&\frac{\partial\nu}{\partial\eta}\bigg|_{\eta=0}=-2\frac{\partial\gamma}{\partial\xi}\frac{\partial\gamma}{\partial\eta}\bigg|_{\eta=0}\cot\xi\,.\end{aligned} \tag{12}$$

The Lapace equation is separable in spheroidal coordinates, leading to a solution for γ in terms of Legendre functions of the first and second kind,

$$\gamma=\sum_{n=0}^{N}\frac{an}{Q_{2n}(0)}Q_{2n}(i\sinh\eta)P_{2n}(\sin\xi)\,. \tag{13}$$

This enables an explicit expression to be written for the surface stress-energy tensor of a disk generating the metric of Eq. (6):

$$\begin{aligned}&S_{\xi\xi}=0\,,\\&S_{\varphi\varphi}=4\rho^2e^{-(\nu+\gamma)}\sum_{n=0}^{N}na_n\left(P_{2n}(\sin\xi)-\frac{P_{2n-1}(\sin\xi)}{\sin\xi}\right)\sum_{m=0}^{N}ia_m\frac{Q_{2m}'(0)}{Q_{2m}(0)}P_{2m}(\sin\xi)\bigg|_{\eta=0}\,,\\&S_{tt}=2e^{-(\nu-\delta\gamma)}\left[1+2\sum_{n=0}^{N}na_n\left(P_{2n}(\sin\xi)-\frac{P_{2n-1}(\sin\xi)}{\sin\xi}\right)\right]\sum_{m=0}^{N}ia_m\frac{Q_{2m}'(0)}{Q_{2m}(0)}P_{2m}(\sin\xi)\bigg|_{\eta=0}\,.\end{aligned} \tag{14}$$

Since $S_{\xi\xi}=0$, there is no radial stress. Hence, the particles must follow circular orbits; and, since there is no net angular momentum, there must be equal numbers of particles moving to left and right. Allowing N to take all values in the range $(1,\infty)$ will give all possible such disks. The statement that the procedure used would generate all counter-rotating disks and none others is thus verified.

There are two points remaining. Examination of (14) reveals that the stress-energy tensor will be singular at the edge of the disk unless the condition

$$\sum_{n=0}^{N}a_n\frac{Q_{2n}'(0)}{Q_{2n}(0)}P_{2n}(0)=0 \tag{15}$$

is satisfied. This means that if $N=0$ such a singularity is unavoidable. It is also easy to verify that in this case $\partial\nu/\partial\eta|_{\eta=0}=0$, so that the only nonzero component of the stress-energy tensor is S_{tt}. The physical picture associated with this is a disk of motionless dust supported against collapse by a singular ring at the rim. This configuration must be eliminated on physical grounds.

Secondly, in order to finish specification of the solutions given, it is necessary to know $\nu(\eta, \xi)$. This is given by integration of the third equation of (8) – however for $N>1$, no closed form is known for these integrals. This difficulty is overcome by observing that once γ is given, the field equations determine $\partial\nu/\partial\eta$ and $\partial\nu/\partial\xi$; and, since ν is to be analytic everywhere off the disk and continuous across the disk, a Taylor expansion of $\nu(\eta, \xi)$ about $\nu(0, \pm\frac{1}{2}\pi)$ is possible, $\nu(0, \frac{1}{2}\pi)$ being known from elementary flatness.

RELATIVISTIC KINETIC THEORY OF SURFACE LAYERS

Complete reviews of relativistic kinetic theory have been given by Ehlers[6] and by Ehlers and Sachs.[7] Hence, only a brief outline of useful results will be given here.

Only a single species of particles is assumed, having unit rest mass. The equations of motion

$$p^a = \frac{dx^a}{ds}\,, \qquad \frac{\Delta p^a}{ds}=0 \tag{16}$$

define a unique vector field, the Liouville field on the one-particle relativistic phase space

$$X=\{(x,p): x\in M,\ p\in T_x(M);\ p^2=-1,\ p \text{ future pointing}\}.$$

The orbits of the Liouville field define the phase flow. In component notation, this field is given by

$$L=p^a\frac{\partial}{\partial x^a}-\Gamma^a_{bc}p^b p^c\frac{\partial}{\partial p^a}. \tag{17}$$

If π is the Lorentz-invariant measure on the mass shell $P_m(x)$, the natural measure on X is $\Omega = d^4x\wedge\pi$; and the natural measure for fluxes across hypersurfaces of X is $\omega=L\cdot\Omega$. Now, by analogy with classical kinetic theory, a scalar distribution function $F(x,p)$ is defined on X such that if Σ is some hypersurface of X, the ensemble average number of occupied orbits of L intersecting Σ is

$$\overline{N}[\Sigma]=\int F(x,p)\,\omega. \tag{18}$$

Conditions for the existence of such a distribution function have been given by Ehlers[6] and Bichteler.[8] For the cases to be considered in which all matter is contained within a spatially compact region, the results of Bichteler are sufficient to guarantee the existence and uniqueness of the distribution for all finite times.

In terms of the distribution function, the stress-energy tensor is

$$T^{ab}=\int_{P_m(x)}p^a p^b F(x,p)\pi. \tag{19}$$

For collision-free dust, $F(x,p)$ must be constant along phase orbits. This leads to the Einstein-Liouville equations of stellar dynamics:

$$L(F)=0\,, \qquad T^{ab}=\int_{P_m(x)}p^a p^b F(x,p)\pi. \tag{20}$$

Any solution of these equations will be a solution of the self-consistent-field problem for a collision-free self-gravitating system.

The explicit form of the distribution function is specified to the extent of the following:

Theorem: Let M be a space-time possessing three or more commuting and linearly independent Killing vectors $\xi^{(\alpha)}$. Then if $Q^{(\alpha)}=\xi^{(\alpha)}\cdot p$ the general solution of $L(F)=0$ having the same symmetries as M is

$$F(x,p)=F(Q^{(\alpha)})\,,$$

in which F is an arbitrary C^1 function.

Proof: Let $X^{(\alpha)}$ be the lifts of the $\xi^{(\alpha)}$ onto X. Then $F(x,p)$ must satisfy

$$L(F)=X^{(\alpha)}(F)=0\,. \tag{21}$$

Written explicitly, the second of these equations is

$$\xi^a_{(\alpha)}\frac{\partial F}{\partial x^a}-C^\sigma_{\alpha\beta}Q_{(\sigma)}\frac{\partial F}{\partial Q^{(\beta)}}=0\,, \tag{22}$$

where the $C^\sigma_{\alpha\beta}$ are the structure constants of the group generated by the $\xi^{(\alpha)}$. Let $\{e^{(\alpha)}_a\}$ be a tetrad adapted to the $\xi^{(\alpha)}$. Then $F(x,p)=F(x,Q^{(\alpha)})$ with the Liouville equation becoming

$$p^a\frac{\partial F}{\partial x^a}+\frac{\partial F}{\partial Q^{(\alpha)}}L(Q^{(\alpha)})=0\,. \tag{23}$$

However, by definition the $Q^{(\alpha)}$ are first integrals of the geodesic equations and hence solutions of the Liouville equation. Thus, since by assumption $C^\sigma_{\alpha\beta}=0$, Eqs. (21) become

$$\xi^a_{(\alpha)}\frac{\partial F}{\partial x^a}=p^a\frac{\partial F}{\partial x^a}=0\,, \tag{24}$$

and locally $\{\xi^a_{(\alpha)}\}$ or $\{\xi^a_{(\alpha)}, p^b\}$ will always form a tetrad. Therefore $\partial F/\partial x^a=0$.

If the matter distribution is to be confined to a surface layer Σ, the theorem remains valid so long as M contains two independent commuting

Killing vectors tangent to Σ.

At this point the general procedure would be to attempt to obtain the solution of the Einstein-Liouville equations. However, for the case of a counter-rotating disk, it is possible to obtain a distribution function without ever solving the Liouville equation.

Since the particle orbits are circular, any function of ξ will be constant along the phase orbits. Taking $\xi_a p^a = E$, it is now ensured that any function $F = F(\xi, E)\delta(p^\xi)$ is a solution of the Liouville equation. In particular, $F(\xi, E)$ may be taken to be separable,

$$F(\xi, E) = g(\xi) f(E) . \tag{25}$$

Use of this in combination with (11) and (19) leads to a distribution function for the set of counter-rotating disks:

$$F(\xi, E) = -\left[2e^{-2(\nu-\gamma)}\left(\frac{\partial\nu}{\partial\eta} - \frac{\partial\gamma}{\partial\eta}\right)\bigg|_{\eta=0}\right] \times \left(\int_{-\infty}^{\infty} f(E)\pi\right)^{-1} f(E) , \tag{26}$$

where $f(E)$ is an arbitrary C^1 function, subject only to the condition that $\int_{-\infty}^{\infty} f(E)\pi$ be finite and nonzero.

STABLE PARTICLE ORBITS: BINDING ENERGY

If J and E are, respectively, $\zeta_a p^a$ and $\xi_a p^a$, the perturbed geodesic equations for a counter-rotating disk are[9]

$$\begin{aligned} &2E^2 e^{2(\nu-3\gamma)} \frac{\partial^2(\delta\xi)}{\partial t^2} + E^2 e^{-2\gamma} \frac{\partial\gamma}{\partial\xi} \frac{\partial(\delta t)}{\partial t} + EJe^{-2\gamma}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial\xi} - \frac{\partial\gamma}{\partial\xi}\right)\frac{\partial(\delta\varphi)}{\partial t} = 0 , \\ &\frac{\partial^2(\delta\varphi)}{\partial t^2} - \frac{Je^{4\gamma}}{\rho^2 E}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial\xi} - \frac{\partial\gamma}{\partial\xi}\right)\frac{\partial(\delta\xi)}{\partial t} = 0 , \\ &\frac{\partial^2(\delta t)}{\partial t^2} - \frac{\partial\gamma}{\partial\xi}\frac{\partial(\delta\xi)}{\partial t} = 0 , \end{aligned} \tag{27}$$

where the replacement of d/ds by $\dot{t}\partial/\partial t$ is correct to first order. The second and third equations of (27) may be integrated once and the results substituted into the first equation to obtain

$$\begin{aligned} &\frac{\partial^2(\delta\xi)}{\partial t^2} + \omega^2(\xi)\delta\xi = 0 , \\ &\omega^2(\xi) = -\tfrac{1}{2}e^{-2(\nu-3\gamma)} \frac{\partial\gamma}{\partial\xi}\bigg|_{\eta=0} \tan\xi . \end{aligned} \tag{28}$$

From (28) it is immediately apparent that the necessary and sufficient condition for the stability of individual particle orbits is

$$\frac{\partial\gamma}{\partial\xi}\bigg|_{\eta=0} < 0 . \tag{29}$$

If the disk is to be stable in any sense, Eq. (29) must certainly be satisfied almost everywhere, but it is equally clear that this is not a sufficient criterion for stability.

Another simple indicator for the discussion of stability is the binding energy. In those cases which have been analyzed[10-12], the binding energy increases with density until some maximum value is reached, after which futher increases in density lead to a decrease in binding energy. In extreme cases, the binding energy may even become negative.

For a disk of mass M, rest mass M_0, the binding energy is $B = M_0 - M$. Let σ be the surface energy density and dA the element of surface area on the disk. Then

$$\begin{aligned} M &= -\int_{\pi/2}^{0} (S_t{}^t - S_\varphi{}^\varphi) dA , \\ M_0 &= 2\int_{\pi/2}^{0} \sigma p_a \xi^a e^{-\gamma} dA . \end{aligned} \tag{30}$$

The quantities p^α and σ are obtained by writing $S^{\alpha\beta}$ as the sum of two counter-rotating dust flows:

$$S^{\alpha\beta} = \sigma(p^\alpha p^\beta + \bar{p}^\alpha \bar{p}^\beta) , \tag{31}$$

with

$$p^\alpha = (0, p^\varphi, p^t)$$

and

$$\bar{p}^\alpha = (0, -p^\varphi, p^t) .$$

Using this,

$$\begin{aligned} \sigma &= \tfrac{1}{2}\mathrm{Tr}(S_{\alpha\beta}) , \\ p^\varphi &= (S^{\varphi\varphi}/2\sigma)^{1/2} , \\ p^t &= (S^{tt}/2\sigma)^{1/2} . \end{aligned} \tag{32}$$

The binding energy is now

$$B = 4\pi \int_0^{\pi/2} \rho e^{-\gamma}\left[1 - \left(1 + 2\frac{\partial\gamma}{\partial\xi}\cot\xi\right)^{1/2} \times \left(1 + \frac{\partial\gamma}{\partial\xi}\cot\xi\right)^{1/2}\right] \frac{\partial\gamma}{\partial\eta}\bigg|_{\eta=0} d\xi . \tag{33}$$

This must be considered in terms of the requirements

$$\frac{\partial\gamma}{\partial\eta}\bigg|_{\eta=0} > 0 , \quad \left(1 + 2\frac{\partial\gamma}{\partial\xi}\cot\xi\right)\bigg|_{\eta=0} > 0 , \tag{34}$$

imposed since $\partial\gamma/\partial\eta|_{\eta=0}$ is proportional to the surface density of matter and the surface energy density is

$$\sigma = \tfrac{1}{2}\left(1 + 2\frac{\partial\gamma}{\partial\xi}\cot\xi\right)\frac{\partial\gamma}{\partial\eta}\bigg|_{\eta=0} .$$

Equations (29) and (34) give the result that regions in which particle orbits are unstable make negative contributions to the binding energy while regions in which the orbits are stable make positive contributions. Hence the requirement $B>0$ is weaker than the stable-orbit requirement.

The strongest stability criterion involving binding energy is that B be a local maximum subject to the constraint that the rest mass remain constant. This is expressed symbolically by requiring

$$\frac{\partial M}{\partial a_n}\,\delta a_n = 0\,. \qquad (35)$$

If Eq. (33) is perturbed and the condition that the perturbation δB be maximal subject to $\delta M_0 = 0$ is imposed, it becomes necessary that

$$\left.\frac{\partial\gamma}{\partial\eta}\,\frac{\partial(\delta\gamma)}{\partial\xi}\right|_{\eta=0} < 0 \qquad (36)$$

(with $\delta\gamma$ the perturbation of γ). Since $\partial\gamma/\partial\eta$ must be positive, this means that particle orbits will tend towards stability under such perturbations. Since unstable orbits will become depopulated, this suggests the breakup of the disk into rings, a form of instability known to be endemic in Newtonian disks.

Further questions of stability as well as those of perturbations in the background metric will be discussed in detail in a subsequent paper.

STRONG FIELDS AND LARGE RED SHIFTS

In 1939 Einstein[10] showed that spherical shells of dust having arbitrarily large red shift cannot be constructed because particle orbital velocities approached the speed of light for a finite value of the red shift (on the order of 0.6). Recently, Morgan and Morgan[11] have demonstrated similar results for a subfamily of counter-rotating disks, raising a question as to whether any physically reasonable distribution of matter can give rise to gravitational red shifts greater than the order of unity.

Based on the results presented in earlier sections, it can be shown that the central red shift of any counter-rotating disk must be finite, although, due to computational difficulties, this limit will be given only for the simplest cases.

A slight digression is in order, however, since it is, in principle, incorrect to impose as an extra condition the requirement that particle velocities be less than light velocity. By the very nature of general relativity, this will always occur; and, hence, its inclusion should not be necessary. The orbital velocity of a particle in a counter-rotating disk is

$$v^2 = -S_\varphi{}^\varphi / S_t{}^t\,. \qquad (37)$$

This means that if the weak energy condition

$$\mathrm{Tr}(S_{\alpha\beta}) > 0 \qquad (38)$$

is imposed, then all particle velocities will automatically be less than that of light. As seen in a previous section, however, Eq. (38) can be replaced by the pair of equations (34). Thus, from (15) and (34) the requisite equations are

$$a_0 = 2a_1 - \sum_{n=2}^{N}(-1)^n\left(\frac{2^{3n}(2n-1)!!(n!)^3}{[(2n)!]^2}\right)a_n\,,$$
$$\sum_{n=0}^{N}\frac{2^{4n+1}(n!)^4}{[(2n)!]^2}a_n P_{2n}(\sin\xi) > 0\,, \qquad (39)$$
$$1 + 4\sum_{n=1}^{N} n a_n\left(P_{2n}(\sin\xi) - \frac{P_{2n-1}(\sin\xi)}{\sin\xi}\right) > 0\,.$$

The central red shift z_c is obtained from

$$\ln(z+1) = \sum_{n=0}^{N} a_n\,, \qquad (40)$$

and from Eq. (39) it is possible to show that the central red shifts of counter-rotating disks will be bounded for all finite values of N. This follows because from (40) the only way that z_c could become arbitrarily large would be if at least one of the a_i did so, and for $N \neq 0$ this cannot occur if all of (39) are to be satisfied. If the first equation of (39) is not imposed, however (i.e., if a singularity is allowed at the edge of the disk), then arbitrary central red shifts may be obtained.[11]

Computer calculations have been made for the maximum central red shift for the cases $N \leq 5$. The greatest red shift is obtained for $N=5$ and is $z_c(\max) = 1.56 \pm 0.05$. This is of interest only in that it is slightly greater than the maximum possible central red shift obtained by Morgan and Morgan for a subset of the set of counter-rotating disks. Unfortunately, computational difficulties have not yet allowed an analysis as extensive as that of Morgan and Morgan and the question as to whether the series $z_c(N)$ of central red shifts converges or not remains unanswered.

THE CASE $N=1$

As an example, those disks obtained by choosing $N=1$ will be briefly studied. For $N=1$, γ and ν are explicitly given by

$$\gamma = -\delta\{\cot^{-1}(\sinh\eta) + \tfrac{1}{4}[(1+3\sinh^2\eta)\cot^{-1}(\sinh\eta) - 3\sinh\eta](3\sin^2\xi - 1)\}\,,$$

$$\nu = \ln R_0 + \tfrac{1}{2}\ln(\cosh^2\eta - \cos^2\xi) + \frac{9\delta^2\cos^2\xi}{4}\left[\frac{1}{4}\left(\frac{\sinh\eta}{1+\sinh\eta} - \cot^{-1}(\sinh\eta)\right)^2\cos^2\xi - (1+\sin^2\xi)[\sinh\eta\cot^{-1}(\sinh\eta) - 1]^2\right.$$

$$-\sinh\eta\cos^2\xi[\sinh\eta\cot^{-1}(\sinh\eta)-1]\left(\frac{\sinh\eta}{1+\sinh\eta}-\cot^{-1}(\sinh\eta)\right)\Bigg] \quad (41)$$

and

$$\left.\frac{\partial\gamma}{\partial\eta}\right|_{\eta=0}=3\delta\sin^2\xi\,,$$
$$\left.\frac{\partial\gamma}{\partial\xi}\right|_{\eta=0}=-\tfrac{3}{4}\pi\delta\sin\xi\cos\xi\,. \quad (42)$$

The nonzero components of the stress-energy tensor are

$$S_{\varphi\varphi}=\tfrac{9}{4}\pi R_0{}^2\delta^2e^{-(\nu+\gamma)}|_{\eta=0}\sin^2\xi\cos^4\xi\,,$$
$$S_{tt}=3\delta e^{-(\nu-3\gamma)}|_{\eta=0}(2-\tfrac{3}{4}\pi\delta\cos^2\xi)\sin^2\xi\,, \quad (43)$$

which agree in the Newtonian limit with the model obtained by Hunter using a Newtonian potential γ.

From (42) all particle orbits are stable, and if the second equation of (28) is expanded in powers of δ, then

$$\omega^2(\xi)\simeq\frac{3\pi\delta}{4R_0{}^2}[1-2\delta(1+\sin^2\xi)+\cdots]\,. \quad (44)$$

Comparison with the Newtonian limit indicates that $v=\pm(\frac{3}{4}\pi\delta)^{1/2}$ is the particle orbital velocity. The second-order term of (44) contains the special relativistic corrections and corrections which may be attributed to the non-Euclidean nature of the space-time geometry (i.e., in the general relativistic case R_0 is the coordinant radius of the disk – not the physical radius).

DISCUSSION

A large class of background models for static disks have been given. Although only the counter-rotating models have been studied in detail, the only obstacle to the study of other types of disks is the computational difficulties which arise in attempts to solve Eq. (7).

The primary question remaining for counter-rotating models is that of the central red shift. Other problems of a more general nature which remain to be resolved involve slowly rotating disks and nonstationary perturbations. In particular, those perturbations involving absorption and emission of gravitational radiation will be of special interest.

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[1]W. Israel, Nuovo Cimento 44B, 1 (1966).

[2]A. Papapetrou and A. Hemoui, Ann. Inst. Henri Poincaré 9, 179 (1968).

[3]J. Ehlers (private communication).

[4]Throughout, Latin indices range from 1 to 4 while Greek indices denote coordinates within Σ.

[5]T. Morgan and L. Morgan, Phys. Rev. D 2, 2756 (1970).

[6]J. Ehlers, in *General Relativity and Cosmology*, edited by R. K. Sachs (Academic, New York, to be published).

[7]J. Ehlers and R. K. Sachs, in *Astrophysics and Relativity*, edited by M. Chrétien, S. Deser, and J. Goldstein (Gordon and Breach, New York, to be published).

[8]K. Bichteler, Commun. Math. Phys. 4, 352 (1967).

[9]These equations assume that the perturbation does not affect the background metric.

[10]A. Einstein, Ann. Math. 40, 924 (1939).

[11]T. Morgan and L. Morgan, Phys. Rev. 183, 1097 (1969).

[12]C. Hunter, Monthly Notices Roy. Astron. Soc. 126, 23 (1963).