[1]R. Carlitz, S. Ellis, P. G. O. Freund, and Satoshi Matsuda, Caltech Report No. CALT-68-260, 1970 (unpublished).

[2]R. P. Feynman, M. Kislinger , and F. Ravndal, Phys. Rev. D 3, 2706 (1971).

[3]R. Delbourgo and P. Rotelli, Phys. Letters 35B, 65 (1971).

[4]For more details of this dual model study see S. D. Ellis, Caltech Ph.D. thesis, 1971 (unpublished).

[5]S. Mandelstam, Phys. Rev. 183, 1374 (1969); K. Bardakci and M. B. Halpern, *ibid.* 183, 1456 (1969).

[6]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[7]This explains why the present model gives different results from that of R. Carlitz and M. Kislinger, Phys. Rev. D 2, 336 (1970).

[8]E. L. Berger and G. C. Fox, Phys. Rev. 188, 2120 (1969).

[9]R. Carlitz and M. Kislinger, Phys. Rev. Letters 24, 186 (1970), and Phys. Rev. D 2, 336 (1970).

[10]The neutralizer function is an essential feature of the model of Ref. 1. It is required in order to prevent the $\not{k}_t/m$ term in the quark propagator from contributing to poles in any other channel except the t channel. If allowed to contribute, this term would not only break the $U(6)_W$ symmetry but also lead to new poles displaced by $\frac{1}{2}$ in the complex angular momentum plane.

[11]See, for example, E. C. Titchmarsh, *Theory of Functions*, 2nd ed. (Oxford Univ. Press, London, 1939), p. 177.

[12]In the language of Ref. 1 this corresponds to

$$\Phi(z)=\int_0^z \frac{dx}{[\ln(1/x)]^2}\,\frac{\exp\{-k^2/4[\ln(1/x)]^2\}}{x}.$$

This function lends itself more easily to numerical calculations than the example given in Ref. 1.

[13]A recent review of data and current theories is given by E. Berger and G. Fox, Nucl. Phys. B26, 1 (1971).

[14]S. Mandelstam, Phys. Rev. D 1, 1720 (1970); 1, 1734 (1970).

[15]G. Frye, C. W. Lee, and L. Susskind, Nuovo Cimento 69A, 497 (1970).

[16]Included in the constants is the fact that there are two independent ways to construct the s, u diagram both of which give the same contribution.

[17]A program of study similar to the one discussed here, but utilizing other dual models with spin, is currently under way. For a review of the current situation in dual models with spin and further references, see M. A. Virasoro, talk at the International Conference on Duality and Symmetry in Hadron Physics, Tel-Aviv, Israel, 1971 (unpublished).

[18]A. Salam, R. Delbourgo, and I. Strathdee, Proc. Roy. Soc. (London) A284, 146 (1965); B. Sakita and K. C. Wali, Phys. Rev. 139, B1355 (1965).

[19]R. Delbourgo and M. A. Rashid, Proc. Roy. Soc. (London) A286, 412 (1965).

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Low-Energy Theorem for $\gamma \to 3\pi$

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A low-energy theorem relating $\gamma \to 3\pi$ to $\pi^0 \to 2\gamma$ is derived by using anomalous Ward identities and by using Schwinger's proper-time technique. We discuss the theoretical significance of this low-energy theorem. In particular, the theorem is meaningful only if the partially conserved axial-vector current anomaly is in fact responsible for the decay $\pi^0 \to 2\gamma$.

I. INTRODUCTION

Recently Adler *et al.*[1] discovered a low-energy theorem relating $\gamma \to 3\pi$ to $\pi \to 2\gamma$. The theorem states that

$$eF^{3\pi} = F^{\pi} f_\pi^{-2}, \qquad (1.1)$$

where $F^{3\pi}$ and F^{π} are "coupling constants," to be defined in Sec. II, describing $\gamma \to 3\pi$ and $\pi \to 2\gamma$, respectively. The theorem rests upon the following assumptions:

(a) gauge invariance;

(b) Gell-Mann's current algebra and the hypothesis of the partial conservation of the axial-vector current (PCAC); and

(c) that the electromagnetic *current* commutes with the neutral axial *charge* at equal times.

We should emphasize that Eq. (1.1) is independent of the nature of chiral-symmetry breaking. It was also shown[1] that the amplitudes for $\gamma+\gamma \to 3\pi^0$ and for $\gamma+\gamma \to \pi^+\pi^-\pi^0$ may be computed up to second order in momenta in terms of one parameter, using only the assumptions (a), (b), and (c) enumerated above. The value of this parameter will give us valuable information on the nature of chiral-symmetry breaking. In particular, its measurement will shed light on the question of whether the chiral-symmetry-breaking interaction belongs to the representation $(\frac{1}{2}, \frac{1}{2})$ of SU(2)⊗ SU(2). These points will be reviewed in detail in Sec. II.

As is well known, a naive application of (a), (b), and (c) leads to the erroneous conclusion[2] that $\pi \to 2\gamma$ is suppressed to order m_π. If this is indeed the case, Eq. (1.1) would not make sense since the PCAC error terms omitted in deriving (1.1) would be of the same order in m_π as the right-hand side. Thus the verification of Eq. (1.1) may be construed as evidence that $\pi^0 \to 2\gamma$ is not suppressed. In this sense, Eq. (1.1) opens up the possibility of checking whether $\pi \to 2\gamma$ proceeds through the anomaly[3-5] mechanism, and to the extent the theory of anomalies correctly describes $\pi^0 \to 2\gamma$ *to any*[6] *finite order* in renormalized perturbation theory Eq. (1.1) affords a unique opportunity to confront renormalized perturbation theory with data.

In Sec. III we investigate the set of Ward identities relating $\langle AAAV\rangle$ to $\langle \partial A \partial A \partial A V\rangle$ in a free massive fermion field theory. It is shown that some of these Ward identities are anomalous. An interesting, albeit technical, point arises in relating $\gamma \to 3\pi$ to $\pi \to 2\gamma$ through Ward identities. The Ward identities involved are in fact *not* anomalous. However, the "surface" term, which is normally dropped in standard applications of current algebra, cannot be dropped here, the reason being that the surface term itself appears in an anomalous Ward identity. Our results in this section are entirely consistent with the general expression for the divergence of the axial-vector current given by Bardeen.[7,8]

In Sec. IV we generalize the powerful proper-time technique developed by Schwinger[9] to a chiral SU(3)⊗ SU(3) theory. This technique enables us to write down a formal expression containing the most general coupling of any number of photons to any number of mesons through a fermion loop. Using this general expression we treat the case of a single photon coupling to any number of mesons and the case of a single meson coupling to any number of photons.

In Sec. V we discuss the experimental prospects of measuring the amplitude for $\gamma \to 3\pi$.

The Appendix contains a phase-space calculation.

II. CURRENT ALGEBRA, $\pi \to 2\gamma$, $\gamma \to 3\pi$, AND $2\gamma \to 3\pi$

For the sake of completeness, we shall review in detail the argument of Adler *et al.*[1] that the twin requirement of current algebra and gauge invariance suffice to determine $\gamma \to \pi^+\pi^-\pi^0$ in terms of $\pi^0 \to 2\gamma$.

Let us write the general amplitudes (not necessarily on-shell), for $\pi^0 \to 2\gamma$ and $\gamma \to \pi^+\pi^-\pi^0$ as

$$M(\gamma(k_1)+\gamma(k_2) \to \pi^0) = i|\epsilon_1\epsilon_2 k_1 k_2|F^\pi(k_1^2, k_2^2, (k_1+k_2)^2), \tag{2.1}$$

$$M(\gamma(k) \to \pi^0(q_0)+\pi^+(q_+)+\pi^-(q_-)) = (-i)|\epsilon q_+ q_- q_0|F^{3\pi}(q_+q_0, q_+q_-; q_+^2, q_-^2, q_0^2, (q_++q_-+q_0)^2), \tag{2.2}$$

where we have introduced the notation $|abcd| \equiv \epsilon_{\mu\nu\sigma\lambda}a^\mu b^\nu c^\sigma d^\lambda$. Our result [Eq. (1.1)] relates $F^\pi \equiv F^\pi(0,0,0)$ to $F^{3\pi} \equiv F^{3\pi}(0,0;0,0,0,0)$. That the approximation of the experimental quantities $F^\pi(0,0,m_\pi^2)$ and $F^{3\pi}(q_+q_0, q_+q_-; m_\pi^2, m_\pi^2, m_\pi^2, (q_++q_-+q_0)^2)$ by F^π and $F^{3\pi}$ [for q_+q_0, q_+q_-, $(q_++q_-+q_0)^2$ small] is a relatively accurate one is the content of the standard PCAC assumption. The observed rate for $\pi^0 \to 2\gamma$ gives[5] $|F^\pi| = (\alpha/\pi)[(0.66\pm 0.08)m_\pi]^{-1}$.

To relate $F^{3\pi}$ to F^π Adler *et al.*[1] consider the processes $\gamma+\gamma \to 3\pi^0$ and $\gamma+\gamma \to \pi^+\pi^-\pi^0$. To begin with, note that the amplitudes

$$A(\gamma(k_1)+\gamma(k_2) \to \pi^0(q_0)+\pi^0(q_0')+\pi^0(q_0'')) \equiv A^{000}$$

and

$$A(\gamma(k_1)+\gamma(k_2) \to \pi^0(q_0)+\pi^+(q_+)+\pi^-(q_-)) \equiv A^{+-0}$$

must vanish when one lets $q_0 \to 0$, keeping the other two pions on-mass-shell. The reason[10] is that at equal times the electromagnetic current commutes with the neutral axial charge. The next observation is that the part of A^{000} and A^{+-0} coming from diagrams with a pion pole occurring therein may be computed exactly in terms of F^π and $F^{3\pi}$. There are two classes of such diagrams: those involving external photon insertions and those involving pion-pion scattering (see Fig. 1). This observation is a powerful one in that these pion-pole diagrams by themselves neither maintain gauge invariance nor vanish in the soft-neutral-pion limit described above, thus calling for chiral and gauge completions. By imposing the

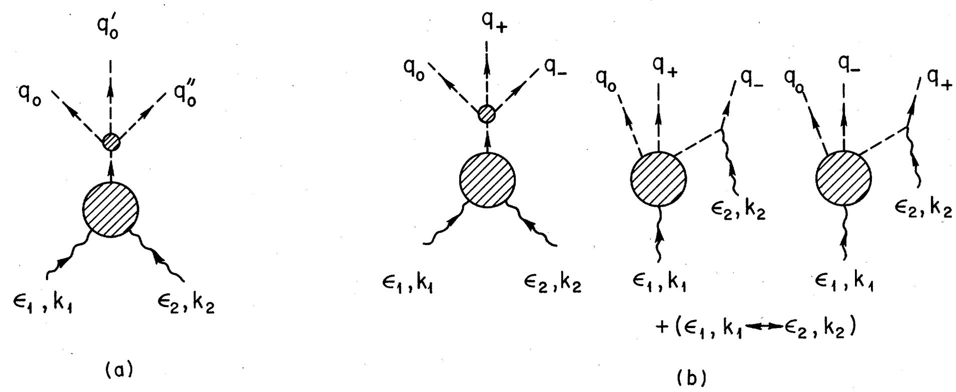

FIG. 1. (a) Pion pole diagram for $2\gamma\rightarrow 3\pi^0$. (b) Pion pole diagrams for $2\gamma\rightarrow\pi^+\pi^-\pi^0$.

twin requirements of gauge invariance and of chirality the amplitudes A^{000} and A^{+-0} may be determined up to and including terms second order in momenta, as we shall see presently.

To compute the pion-pole diagrams of Fig. 1 we need the amplitude for pion-pion scattering. Following Weinberg[11] we write the general off-shell amplitude up to second order in momenta as

$$M(\pi^a+\pi^b\rightarrow\pi^c+\pi^d)=i\delta^{ab}\delta^{cd}[A+B(t+u)+Cs]+i\delta^{ac}\delta^{bd}[A+B(u+s)+Ct]+i\delta^{ad}\delta^{bc}[A+B(s+t)+Cu], \tag{2.3}$$

where $s=(q_a+q_b)^2$, $t=(q_a-q_c)^2$, $u=(q_a-q_d)^2$. Note that by Bose symmetry the variables p_a^2, p_b^2, p_c^2, and p_d^2 can only occur in the combination $p_a^2+p_b^2+p_c^2+p_d^2$, and hence the above amplitude also holds off-shell. Imposing the requirements of current algebra and the Adler consistency[12] condition we obtain

$$M(\pi^a+\pi^b\rightarrow\pi^c+\pi^d)=i\delta^{ab}\delta^{cd}\left(\frac{1}{f_\pi^2}(s-m_\pi^2)+B(s+t+u-3m_\pi^2)\right)+\text{permutations}. \tag{2.4}$$

We recover Weinberg's original amplitude if we introduce the additional assumption that the σ term is an isoscalar and which fixes $B=0$. We are now able to write down the amplitudes A^{000} and A^{+-0} (the pions are not necessarily on shell) to second order in momenta:

$$A^{000}=-iF^\pi f_\pi^{-2}|\epsilon_1\epsilon_2k_1k_2|\left(\frac{(1+3f^2B)[(q_0+q_0')^2+(q_0+q_0'')^2+(q_0'+q_0'')^2-3m_\pi^2]}{(q_0+q_0'+q_0'')^2-m_\pi^2}+C_1\right), \tag{2.5}$$

$$\begin{aligned}A^{+-0}=&-iF^\pi f_\pi^{-2}|\epsilon_1\epsilon_2k_1k_2|\left(\frac{(q_++q_-)^2-m_\pi^2+f_\pi^2B[(q_++q_-)^2+(q_++q_0)^2+(q_0+q_-)^2-3m_\pi^2]}{(q_++q_-+q_0)^2-m_\pi^2}+C_2\right)\\&+ieF^{3\pi}\left(|\epsilon_1(q_+-k_2)q_-q_0|\frac{\epsilon_2(2q_+-k_2)}{(q_+-k_2)^2-m_\pi^2}-|\epsilon_1q_+(q_--k_2)q_0|\frac{\epsilon_2(2q_--k_2)}{(q_--k_2)^2-m_\pi^2}+(\epsilon_1\leftrightarrow\epsilon_2,k_1\leftrightarrow k_2)+h\right).\end{aligned} \tag{2.6}$$

Here h is a function of ϵ_1, ϵ_2, and the available momenta. C_1 and C_2 are numerical constants. The requirement that $A^{000}\rightarrow 0$ as $q_0\rightarrow 0$ with $(q_0')^2=(q_0'')^2=m_\pi^2$ fixes C_1 to be $-(1+3f_\pi^2B)$. Similarly, $C_2=-(1+f_\pi^2B)$. Gauge invariance asserts that under the variation $\epsilon_2\rightarrow\epsilon_2+\lambda k_2, h\rightarrow h+|\epsilon_1k_2q_0(q_++q_-)|$. The unique solution respecting Bose symmetry is $h=|\epsilon_1\epsilon_2q_0(k_1-k_2)|$.[13] A^{000} and A^{+-0} are now determined up to second order in momentum in terms of F^π and $F^{3\pi}$. These amplitudes for $2\gamma\rightarrow 3\pi$ have been presented and discussed by Adler *et al.*[1]

In order to fix $F^{3\pi}$ in terms of F^π we write down the most general expansion for the amplitude

$$A^{abc}(\gamma(k_1)+\gamma(k_2)\rightarrow\pi^a+\pi^b+\pi^c)$$

up to second order in momenta and respecting Bose symmetry and G parity:

$$A^{abc}=(\delta^{c3}\delta^{ab}+\delta^{a3}\delta^{bc}+\delta^{b3}\delta^{ac})\alpha_1|\epsilon_1\epsilon_2k_1k_2|+\alpha_2|\epsilon_1,\epsilon_2,k_1-k_2,\delta^{c3}\delta^{ab}(q_a+q_b)+\delta^{a3}\delta^{bc}(q_b+q_c)+\delta^{b3}\delta^{ac}(q_c+q_a)|. \tag{2.7}$$

Here α_1 and α_2 are arbitrary constants. (Note that gauge invariance may not be imposed.) Demanding that Eqs. (2.5) and (2.6) be consistent with Eq. (2.7) we obtain immediately the relation

$$eF^{3\pi}=F^\pi f_\pi^{-2} \tag{2.8}$$

which is the desired result. A most remarkable feature of the relation in Eq. (2.8) is its independence of the parameter B and hence of any assumption about the isospin of the σ term. In contrast, the measurement of $2\gamma \to 3\pi$ would yield a value for B. The popular $(\frac{1}{2}, \frac{1}{2})$ model of chiral-symmetry breaking predicts that $B = 0$.

As mentioned in the Introduction, Eq. (2.8) depends on three assumptions referred to as (a), (b), and (c). Assumption (c) represents the local generalization of the self-evident statement that the electromagnetic charge commutes with the neutral axial charge. It appears to be not inconsistent with existing photoproduction data.[14] On the other hand, it leads to two[15] unhappy predictions: (1) that[16] the decay $\eta \to 3\pi$ is forbidden, and (2) that[17] $m_{K^+}{}^2 - m_{K^0}{}^2 = m_{\pi^+}{}^2 - m_{\pi^0}{}^2$. An independent confirmation or negation of assumption (c) would thus be of considerable interest.

III. WARD IDENTITIES

In this section we employ the traditional techniques of Ward identities to discuss the process $\gamma \to 3\pi$. Before we proceed we define a collection of time-ordered products of currents.

$$\epsilon^{abc}\overline{Q}T_{\mu\nu\sigma\lambda}(k,p,q) = i\int d^4x d^4y d^4z\, e^{i(kx+py+qz)}\langle 0|TA^a_\mu(x)A^b_\nu(y)A^c_\sigma(z)V_\lambda(0)|0\rangle, \tag{3.1}$$

$$\epsilon^{abc}\overline{Q}T_{\lambda}(k,p,q) = i\int d^4x d^4y d^4z\, e^{i(kx+py+qz)}\langle 0|T\partial A^a(x)\partial A^b(y)\partial A^c(z)V_\lambda(0)|0\rangle, \tag{3.2}$$

$$\epsilon^{abc}\overline{Q}S_{\mu\lambda}(k,p,q) = i\int d^4x d^4y d^4z\, e^{i(kx+py+qz)}\langle 0|TA^a_\mu(x)\partial A^b(y)\partial A^c(z)V_\lambda(0)|0\rangle, \tag{3.3}$$

$$\epsilon^{abc}\overline{Q}P_{\mu\nu\lambda}(k,p,q) = i\int d^4x d^4y d^4z\, e^{i(kx+py+qz)}\langle 0|TA^a_\mu(x)A^b_\nu(y)\partial A^c(z)V_\lambda(0)|0\rangle, \tag{3.4}$$

$$\delta^{ab}\overline{Q}T_{\mu\nu\lambda}(k,p) = i\int d^4x d^4y\, e^{i(kx+py)}\langle 0|TV^a_\mu(x)A^b_\nu(y)V_\lambda(0)|0\rangle, \tag{3.5}$$

$$\delta^{ab}\overline{Q}R_{\mu\lambda}(k,p) = i\int d^4x d^4y\, e^{i(kx+py)}\langle 0|TV^a_\mu(x)\partial A^b(y)V_\lambda(0)|0\rangle. \tag{3.6}$$

We work in the free massive fermion field theory and define the axial-vector current as $A^a_\mu(x) \equiv \overline{\psi}\frac{1}{2}\tau^a\gamma_\mu\gamma_5\psi$ and the electromagnetic current as $V_\mu(x) \equiv \overline{\psi}\lambda_Q\gamma_\mu\psi$, where $\tau_a = \lambda_a$ for $a = 1, 2, 3$,

$$\lambda_Q = \begin{pmatrix} Q & & \\ & Q-1 & \\ & & Q-1 \end{pmatrix},$$

and $\overline{Q} \equiv Q - \frac{1}{2}$. We have in mind a theory with a triplet of fermion fields having the same isospin and hypercharge assignment as the SU(3) quark model. In order to discuss $\pi^0 \to 2\gamma$ and $\gamma \to 3\pi$ we ought to work in a theory with pions and incorporating chirality, such as the σ model of Gell-Mann and Lévy.[18] However, it would become apparent that the Feynman amplitudes describing $\pi^0 \to 2\gamma$ and $\gamma \to 3\pi$ in the σ model are intimately related to the time-ordered products $R_{\mu\lambda}$ and T_λ as evaluated in the free fermion field theory. On the other hand, we were instructed by Ref. 1 that it does not suffice to consider the free massive fermion field theory in a discussion of $2\gamma \to 3\pi$.

Standard techniques and Gell-Mann's current algebra lead to the following *naive* Ward identities:

$$q^\sigma T_{\mu\nu\sigma\lambda}(k,p,q) = iP_{\mu\nu\lambda}(k,p,q) + T_{\nu\mu\lambda}(p+q,k) - T_{\mu\nu\lambda}(k+q,p), \tag{3.7}$$

$$p^\nu P_{\mu\nu\lambda}(k,p,q) = S_{\mu\lambda}(k,p,q) + R_{\mu\lambda}(k+p,q), \tag{3.8}$$

$$k^\mu S_{\mu\lambda}(k,p,q) = -T_\lambda(k,p,q), \tag{3.9}$$

$$p^\nu T_{\mu\nu\lambda}(k,p) = iR_{\mu\lambda}(k,p). \tag{3.10}$$

Note that the σ term does not play a role here. The Ward identity in Eq. (3.10) is responsible for the famous false statement[2] that $\pi^0 \to 2\gamma$ is forbidden by PCAC and current algebra. Let us briefly review the reasoning which led to the preceding statement. Define

$$R_{\mu\lambda}(k,p) = i\epsilon_{\mu\nu\sigma\lambda}p^\nu k^\sigma f(k^2, p^2, (k+p)^2).$$

The invariant amplitude describing $\pi^0 \to 2\gamma$ is proportional to $f(0, m_\pi{}^2, 0)$. PCAC assumes that $f(0, m_\pi{}^2, 0) \simeq f(0,0,0)$. We now determine $f(0,0,0)$ as follows. In the limit $k, p \to 0$ crossing symmetry (and parity) imply that $T_{\mu\nu\lambda}(k,p) \to A\epsilon_{\mu\nu\sigma\lambda}(2k+p)^\sigma$

for some constant A. Gauge invariance $[k^\mu T_{\mu\nu\lambda}(k,p)=0]$ demands that A vanishes. Equation (3.10) now states that $f(0,0,0)=2A=0$. This conclusion is in fact false as is shown by an elementary calculation of $f(0,0,0)$ in the free massive fermion field theory.

Similarly, we now show that the Ward identities (3.7), (3.8), (3.9), and (3.10) imply that $\gamma\to 3\pi$ is forbidden. We put the Ward identities (3.7), (3.8), (3.9), and (3.10) together to obtain

$$k^\mu p^\nu q^\sigma T_{\mu\nu\sigma\lambda}(k,p,q) = -iT_\lambda(k,p,q)+ik^\mu R_{\mu\lambda}(k+p,q) + ip^\nu R_{\nu\lambda}(p+q,k)-ik^\mu R_{\mu\lambda}(k+q,p)\,. \tag{3.11}$$

To lowest order in momentum we must have $T_{\mu\nu\lambda\sigma}(k,p,q)\to B\epsilon_{\mu\nu\sigma\lambda}$ as $k,p,q\to 0$ with B some constant. Hence Eq. (3.11) implies that $-iT_\lambda(k,p,q)\to B\epsilon_{\mu\nu\sigma\lambda}k^\mu p^\nu q^\sigma$. But gauge invariance requires that

$$(k+p+q)^\lambda T_{\mu\nu\sigma\lambda}(k,p,q)=0\,,$$

so that $B=0$. The amplitude for a virtual photon to produce three pions,

$$F(m_\pi{}^2, m_\pi{}^2, m_\pi{}^2, kp, kq, pq)\,,$$

is defined by

$$T_\lambda(k,p,q)\equiv F(k^2,p^2,q^2,kp,kq,pq)\epsilon_{\mu\nu\sigma\lambda}k^\mu p^\nu q^\sigma\,.$$

The low-energy theorem just derived states that $F(0,0,0,0,0,0)=0$.

This conclusion is also false as is shown by an elementary calculation of T_λ in the free massive fermion field theory. One finds that

$$T_\lambda(k,p,q)\to \frac{-i}{2\pi^2}\epsilon_{\mu\nu\sigma\lambda}k^\mu p^\nu q^\sigma$$

as $k,p,q\to 0$. We now wish to go back to the Ward identities in Eqs. (3.7), (3.8), (3.9), and (3.10) and to study their individual validity. We shall use techniques that have been discussed extensively[2] by Adler and others.

In the free massive fermion field theory quantities such as $T_{\mu\nu\sigma\lambda}$ are given by loop diagrams (Fig. 2). Thus we may write down the Feynman expression

$$T^F_{\mu\nu\sigma\lambda}=\tfrac{1}{4}\int\frac{d^4r}{(2\pi)^4}\,\mathrm{tr}S(r+q)\gamma_\mu\gamma_5 S(r+k+q)\gamma_\lambda \times S(r-p)\gamma_\nu\gamma_5 S(r)\gamma_\sigma\gamma_5 + (\text{five other terms by permutations}), \tag{3.12a}$$

which is reputedly a valid representation of $T_{\mu\nu\sigma\lambda}$.

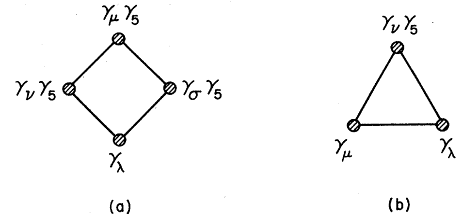

FIG. 2. (a) The diagram which together with its permutations describes $T_{\mu\nu\sigma\lambda}$ in a free-fermion theory. (b) The diagram which together with its permutations describes $T_{\mu\nu\lambda}$ in a free-fermion theory.

Note that $T^F_{\mu\nu\sigma\lambda}$ is superficially only logarithmically divergent and so there can be *no* question on how the loop momentum is to be labeled. In fact, $T^F_{\mu\nu\sigma\lambda}$ is finite. To establish this fact we introduce a regulator field of mass Λ and write the logarithmically divergent part of $T^F_{\mu\nu\sigma\lambda}$ as

$$\tfrac{1}{4}\int\frac{d^4r}{(2\pi)^4}\frac{1}{(r^2-m^2)^4}\,\mathrm{tr}\,\not r\gamma_\mu\gamma_5\not r\gamma_\lambda\not r\gamma_\nu\not r\gamma_\sigma-(m\to\Lambda) \tag{3.12b}$$

$$=-\int_{m^2}^{\Lambda^2}dm^2\int\frac{d^4r}{(2\pi)^4}\frac{1}{(r^2-m^2)^5} \times\mathrm{tr}\gamma_5\not r\gamma_\lambda\not r\gamma_\nu\not r\gamma_\sigma\not r\gamma_\mu \tag{3.12c}$$

$$=0\,. \tag{3.12d}$$

The last equality holds because of the identities

$$\int d^4r(r^2-m^2)^{-5}r_\mu r_\nu r_\lambda r_\sigma \propto g_{\mu\nu}g_{\lambda\sigma}+g_{\mu\lambda}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\lambda}\,,$$

$$\gamma^\mu\not a\gamma_\mu=-2\not a\,,$$

$$\gamma^\mu\not a\not b\gamma_\mu=4a\cdot b\,,$$

and

$$\gamma^\mu\not a\not b\not c\gamma_\mu=-2\not c\not b\not a\,.$$

Hence $T^F_{\mu\nu\sigma\lambda}$ is finite and we find that

$$T^F_{\mu\nu\sigma\lambda}\to\frac{1}{16\pi^2}\epsilon_{\mu\nu\sigma\lambda}$$

as $k,p,q\to 0$, which means that $T^F_{\mu\nu\sigma\lambda}$ is not gauge-invariant. A subtraction is necessary, and a good representation of $T_{\mu\nu\sigma\lambda}$ is provided by

$$T^{\mathrm{CSGI}}_{\mu\nu\sigma\lambda}(k,p,q)\equiv T^F_{\mu\nu\sigma\lambda}-\frac{1}{16\pi^2}\epsilon_{\mu\nu\sigma\lambda}\,. \tag{3.13}$$

(The letters CSGI denote a crossing-symmetric and gauge-invariant amplitude.)

[Alternatively, one may compute $(k+p+q)^\lambda T^F_{\mu\nu\sigma\lambda}$, which is given by pairs of linearly divergent Feynman integrals which would cancel pairwise if one is allowed to shift the integration variable. In fact these shifts are forbidden and one finds

$(k+p+q)^{\lambda}T^F_{\mu\nu\sigma\lambda}(k,p,q)=(1/16\pi^2)\epsilon_{\mu\nu\sigma\lambda}(k+p+q)^{\lambda}$.]

Similarly, one finds that

$$T^{\rm CSGI}_{\mu\nu\lambda}(k,p)\equiv T^F_{\mu\nu\lambda}(k,p)+\frac{1}{16\pi^2}\epsilon_{\mu\nu\sigma\lambda}(p^{\sigma}+2k^{\sigma}) \tag{3.14}$$

(but not

$$T^F_{\mu\nu\lambda}(k,p)=-\tfrac{1}{2}\int\frac{d^4r}{(2\pi)^4}[{\rm tr}S(r+k)\gamma_\lambda S(r-p)\gamma_\nu\gamma_5 S(r)\gamma_\mu + {\rm tr}S(r-k-p)\gamma_\mu S(r-p)\gamma_\nu\gamma_5 S(r)\gamma_\lambda]), \tag{3.15}$$

provides a good representation of $T_{\mu\nu\lambda}$. On the other hand, one verifies that $P^F_{\mu\nu\lambda}$, $S^F_{\mu\lambda}$, $R^F_{\mu\lambda}$, and T^F_λ are good representations of $P_{\mu\nu\lambda}$, $S_{\mu\lambda}$, $R_{\mu\lambda}$, and T_λ, respectively, which confirms one of Bardeen's conclusions.[7] Putting these facts together, we see that the Ward identities of Eqs. (3.8) and (3.9) are not modified while the Ward identity of Eq. (3.10) is modified in the well-known manner. We now discuss how the Ward identity of Eq. (3.7) must be modified.

Let us next consider $q^{\sigma}T^F_{\mu\nu\sigma\lambda}$ which consists of 18 terms. Out of these there are four that may be identified as $\langle 0|T(A_\mu A_\nu A_\lambda)|0\rangle$ even though they should not be there since the isoscalar part of the electromagnetic current is supposed to commute with the time component of the axial-vector current A^c_μ at equal time. They sum up to give a contribution of $(-1/32\pi^2)\epsilon_{\mu\nu\sigma\lambda}(k+p)^{\sigma}$ to $q^{\sigma}T^F_{\mu\nu\sigma\lambda}$. A second source of anomaly arises because those terms that may be identified as $\langle 0|T(V_\mu A_\nu V_\lambda)|0\rangle$ and $\langle 0|T(V_\nu A_\mu V_\lambda)|0\rangle$ differ from the crossing-symmetric amplitude $T^F_{\mu\nu\lambda}$ and $T^F_{\nu\mu\lambda}$ as defined in Eq. (3.15) by a shift in the loop momentum. Including these two sources of anomalous terms we may write

$$-q^{\sigma}T^F_{\mu\nu\sigma\lambda}(k,p,q)=-iP^F_{\mu\nu\lambda}(k,p,q)+T^F_{\mu\nu\lambda}(k+q,p) - T^F_{\nu\mu\lambda}(p+q,k)+\frac{1}{32\pi^2}\epsilon_{\mu\nu\sigma\lambda}(k+p)^{\sigma} - \frac{1}{32\pi^2}\epsilon_{\mu\nu\sigma\lambda}(3p+3k+2q)^{\sigma}. \tag{3.16}$$

For completeness we collect together the set of correct Ward identities:

$$q^{\sigma}T^{\rm CSGI}_{\mu\nu\sigma\lambda}(k,p,q)=iP^{\rm CSGI}_{\mu\nu\lambda}(k,p,q)+T^{\rm CSGI}_{\nu\mu\lambda}(p+q,k) - T^{\rm CSGI}_{\mu\nu\lambda}(k+q,p)+\frac{1}{4\pi^2}\epsilon_{\mu\nu\sigma\lambda}(k+p+q)^{\sigma}, \tag{3.7$'$}$$

$$p^{\nu}P^{\rm CSGI}_{\mu\nu\lambda}(k,p,q)=S^{\rm CSGI}_{\mu\lambda}(k,p,q)+R^{\rm CSGI}_{\mu\lambda}(k+p,q), \tag{3.8$'$}$$

$$k^{\mu}S^{\rm CSGI}_{\mu\lambda}(k,p,q)=-T^{\rm CSGI}_{\lambda}(k,p,q), \tag{3.9$'$}$$

$$p^{\nu}T^{\rm CSGI}_{\mu\nu\lambda}(k,p)=iR^{\rm CSGI}_{\mu\lambda}(k,p)-\frac{1}{4\pi^2}\epsilon_{\mu\nu\sigma\lambda}k^{\nu}p^{\sigma}. \tag{3.10$'$}$$

This set of equations replaces Eqs. (3.7), (3.8), (3.9), and (3.10). (We have introduced the notation $P^{\rm CSGI}_{\mu\nu\lambda}=P^F_{\mu\nu\lambda}$, $S^{\rm CSGI}_{\mu\lambda}=S^F_{\mu\lambda}$, $R^{\rm CSGI}_{\mu\lambda}=R^F_{\mu\lambda}$, and $T^{\rm CSGI}_\lambda=T^F_\lambda$.) This set of Ward identities leads to the equation

$$k^{\mu}p^{\nu}q^{\sigma}T^{\rm CSGI}_{\mu\nu\sigma\lambda}=-iT^{\rm CSGI}_\lambda+ik^{\mu}R^{\rm CSGI}_{\mu\lambda}(k+p,q) - ik^{\mu}R^{\rm CSGI}_{\mu\lambda}(k+q,p)+ip^{\mu}R^{\rm CSGI}_{\mu\lambda}(p+q,k) - \frac{3}{4\pi^2}k^{\mu}p^{\nu}q^{\sigma}\epsilon_{\mu\nu\sigma\lambda}, \tag{3.11$'$}$$

which is to be contrasted with Eq. (3.11). The Ward identity in Eq. (3.10′) tells us that $R^{\rm CSGI}_{\mu\lambda}$ is second order in momentum in the small momentum limit. Hence we obtain from Eq. (3.11)

$$T^{\rm CSGI}_\lambda(k,p,q)\to\frac{-i}{2\pi^2}\epsilon_{\mu\nu\sigma\lambda}k^{\mu}p^{\nu}q^{\sigma} \tag{3.17}$$

in agreement with the result of direct calculation and with the result of Sec. II. Needless to say, the set of Ward identities in Eqs. (3.7′), (3.8′), (3.9′), and (3.10′) is entirely consistent with the expression for the divergence of the axial-vector current given by Bardeen.[7]

It is instructive to view the question from a slightly different viewpoint. Heuristically one might have thought that by "contracting" the charged axial-vector currents in $\langle 0|T(\partial A^+\partial A^-\partial A^0 V_\lambda)|0\rangle$ one obtains an amplitude proportional to $\langle 0|T(V_\mu\partial A^0 V_\lambda)|0\rangle$, thus relating $\gamma\to\pi^+\pi^-\pi^0$ to $\pi^0\to 2\gamma$ without further ado. This "contraction" may be expressed by combining the Ward identities in Eqs. (3.8′) and (3.9′) to give

$$k^{\mu}p^{\nu}P^{\rm CSGI}_{\mu\nu\lambda}(k,p,q)=-T^{\rm CSGI}_\lambda(k,p,q)+k^{\mu}R^{\rm CSGI}_{\mu\lambda}(k+p,q). \tag{3.18}$$

Note that the two Ward identities we used are both free from anomalies. In standard applications of current algebra the so-called "surface term" $k^{\mu}p^{\nu}P^{\rm CSGI}_{\mu\nu\lambda}$ is usually dropped,[19] thus relating $T^{\rm CSGI}_\lambda$, the amplitude for $\gamma\to\pi^+\pi^-\pi^0$, to $R^{\rm CSGI}_{\mu\lambda}$, the amplitude for $\pi^0\to 2\gamma$. Here, however, we must keep terms to third order in momentum. Now, if we used the naive Ward identity in Eq. (3.7) we would have ascertained that $P_{\mu\nu\lambda}$ is at least second order in momentum and hence may be dropped in Eq. (3.18). However, the appearance of an anomaly in the correct Ward identity (3.7′) informs us that in actuality

$$P^{\rm CSGI}_{\mu\nu\lambda}\to\frac{+i}{4\pi^2}\epsilon_{\mu\nu\sigma\lambda}(k+p+q)^{\sigma}$$

as $k, p, q \to 0$ and hence cannot be dropped from Eq. (3.18) if we wish to determine T_λ^{CSGI} to third order in momentum. Incidentally, the anomaly in the Ward identity of Eq. (3.7′) does not vanish if any one of the three pion momenta approaches zero. This is quite different from the behavior of the more familiar anomaly present in Eq. (3.10′).

We feel it worthwhile to emphasize again that quantities such as $T^F_{\mu\nu\sigma\lambda}$ and T^F_λ are in fact finite even though they appear to diverge logarithmically. This is because gauge invariance requires that the photon field couples through the field tensor $F^{\mu\nu} \sim \epsilon^\mu k^\nu - \epsilon^\nu k^\mu$. Similar remarks apply to the $\pi^0 \to 2\gamma$ discussion.

Why go through with this long discussion of Ward identities in the free fermion theory when the actual computation of T_λ and $R_{\mu\lambda}$ in the low-energy limit is (trivially) simple? The reason is that it provides the framework within which one can deduce general statements about T_λ and $R_{\mu\lambda}$ in the presence of interactions. Typical diagrams that contribute to T_λ and $R_{\mu\lambda}$ are shown in Fig. 3. Adler and Bardeen[20,21] assert that to *any finite order* in renormalized perturbation theory the amplitude for $\gamma \to 3\pi$ and $\pi \to 2\gamma$ are given by the lowest order diagram only (Fig. 2). This conclusion is clearly essential if the PCAC anomaly is to have physical significance. This is an extraordinary assertion for it tells us that PCAC and gauge invariance imply the existence of a spectacular cancellation among the infinite[22] collection of Feynman diagrams, thus providing a unique opportunity to decide whether field theory is in fact relevant to hadron physics. In fact, decades ago people had already computed the fermion loop diagrams for $\pi \to 2\gamma$ and $\gamma \to 3\pi$ using the nucleons as the fundamental fermions.[23] However, the justification for calculating only one Feynman diagram in strong interaction physics, provided nowadays by Sutherland[2] and Adler and Bardeen,[6] was totally lacking then. This point does not appear to be universally appreciated.

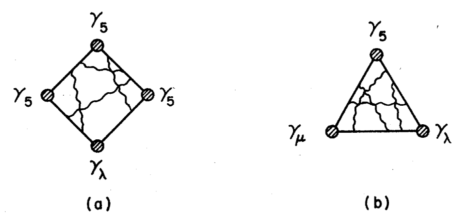

FIG. 3. (a) Typical diagram contributing to $\gamma \to 3\pi$ in a renormalizable theory. (b) Typical diagram contributing to $\pi \to 2\gamma$ in a renormalizable theory. The wavy lines represent vector gluons, photons, and possibly other particles.

IV. EFFECTIVE-LAGRANGIAN FORMULATION

We consider a theory of interacting "quarks," octet mesons, and photons, with the interaction Lagrangian density

$$\mathcal{L}^I = -g\bar{q}(x)[\sigma(x) + i\gamma_5\phi(x)]q(x) + e\bar{q}(x)Q\gamma_\mu q(x)A^\mu(x), \tag{4.1}$$

where

$$\phi = \frac{1}{\sqrt{2}}\sum_{a=0}^{8}\varphi_a\lambda_a$$

and (4.2)

$$\sigma = \frac{1}{\sqrt{2}}\sum_{a=1}^{8}\sigma_a\lambda_a .$$

We demand that $SU_3 \otimes SU_3$ be realized nonlinearly, so that σ_a are not independent fields, but are determined by the matrix equation[24]

$$\sigma^2 + \phi^2 = f^2 . \tag{4.3}$$

The generator of all quark loop diagrams is the matrix element of the S operator between quark vacuum states (i.e., no quarks, but an arbitrary number of mesons and photons)

$$\langle S\rangle = \left\langle T\left(\exp\left[+i\int \mathcal{L}^I(y)d^4y\right]\right)\right\rangle . \tag{4.4}$$

The response to arbitrary variation of the vector potential A_μ is

$$\delta\langle S\rangle = ie\int\left\langle T\left(\bar{q}(x)Q\gamma_\mu q(x) \times \exp\left[+i\int\mathcal{L}^I(y)d^4y\right]\right)\right\rangle \delta A^\mu(x)d^4x = e\int \mathrm{tr}[\gamma_\mu Q S^F(x,x)]\delta A^\mu(x)d^4x , \tag{4.5}$$

where $S^F(x, x)$ is the Feynman propagator of the quark field, in the presence of meson and electromagnetic fields:

$$S^F(x,y) \equiv -i\langle T(\hat{q}(x)\hat{\bar{q}}(y))\rangle , \tag{4.6}$$

where $\hat{q}(x)$ is in the Heisenberg picture. $W^{\text{eff}} \equiv -i\langle S\rangle$ can be thought of as effective action for the photon-mesons couplings. The derived electromagnetic current density is

$$J_\mu(x) = \frac{\delta W^{\text{eff}}}{\delta A^\mu(x)} = -ie\,\mathrm{tr}[\gamma_\mu Q S^F(x,x)] . \tag{4.7}$$

Our problem is, thus, to calculate the functional dependence of the propagator function on the electromagnetic and the meson fields. Mathematically, the equation to be solved is of the Green's function type:

$$\{-\gamma_\mu[i\partial_x^\mu + eQA^\mu(x)] + g[\sigma(x) + i\gamma^5\phi(x)]\}S^F(x,y) = -\delta^4(x-y)1\,.$$

Here 1 is the unit matrix in $SU_3 \otimes SU_3$ and Dirac-algebra representation spaces. It is crucial, however, that the local current is calculated at the singular point $x=y$. As is well known, improper mathematical operations might lead to gauge-non-invariant results. In order to avoid such pitfalls we follow the suggestion of Schwinger.[9] Essentially Schwinger separates the points x and y and maintains gauge invariance for all the pairs, with the result that the A_μ dependence (apart from explicit electromagnetic field strength dependence) appears through a *phase* which vanishes in all physical, diagonal quantities, like $J_\mu(x)$.

For brevity, we regard all quantities below as operators in Minkowski, Dirac and $SU_3 \otimes SU_3$ representation spaces. We have

$$(\not\pi + m^{(+)})S^F = -1\,, \tag{4.8}$$

where

$$\pi_\mu \equiv -i\partial_\mu - eQA_\mu\,, \quad \not\pi = \gamma^\mu \pi_\mu\,, \tag{4.9}$$

$$m^{(\pm)} \equiv g(\pm\sigma + i\gamma^5\phi)\,. \tag{4.10}$$

In other words, we introduce state vectors $|x\rangle$ and an operator S^F such that $\langle x|S^F|y\rangle = S^F(x,y)$ and $\langle x|y\rangle = \delta^4(x-y)$. Using the anticommutativity of γ^5, we have

$$\not\pi m^{(+)} + m^{(-)}\not\pi = im_\mu^{(-)}\gamma^\mu + e[Q, m^{(-)}]A_\mu\gamma^\mu\,. \tag{4.11}$$

Here we introduced the meson-field derivative matrices:

$$m_\mu^{(\pm)} \equiv g(\pm\partial_\mu\sigma + i\gamma^5\partial_\mu\phi) = g\frac{1}{\sqrt{2}}\sum_{a=0}^{8}(\pm\partial_\mu\sigma_a + i\gamma^5\partial_\mu\varphi_a)\lambda_a\,. \tag{4.12}$$

Multiplying Eq. (4.8) by $\not\pi + m^{(-)}$ and using (4.3) and (4.11) we have

$$\{\not\pi^2 - g^2f^2 + (im_\mu^{(-)} + e[Q, m^{(-)}]A_\mu)\gamma^\mu\}S^F = -(\not\pi + m^{(-)})\,. \tag{4.13}$$

The advantage of this form is that we have separated the "perturbation" $(im_\mu^{(-)} + e[Q, m^{(-)}]A_\mu)\gamma^\mu$: Eventually we shall consider the soft-meson limit (with restricted number of derivatives), for processes involving a small number of photons. Apart from the explicit A_μ dependence, gauge invariance will be maintained by dealing directly with the operators π_μ. We have

$$\not\pi^2 = \pi_\mu\pi^\mu + \tfrac{1}{2}eQ\sigma_{\alpha\beta}F^{\alpha\beta}\,. \tag{4.14}$$

Once the matrix elements of π_μ are known, the operator S^F is given by

$$S^F = i\int_0^\infty ds\,\exp(i\{\not\pi^2 - g^2f^2 + (im_\mu^{(-)} + e[Q, m^{(-)}]A_\mu)\gamma^\mu\}s)(\not\pi + m^{(-)}) \tag{4.15}$$

or

$$S^F = i\int_0^\infty ds\, e^{-ig^2f^2s}\, e^{-i(h+h^I)s}(\not\pi + m^{(-)}) \tag{4.16}$$

with the definitions

$$h \equiv -\not\pi^2\,, \tag{4.17a}$$

$$h^I = h_1 + h_2\,, \tag{4.17b}$$

$$h_1 = -im_\mu^{(-)}\gamma^\mu\,, \tag{4.17c}$$

$$h_2 = -e[Q, m^{(-)}]A_\mu\gamma^\mu\,. \tag{4.17d}$$

In this formalism, the variable s plays the role of a "proper time" while the operators h and h^I are highly suggestive of the free and interaction Hamiltonian, respectively.

Using the general formula

$$e^{A+B} = e^A + \int_0^1 du_1\, e^{A(1-u_1)}B\, e^{Au_1} + \int_0^1 u_1du_1\int_0^1 du_2\, e^{A(1-u_1)}B\, e^{Au_1(1-u_2)}B\, e^{Au_1u_2} + \cdots + \int_0^1 u_1^{\,n-1}du_1\int_0^1 u_2^{\,n-2}du_2\cdots\int_0^1 du_n\, e^{A(1-u_1)}B\, e^{Au_1(1-u_2)}B\cdots e^{Au_1\cdots u_n} + \cdots, \tag{4.18}$$

we can generate S^F as a series in h^I. At this point we introduce Schwinger's proper-time parameter transformation: All operators and states are transformed into the "interacting picture"

$$O \to \hat{O}(S) \equiv e^{ihs} O\, e^{-ihs},$$
$$|x\rangle \to |x(s)\rangle\!\rangle \equiv e^{ihs} |x\rangle . \tag{4.19}$$

(Internal indices are suppressed.) The "equation of motion" reads

$$\frac{d\hat{O}(s)}{ds} = -i[\hat{O}(s), h]. \tag{4.20}$$

In particular, the Heisenberg commutation relations lead to

$$\begin{aligned} \frac{d\hat{x}_\mu(s)}{ds} &= -2\hat{\pi}_\mu(s), \\ \frac{d\hat{\pi}_\mu}{ds} &= -2eQ\hat{F}_{\mu\nu}\hat{\pi}^\nu(s) + ieQ\partial^\nu \hat{F}_{\mu\nu} + \tfrac{1}{2}eQ\sigma_{\alpha\beta}\partial_\mu \hat{F}^{\alpha\beta}, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} &i\partial_s \langle x(s)|y(0)\rangle\!\rangle = \langle x(s)|h|y(0)\rangle\!\rangle, \\ &[i\partial_\mu - eQA_\mu(x)]\langle x(s)|y(0)\rangle\!\rangle = \langle x(s)|\hat{\pi}_\mu(s)|y(0)\rangle\!\rangle, \\ &\left(i\frac{\partial}{\partial y^\mu} - eQA_\mu(y)\right)\langle x(s)|y(0)\rangle\!\rangle = \langle x(s)|\hat{\pi}_\mu(0)|y(0)\rangle\!\rangle . \end{aligned} \tag{4.22}$$

This set of equations is supplemented by the boundary condition: $\langle x(s)|y(0)\rangle\!\rangle \to \delta^4(x-y)$ as $s \to 0$. These equations exhibit the x, y separation mentioned above. The solution of these equations (for *constant* electromagnetic field strength) has been given by Schwinger[9]:

$$\langle x(s)|y(0)\rangle\!\rangle = \frac{-i}{16\pi^2}\phi(x,y)\, e^{-L(s)} s^{-2} \exp\{-\tfrac{1}{4}i(x-y)^\alpha eQF_{\alpha\beta}[\coth(eQFs)]^{\beta\gamma}(x-y)_\gamma\}\exp(\tfrac{1}{2}ieQ\sigma_{\mu\nu}F^{\mu\nu}s), \tag{4.23}$$

where

$$\begin{aligned} \phi(x,y) &= \exp\left(ieQ\int_y^x dz^\mu A_\mu(z)\right), \\ L(s) &= \tfrac{1}{2}\operatorname{tr}\ln\left(\frac{\sinh(eQFs)}{eQFs}\right), \\ F &\equiv F_{\mu\nu}. \end{aligned} \tag{4.24}$$

The matrix elements of π_μ are then found to be

$$\langle x(s)|\hat{\pi}_\mu(s)|y(0)\rangle\!\rangle = -\tfrac{1}{2}eQF_{\mu\nu}[\coth(eQFs) - 1]^{\nu\rho}(x-y)_\rho \langle x(s)|y(0)\rangle\!\rangle, \tag{4.25}$$
$$\langle x(s)|\hat{\pi}_\mu(0)|y(0)\rangle\!\rangle = -\tfrac{1}{2}eQF_{\mu\nu}[\coth(eQFs) + 1]^{\nu\rho}(x-y)_\rho \langle x(s)|y(0)\rangle\!\rangle . \tag{4.26}$$

The propagator functional operator can be rewritten in the transformed language:

$$\begin{aligned} S^F = i\int_0^\infty ds\, e^{ig^2f^2s}\Big(& e^{-ihs} + (-is)\int_0^1 du_1\, e^{-ih(1-u_1)s}h^I e^{-ihu_1 s} \\ &+ (-is)^2 \int_0^1 u_1 du_1 \int_0^1 du_2\, e^{-ih(1-u_1)s}h^I e^{-ihu_1(1-u_2)s}h^I e^{-ihu_1u_2s} + \cdots \\ &+ (-is)^n \int_0^1 u_1^{\,n-1}du_1 \int_0^1 u_2^{\,n-2}du_2 \cdots \int_0^1 du_n\, e^{-ih(1-u_1)s}h^I e^{-ihu_1(1-u_2)s}h^I \cdots h^I e^{-ihu_1\cdots u_n} + \cdots\Big)(\not{\pi} + m^{(-)}) \\ = i\int_0^\infty ds\, e^{-ig^2f^2s}\Big(& 1 + (-is)\int_0^1 du_1 \hat{h}^I((u_1-1)s) + (-is)^2\int_0^1 u_1du_1\int_0^1 du_2 \hat{h}^I((u_1-1)s)\hat{h}^I((u_1u_2-1)s) + \cdots \\ &+ (-is)^n \int_0^1 u_1^{\,n-1}du_1\int_0^1 u_2^{\,n-2}du_2\cdots\int_0^1 du_n \hat{h}^I((u_1-1)s)\hat{h}^I((u_1u_2-1)s)\cdots \hat{h}^I((u_1u_2\cdots u_n-1)s) + \cdots\Big) \\ &\times e^{-ihs}(\not{\pi} + m^{(-)}). \end{aligned} \tag{4.27}$$

The matrix elements

$$S^F(x,y) \equiv \langle x|S^F|y\rangle = \sum_n S_n^F(x,y), \tag{4.28}$$

where

$$S_n^F(x,y)=i\int_0^\infty ds\, e^{-ig^2f^2s}(-is)^n\int_0^1 u_1^{\,n-1}du_1\cdots\int_0^1 d^4z\langle x|\hat{h}^I((u_1-1)s)\hat{h}^I((u_1u_2-1)s)\cdots\hat{h}^I((u_1\cdots u_n-1)s)|z\rangle$$
$$\times\langle z(s)|\not{\pi}+m^{(-)}|y(0)\rangle\,, \qquad (4.29)$$

are determined by Eqs. (4.23), (4.24), (4.25), and (4.26). Equation (4.29) enables us to calculate the general coupling of any number of photons to any number of mesons through a single fermion loop. (As emphasized in Ref. 1, in order to obtain the correct amplitude for a given photon-meson process terms corresponding to pole diagrams must be added.) In practice, the calculation of the general coupling involving any number of photons and mesons is a rather tedious task. In what follows we content ourselves with treating two special cases: (A) single photon coupling to any number of mesons, and (B) single meson coupling to any number of photons.

Case A: Single-Photon Couplings

In the case of a single photon coupling to any number of mesons it suffices to evaluate our equations only to zeroth order in e, resulting in drastic simplifications. h and h^I are approximated by [cf. Eq. (4.17)]

$$h\equiv h_0=\not{\partial}^2\,,\quad h^I=h_1=im_\mu^{(-)}\gamma^\mu\,. \qquad (4.30)$$

The transformation function reads [cf. Eq. (4.23)]

$$\langle x(s)|y(0)\rangle_0=\frac{-i}{16\pi^2}\frac{1}{s^2}e^{-(i/4s)(x-y)^2} \qquad (4.31)$$

and the momentum matrix elements are [cf. Eq. (4.26)]

$$\langle x(s)|\pi_\mu|y(0)\rangle=-\frac{1}{2s}(x-y)_\mu\langle x(s)|y(0)\rangle_0\,. \qquad (4.32)$$

The general term in Eq. (4.27) is a product of matrix elements of $\hat{h}^I$ which have the following structure for local h^I:

$$\langle y|\hat{h}^I(\alpha)|z\rangle_0=\int\int d^4\xi\, d^4\eta\langle y|e^{ih_0\alpha}|\xi\rangle\langle\xi|h_1|\eta\rangle\langle\eta|e^{-ih_0\alpha}|z\rangle$$
$$=\int d^4\xi\langle y(-\alpha)|\xi(0)\rangle_0 h_1(\xi)\langle\xi(\alpha)|z(0)\rangle_0\,.$$

$h_1(\xi)$ is a c-number matrix function of the four-vector ξ,

$$h_1(\xi)\equiv -i\frac{\partial}{\partial\xi^\mu}m^{(-)}(\xi)\gamma^\mu\,, \qquad (4.33)$$

and should not be confused with the operator $\hat{h}^I(\alpha)$ which is a function of the parameter α. The dependence of $\langle y(-\alpha)|\xi(0)\rangle$ on the pair of points y, ξ is given explicitly by (4.31), so that we may use the identity

$$\xi_\alpha\langle y(-\alpha)|\xi(0)\rangle_0=(y+2i\alpha\partial_y)\langle y(-\alpha)|\xi(0)\rangle_0 \qquad (4.34)$$

to shift the ξ dependence into an external-coordinate dependence

$$\langle y(-\alpha)|\xi(0)\rangle_0 h_1(\xi)=h_1[D_y(\alpha)]\langle y(-\alpha)|\xi(0)\rangle_0 \qquad (4.35)$$

with the four-vector differential operator

$$D_y(\alpha)\equiv y+2i\alpha\partial_y \qquad (4.36)$$

as the argument of the h_1 function. The matrix element in question is thus

$$\langle y|\hat{h}^I(\alpha)|z\rangle_0=h_1[D_y(\alpha)]\delta^4(y-z)\,. \qquad (4.37)$$

The appropriate product of n matrix elements is a straightforward generalization

$$\langle x|\hat{h}^I((u_1-1)s)\hat{h}^I((u_1u_2-1)s)\cdots\hat{h}^I((u_1u_2\cdots u_n-1)s)|y\rangle$$
$$=h_1[D_x((u_1-1)s)]h_1[D_x((u_1u_2-1)s)]\cdots h_1[D_x((u_1u_2\cdots u_n-1)s)]\delta^4(x-y) \qquad (4.38)$$

and therefore

$$S_n^F(x,y)=i\int_0^\infty ds\, e^{-ig^2f^2s}(-is)^n\int_0^1 u_1^{n-1}du_1\cdots\int_0^1 du_n h_1[D_x((u_1-1)s)]\times\cdots$$
$$\times h_1[D_x((u_1u_2\cdots u_n-1)s)]\left(-\frac{1}{2s}(x-y)_\mu\gamma^\mu+m^{(-)}(y)\right)\langle x(s)|y(0)\rangle_0 . \tag{4.39}$$

Having low-energy approximations in mind, we expand

$$h_1[D_x(\alpha)]=-i[m_\mu^{(-)}(x)+2i\alpha m_{\mu\nu}^{(-)}\partial_x^\nu+2i\alpha m_{\mu\nu\rho}^{(-)}x^\nu x^\rho+\tfrac{1}{2}(2i\alpha)^2 m_{\mu\nu\rho}^{(-)}\partial_x^\nu\partial_x^\rho]\gamma^\mu , \tag{4.40}$$

where

$$m_{\mu\nu}^{(-)}\equiv g[-\partial_\mu\partial_\nu\sigma(x)+i\gamma^5\partial_\mu\partial_\nu\phi(x)]_{x=0},$$
$$m_{\mu\nu\rho}^{(-)}\equiv g[-\partial_\mu\partial_\nu\partial_\rho\sigma(x)+i\gamma^5\partial_\mu\partial_\nu\partial_\rho\phi(x)]_{x=0} \tag{4.41}$$

are totally symmetric tensors. Keeping terms up to and including third-order derivatives of the meson fields, we have for the propagator function

$$S^F(x,y)=\frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s^2}e^{-ig^2f^2s}\bigg(1-s\int_0^1 du_1[m_\alpha^{(-)}(x)+2i(u_1-1)s m_{\alpha\beta}^{(-)}\partial_x^\beta]\gamma^\alpha$$
$$+s^2\int_0^1 u_1du_1\int_0^1 du_2[m_\alpha^{(-)}(x)+2i(u_1-1)sm_{\alpha\beta}^{(-)}\partial_x^\beta]\gamma^\alpha[m_\delta^{(-)}(x)+2i(u_1u_2-1)sm_{\delta\tau}^{(-)}\partial_x^\tau]\gamma^\delta$$
$$-s^3\int_0^1 u_1^2du_1\int_0^1 u_2du_2\int_0^1 du_3 m_\alpha^{(-)}\gamma^\alpha m_\beta^{(-)}\gamma^\beta m_\delta^{(-)}\gamma^\delta\bigg)\left(-\frac{1}{2s}(x-y)_\nu\gamma^\nu+m^{(-)}(y)\right)e^{-(i/4s)(x-y)^2}$$
$$=\frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s^2}e^{-ig^2f^2s}\bigg\{\bigg(1-s\int_0^1 du_1 m_\alpha^{(-)}(x)\gamma^\alpha+s^2\int_0^1 u_1du_1\int_0^1 du_2 m_\alpha^{(-)}(x)\gamma^\alpha m_\beta^{(-)}(x)\gamma^\beta$$
$$-s^3\int_0^1 u_1^2du_1\int_0^1 u_2du_2\int_0^1 du_3 m_\alpha^{(-)}\gamma^\alpha m_\beta^{(-)}\gamma^\beta m_\delta^{(-)}\gamma^\delta\bigg)\left(-\frac{1}{2s}(x-y)_\nu\gamma^\nu+m^{(-)}(y)\right)$$
$$+\bigg(is\int_0^1 du_1(u_1-1)m_{\alpha\beta}^{(-)}\gamma^\alpha-is^2\int_0^1 u_1du_1\int_0^1 du_2[(u_1u_2-1)m_\delta^{(-)}\gamma^\delta m_{\alpha\beta}^{(-)}\gamma^\alpha+(u_1-1)m_{\alpha\beta}^{(-)}\gamma^\alpha m_\delta^{(-)}\gamma^\delta]\bigg)$$
$$\times\left[\left(-\frac{1}{2s}(x-y)_\nu\gamma^\nu+m^{(-)}(y)\right)(x-y)^\beta+\gamma^\beta\right]e^{-(i/4s)(x-y)^2}\bigg\}. \tag{4.42}$$

At this point we take the limit $x=y$. Divergences, if any, are left in the form of integrals. Performing the u_1 integrations, we end up with

$$S^F(x,x)=\frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s^2}e^{-g^2f^2s}\{[1-sm_\alpha^{(-)}\gamma^\alpha+\tfrac{1}{2}s^2m_\alpha^{(-)}\gamma^\alpha m_\beta^{(-)}\gamma^\beta-\tfrac{1}{6}s^3m_\alpha^{(-)}\gamma^\alpha m_\beta^{(-)}\gamma^\beta m_\delta^{(-)}\gamma^\delta]m^{(-)}(x)$$
$$+i[-\tfrac{1}{2}sm_{\alpha\beta}^{(-)}\gamma^\alpha+\tfrac{1}{2}s^2(m_\delta^{(-)}\gamma^\delta m_{\alpha\beta}^{(-)}\gamma^\alpha+m_{\alpha\beta}^{(-)}\gamma^\alpha m_\delta^{(-)}\gamma^\delta)]\gamma^\beta\}. \tag{4.43}$$

Finally we calculate the trace required in (4.7), and the s integration. The resulting induced electromagnetic current (to the first order in e) is

$$J^\mu(x)=ie\,\mathrm{tr}\{Q[G(\sigma^\mu\sigma+\phi^\mu\phi)+\Phi_{\rho\nu\sigma}^{(1)}S^{\mu\rho\nu\sigma}+\Phi_{\rho\nu\sigma}^{(2)}\epsilon^{\mu\rho\nu\sigma}]\}, \tag{4.44}$$

where $\sigma^\mu\equiv\partial^\mu\sigma$, and similarly for higher derivatives. The other quantities involved are

$$\Phi_{\rho\nu\sigma}^{(1)}=-\frac{1}{8\pi^2f^2}(\sigma_\rho\sigma_{\nu\sigma}+\phi_\rho\phi_{\nu\sigma}+\sigma_{\rho\sigma}\sigma_\nu+\phi_{\rho\sigma}\phi_\nu)$$
$$-\frac{1}{24\pi^2f^4}(\sigma_\rho\sigma_\nu\sigma_\sigma\sigma+\sigma_\rho\sigma_\nu\phi_\sigma\phi-\sigma_\rho\phi_\nu\sigma_\sigma\phi+\sigma_\rho\phi_\nu\phi_\sigma\sigma+\phi_\rho\sigma_\nu\sigma_\sigma\phi-\phi_\rho\sigma_\nu\phi_\sigma\sigma+\phi_\rho\phi_\nu\sigma_\sigma\sigma+\phi_\rho\phi_\nu\phi_\sigma\phi), \tag{4.45}$$

$$\Phi_{\rho\nu\sigma}^{(2)}=\frac{-1}{24\pi^2f^4}(-\sigma_\rho\sigma_\nu\sigma_\sigma\phi+\sigma_\rho\sigma_\nu\phi_\sigma\sigma-\sigma_\rho\phi_\nu\sigma_\sigma\sigma-\sigma_\rho\phi_\nu\phi_\sigma\phi+\phi_\rho\sigma_\nu\sigma_\sigma\sigma+\phi_\rho\sigma_\nu\phi_\sigma\phi-\phi_\rho\phi_\nu\sigma_\sigma\phi+\phi_\rho\phi_\nu\phi_\sigma\sigma), \tag{4.46}$$

$$S^{\mu\rho\nu\sigma}\equiv g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\nu}, \tag{4.47}$$

and

$$G=\frac{g^2}{4\pi^2}\int_0^\infty \frac{ds}{s}\,e^{-ig^2f^2s}=\frac{g^2}{4\pi^2}\int_0^\infty\frac{d\lambda}{\lambda}\,e^{-i\lambda} \tag{4.48}$$

is a dimensionless, logarithmically divergent, renormalization constant. We add the electromagnetic current of the free meson fields, $ie\,\mathrm{tr}[Q(\sigma^\mu\sigma+\phi^\mu\phi)]$, so that by renormalizing ϕ, σ, and f by $(1+G)^{1/2}$ we can write the total electromagnetic current as

$$J^\mu(x)=ie\,\mathrm{tr}\{Q[\sigma^\mu\sigma+\phi^\mu\phi+\Phi^{(1)}_{\rho\nu\sigma}S^{\mu\rho\nu\sigma}+\Phi^{(2)}_{\rho\nu\sigma}\epsilon^{\mu\rho\nu\sigma}]\}. \tag{4.49}$$

Here $\Phi^{(1)}$ and $\Phi^{(2)}$ are given by (4.45) and (4.46) in terms of the renormalized quantities. The three meson couplings are generated from the last term of $\Phi^{(2)}_{\rho\nu\sigma}$, i.e.,

$$J^\mu(x)=\frac{ie}{24\pi^2f^4}\,\mathrm{tr}(Q\phi_\rho\phi_\nu\phi_\sigma\sigma)\epsilon^{\mu\rho\nu\sigma}. \tag{4.50}$$

Explicitly

$$J^\mu(x)=\frac{ie}{12\pi^2(\sqrt{2}\,f)^3}\left[(\pi^+_\rho\pi^-_\nu+K^+_\rho K^-_\nu)\left(\pi^0_\sigma+\frac{1}{\sqrt{3}}\eta_\sigma+\left(\frac{2}{3}\right)^{1/2}\eta'_\sigma\right)+K^0_\rho\overline{K}^0_\nu(\pi^0_\sigma-\sqrt{3}\,\eta_\sigma)\right]\epsilon^{\mu\rho\nu\sigma}. \tag{4.51}$$

The effective action function is $W^{\mathrm{eff}}=\int J^\mu(x)A_\mu(x)d^4x$ in this case.

Case B: Single-Meson Couplings

We need the matrix element

$$S_1^F(x,y)=\int_0^\infty sds\,e^{-ig^2f^2s}\int_0^1 du_1d^4z\langle x|\hat{h}^I((u_1-1)s)|z\rangle\langle 2(s)|(\not{\pi}+m^{(-)})|y\rangle. \tag{4.52}$$

Since the meson matrix commutes (in this case) with Q, we actually have

$$h^I(\alpha)=e^{ih\alpha}h_1e^{-ih\alpha},\quad h=-\not{\pi}^2,\quad h_1=-igi\gamma^5\phi_\mu\gamma^\mu, \tag{4.53}$$

so that

$$\langle x|\hat{h}^I((u_1-1)s)|z\rangle=\int\langle x(-(u_1-1)s)|\,\xi(0)\rangle h_1(\xi)\langle\xi(\alpha)|z(0)\rangle d^4\xi \tag{4.54}$$

with $\langle x(-(u_1-1)s)|\,\xi(0)\rangle$ given by (4.23). We note, however, that in $S_1^F(x,x)$ the phase factors $\phi(x,y)$ cancel out, so that the following identity holds,

$$\langle x(-\alpha)|\,\xi(0)\rangle h_1(\xi)=h_1[x-G^{-1}(\alpha)\partial_x]\langle x(-\alpha)|\,\xi(0)\rangle, \tag{4.55}$$

where $G^{-1}(\alpha)$ is the inverse of the symmetric matrix

$$G(\alpha)_{\mu\nu}=\tfrac{1}{2}iF_{\mu\rho}(\coth eQ\alpha F)^\rho{}_\nu \tag{4.56}$$

so that in $S_1^F(x,x)$ we can write

$$\langle x|\hat{h}^I((u_1-1)s)|z\rangle=g\gamma^5\gamma^\mu\phi_\mu[x-G^{-1}((u_1-1)s)\partial_x]\delta^4(x-z). \tag{4.57}$$

Therefore,

$$S_1^F(x,x)=g\int_0^\infty sds\,e^{-ig^2f^2s}\int_0^1 du_1\gamma^5\gamma^\mu\phi_\mu[x-G^{-1}((u_1-1)s)\partial_x]\langle x(s)|\not{\pi}+m^{(-)}|y(0)\rangle|_{y=x}. \tag{4.58}$$

To the first order in the mesons' fields and their derivatives, we have

$$J^\mu(x)=i\frac{eg^2f}{16\pi^2}\int_0^\infty\frac{ds}{s}\,e^{-ig^2f^2s}\,e^{-L(s)}\,\mathrm{tr}[Q\phi_\nu\gamma^\mu\gamma^5\gamma^\nu\exp(i\tfrac{1}{2}esQ\sigma_{\alpha\beta}F^{\alpha\beta})] \tag{4.59}$$

which contains the coupling of single mesons to an arbitrary number of photons. There is no term of the first order in e, of course. The second-order term is

$$\begin{aligned}J^\mu(x)&=-\frac{e^2g^2f}{32\pi^2}\int_0^\infty e^{-ig^2f^2s}\,\mathrm{tr}Q^2\phi_\nu i\gamma^5\gamma^\mu\gamma^\nu\sigma^{\alpha\beta}F_{\alpha\beta}\\&=\frac{e^2}{8\pi^2(\sqrt{2}f)}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}\frac{1}{3}\left[\pi^0_\nu+\frac{1}{\sqrt{3}}\eta_\nu+2\left(\frac{2}{3}\right)^{1/2}\eta'_\nu\right].\end{aligned} \tag{4.60}$$

Using

$$\int \epsilon^{\alpha\beta\mu\nu}\partial_\alpha A_\beta \delta A_\mu(x)\pi^0_\nu d^4x = \tfrac{1}{2}\delta \int \epsilon^{\alpha\beta\mu\nu}\partial_\alpha A_\beta A_\mu \pi^0_\nu d^4x, \tag{4.61}$$

we see that the induced effective Lagrangian is

$$\mathcal{L}^{\rm eff} = \frac{e^2}{16\pi^2(2f)^{1/2}}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}\frac{1}{3}\left(\pi^0_\nu + \frac{1}{\sqrt{3}}\eta_\nu + 2\left(\frac{2}{3}\right)^{1/2}\eta'_\nu\right)A_\mu \tag{4.62}$$

and

$$W^{\rm eff} = \int \mathcal{L}^{\rm eff}(x)d^4x\,. \tag{4.63}$$

From (4.51) and (4.62) we have for the matrix elements of our interest:

$$m(\gamma(k_1)+\gamma(k_2)\to\pi^0) = iF^\pi k_1^\alpha k_2^\beta \epsilon_1^\gamma \epsilon_2^\delta \epsilon_{\alpha\beta\nu\delta} \tag{4.64}$$

and

$$m(\gamma(k_1)\to\pi^0+\pi^+(p_+)+\pi^-(p_-)) = iF^{3\pi}k_1^\alpha \epsilon^\beta p_+^\gamma p_-^\delta \epsilon_{\alpha\beta\gamma\delta} \tag{4.65}$$

with

$$\frac{eF^{3\pi}}{F^\pi} = \frac{1}{(\sqrt{2}f)^2} \tag{4.66}$$

agreeing with the result of Sec. II.

Recently, Wess and Zumino,[8] using elegant variational techniques and exploiting a judicious choice of pion field, have also constructed an effective Lagrangian which leads to Bardeen's[7] expression for PCAC anomalies.

V. REMARKS AND DISCUSSION

In principle, the colliding-beam experiments $e^+e^-\to\gamma\to 3\pi$ provide the cleanest determination of $F^{3\pi}$. A detailed phase-space calculation is given in the Appendix. Here we only give the approximate expression near threshold

$$\sigma(W) \simeq \frac{\alpha}{24\pi}|F^{3\pi}|^2\left(\frac{(m_1m_2m_3)^3}{(m_1+m_2+m_3)^9}\right)^{1/2} \times (W-m_1-m_2-m_3)^4, \tag{5.1}$$

where $W\equiv 2E$ is equal to twice the beam energy in the e^+e^- center-of-mass frame. (We have written the formula with three unequal meson masses as we will mention $e^+e^-\to\gamma\to K\bar{K}\pi$ later.) The low-energy theorem Eq. (5.1) gives

$$|F^{3\pi}| \simeq \frac{3e}{4\pi^2}\frac{1}{m_\pi^{\,3}}$$

[using $f_\pi \simeq (1/\sqrt{2})0.96m_\pi$ and $|F^\pi| \simeq (3\alpha/2\pi)(1/m_\pi)$]. This leads to the estimate

$$\sigma(W) \simeq 3.7\times10^{-10}m_\pi^{\,-2}\left(\frac{W-3m_\pi}{m_\pi}\right)^4 \tag{5.2}$$

giving for example $\sigma(W=4m_\pi)\simeq 10^{-35}$ cm^2. We hope that this will be within reach of the colliding-beam machines in the near future. The experiment will be hard, but as we have indicated in this paper the information is of considerable theoretical interest.

An indirect determination of $F^{3\pi}$ has been attempted by Donnachie and Shaw[25] (DS) who analyze single-pion photoproduction data. The double-pion exchange diagram is included in the calculation with the parameter $\Lambda \equiv (m_\pi^{\,3}/e)F^{3\pi}$. The analysis gives $\Lambda^{\rm DS} = 0.04\pm0.15$. That does not contradict the theoretical prediction[26] of this paper

$$\Lambda = \frac{2}{\alpha}\left(\frac{m_\pi^{\,3}\Gamma(\pi^0\to2\gamma)}{\pi f_\pi^{\,4}}\right)^{1/2} \simeq 0.076\,. \tag{5.3}$$

As is well known,[5] in order to bring the predicted rate for $\pi^0\to2\gamma$ into agreement with experiment one has to set the charge matrix

$$\lambda_Q = \begin{pmatrix}1 & & \\ & 0 & \\ & & 0\end{pmatrix}$$

which amounts to adding an extra component ("charm" current) to the Gell-Mann–Nishijima construction so that $J_\mu^{\rm em} = V_\mu^3 + (1/\sqrt{3})V_\mu^8 + V_\mu^c$. The charm current V_μ^c is usually taken to be an SU(3) singlet and of the form $\frac{1}{3}\bar{\psi}\gamma_\mu\psi$ in the quark model so that the assumption (c) of Sec. II remains valid, although more perverse possibilities may be readily envisioned.

Whether the electromagnetic current is purely an SU(3) octet or a mixture of SU(3) octet and singlet is clearly an important question, quite apart from the context of this paper. The natural way to decide this question is to look for threshold production of $K^+K^-\pi^0, K^0\overline{K}^0\pi^0$, and $K^+\overline{K}^0\pi^-$ via single-photon annihilation in e^+e^- collision. One concludes that the matrix elements for $e^+e^- \to \gamma \to K^+K^-\pi^0, K^0\overline{K}^0\pi^0$, and $K^+\overline{K}^0\pi^-$, respectively, stand in the ratio

$$(3Q-1):-3(Q-1):(3Q-2),$$

where Q is defined as in Sec. III by assigning the charge $(Q, Q-1, Q-1)$ to the fermion triplet. [More generally, Q represents the parameter that measures the relative strength of the SU(3)-singlet and -octet components in J_μ^{em}.] We note that for the case of $Q=\frac{2}{3}$ (J_μ^{em} is purely octet) the $K^+\overline{K}^0\pi^-$ mode is suppressed while for $Q=1$ (J_μ^{em} has an SU(3)-singlet component) the $K^0\overline{K}^0\pi^0$ is suppressed. Fortunately, the availability of three final states allows us to test if SU(3)-symmetry-breaking effects are important by checking the Q-independent linear relation between the three production modes. [One may also compare with $e^+e^- \to \gamma \to 3\pi$ if one is willing to make additional assumptions about SU(3) asymmetric phase spaces.] In any case, one would not be misled by SU(3)-symmetric-breaking effects. We mention that Okubo and Sakita[27] had also considered $\eta \to \pi^+\pi^-\gamma$ as related to $\gamma \to \pi^+\pi^-\pi^0$ by SU(3). This is probably an unreliable way to determine $\gamma \to \pi^+\pi^-\pi^0$ as the SU(3) relation between $\eta \to 2\gamma$ and $\pi \to 2\gamma$ is known to be violated badly. Similar comments apply to attempts to exploit SU(3) symmetry further by relating $\gamma \to \pi^+\pi^-\pi^0$ to the contribution of the vector current to K_{l4} decay.[28] It is clear that the Sutherland argument also applies to this vertex to the extent that the strangeness-changing vector current is conserved. However, K_{l4} decay differs from $\gamma \to \pi^+\pi^-\pi^0$ in one significant aspect: for the electromagnetic process one may change the predicted rate by changing the charge assignment of the underlying fermions or equivalently by adding an extra piece to the electromagnetic current, while for the weak process this freedom is lacking. In other words, the SU(3) rotation relates the vector part of the K_{l4} to the contribution of the Gell-Mann–Nishijima current to $\gamma \to 3\pi$.

Note Added. After this work was completed, it was brought to our attention that Terentiev (Ref. 1) has independently discovered some of the material in this paper.

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APPENDIX: $e^+e^- \to 3$ MESONS NEAR THRESHOLD

For completeness we include here the detailed calculation of the cross section for

$$e^+(p^+)+e^-(p^-) \to \gamma \to M_1(q_1)+M_2(q_2)+M_3(q_3), \tag{A1}$$

where M_i are three pseudoscalar mesons. We do *not* assume that the mesons have equal mass. The matrix element for the process is given by

$$\mathfrak{M} = ieF^{3\pi}\frac{1}{Q^2}\,|\,l q_1 q_2 q_3\,|, \tag{A2}$$

where $l_\mu \equiv \bar{v}(p', s')\gamma_\mu u(p, s)$. The cross section is given by

$$\sigma = \frac{1}{(2\pi)^5}\frac{m_e^{\,2}}{[(p^+p^-)^2 - m_e^{\,4}]^{1/2}}\int \frac{1}{4}\sum_{s,s'}|\mathfrak{M}|^2 \frac{d^3q_1}{2\omega_1}\frac{d^3q_2}{2\omega_2}\frac{d^3q_3}{2\omega_3}\delta^4(Q-q_1-q_2-q_3), \tag{A3}$$

where $Q=p^+ + p^-$. Hence

$$\sigma = \frac{1}{(2\pi)^5}\frac{m_e^{\,2}}{[(p^+p^-)^2 - m_e^{\,4}]^{1/2}}\left(\frac{eF^{3\pi}}{Q^2}\right)^2 \frac{1}{4}\sum_{s,s'} l_\alpha l_\beta^\dagger T^{\alpha\beta}, \tag{A4}$$

where

$$T^{\alpha\beta} = \int \epsilon^{\alpha\nu\rho\sigma}\epsilon^{\beta\nu'\rho'\sigma'} q_{1\nu}q_{2\rho}q_{3\sigma}q_{1\nu'}q_{2\rho'}q_{3\sigma'}\frac{d^3q_1 d^3q_2 d^3q_3}{8\omega_1\omega_2\omega_3}\delta^4(Q-q_1-q_2-q_3). \tag{A5}$$

Since

$$Q^\alpha T_{\alpha\beta} = Q^\beta T_{\alpha\beta} = 0\,,$$

we have

$$T_{\alpha\beta} = I(Q^2)\left(g_{\alpha\beta} - \frac{Q_\alpha Q_\beta}{Q^2}\right), \tag{A6}$$

with $I(Q^2) = \frac{1}{3} T^\alpha{}_\alpha$. Also $Q^\alpha l_\alpha = 0$, so that

$$\sigma = \frac{1}{(2\pi)^5} \frac{m_e^2}{[(p^+p^-)^2 - m_e^4]^{1/2}} \left(\frac{eF^{3\pi}}{Q^2}\right)^2 \left(\frac{1}{4}\sum_{s,s'} l_\alpha l_\beta^\dagger g^{\alpha\beta}\right) I(Q^2)\,. \tag{A7}$$

We now evaluate $I(Q^2)$. Define

$$J_{\mu\nu} \equiv \int \epsilon^\alpha{}_{\sigma\rho\mu} \epsilon_{\alpha\sigma'\rho'\nu} q_1^\sigma q_2^\rho q_1^{\sigma'} q_2^{\rho'} \frac{d^3q_1}{2\omega_1} \frac{d^3q_2}{2\omega_2} \delta^4(Q - q_1 - q_2 - q_3) \tag{A8}$$

so that

$$I(Q^2) = \frac{1}{3} \int \frac{d^3q_3}{2\omega_3} q_3^\mu q_3^\nu J_{\mu\nu}\,. \tag{A9}$$

$J_{\mu\nu}$ is a function of $K \equiv Q - q_3$ and $K^\mu J_{\mu\nu} = K^\nu J_{\mu\nu} = 0$ so that $J_{\mu\nu} = J(K^2)(K^2 g_{\mu\nu} - K_\mu K_\nu)$. The function $J(K^2)$ is easily evaluated,

$$\begin{aligned} J(K^2) &= \frac{1}{3K^2} J_\mu{}^\mu = \frac{2}{3K^2} \int [q_1^2 q_2^2 - (q_1\cdot q_2)^2] \frac{d^3q_1}{2\omega_1} \frac{d^3q_2}{2\omega_2} \delta^4(K - q_1 - q_2) \\ &= -\frac{\pi}{12K^4} \lambda(K^2, m_1^2, m_2^2)^{3/2}\,, \end{aligned} \tag{A10}$$

where we used the two-body phase space formula

$$\int \frac{d^3k_1}{2\omega_1} \frac{d^3k_2}{2\omega_2} \delta^4(K - k_1 - k_2) = \frac{\pi}{2K^2} [\lambda(K^2, m_1^2, m_2^2)]^{1/2}\,, \tag{A11}$$

with $\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. Then

$$\begin{aligned} I(Q^2) &= \frac{1}{3} \int \frac{d^3q_3}{2\omega_3} [K^2 m_3^2 - (q_3\cdot K)^2] J(K^2) \\ &= \frac{1}{3} \int d\alpha^2 \int \frac{d^3q_3}{2\omega_3} [\alpha^2 m_3^2 - (q_3\cdot K)^2] J(\alpha^2) \delta(\alpha^2 - K^2) \\ &= \frac{1}{3} \int d\alpha^2 J(\alpha^2) \int\int \frac{d^3q_3}{2\omega_3} \frac{d^3q'}{2\omega'} [\alpha^2 m_3^2 - (q_3 q')^2] \delta^4(q_3 + q' - Q)\,. \end{aligned} \tag{A12}$$

Here we introduced[29] an auxiliary four-vector q' with mass α^2 and $\omega' \equiv [q'^2 + \alpha^2]^{1/2}$. Once again, we have a two-body phase space. Using (A10) we obtain

$$\begin{aligned} I(Q^2) &= \frac{1}{3} \frac{\pi}{2Q^2} \int d\alpha^2 J(\alpha^2) [\lambda(Q^2, \alpha^2, m_3^2)]^{1/2} [\alpha^2 m_3^2 - \tfrac{1}{4}(Q^2 - m_3^2 - \alpha^2)^2] \\ &= \frac{\pi^2}{288 Q^2} \int \frac{d\alpha^2}{\alpha^4} \lambda(\alpha^2, m_1^2, m_2^2)^{3/2} \lambda(Q^2, \alpha^2, m_3^2)^{3/2}\,. \end{aligned} \tag{A13}$$

Energy-momentum conservation fixes the integration limits to be

$$(m_1 + m_2)^2 \le \alpha^2 \le (W - m_3)^2\,,$$

where $W \equiv \sqrt{Q^2}$. With the change of variable

$$\alpha^2 = \tfrac{1}{2}[(W - m_3)^2 + (m_1 + m_2)^2] + \tfrac{1}{2}[(W - m_3)^2 - (m_1 + m_2)^2]\cos\phi\,, \tag{A14}$$

we obtain

$$I(W^2) = \frac{\pi^2}{288 W^2} F(W^2) \int_0^\pi d\phi \frac{\sin^4\phi}{(1 + \epsilon\cos\phi)^2} (1 - \epsilon'\cos\phi)^{3/2} (1 + \epsilon''\cos\phi)^{3/2}\,, \tag{A15}$$

where

$$F(W^2) \equiv \tfrac{1}{32}[(W-m_3)^2-(m_1+m_2)^2]^4[(W-m_3)^2-(m_1{}^2+m_2{}^2-6m_1m_2)]^{3/2}[W^2+6Wm_3+m_3{}^2-(m_1+m_2)^2]^{3/2} \times[(W-m_3)^2+(m_1+m_2)^2]^{-2}, \tag{A16}$$

$$\epsilon \equiv \frac{(W-m_3)^2-(m_1+m_2)^2}{(W-m_3)^2+(m_1+m_2)^2}, \tag{A17}$$

$$\epsilon' \equiv \frac{(W-m_3)^2-(m_1+m_2)^2}{[W^2+6Wm_3+m_3{}^2-(m_1+m_2)^2]}, \tag{A18}$$

$$\epsilon'' \equiv \frac{(W-m_3)^2-(m_1+m_2)^2}{[(W-m_3)^2-(m_1{}^2+m_2{}^2-6m_1m_2)]}. \tag{A19}$$

This is our final expression for $I(W^2)$. The integral in (A15) is amenable to numerical methods.

Near threshold, a remarkably good approximation is to set $\epsilon = \epsilon' = \epsilon'' = 0$. Then

$$I(W^2) = \tfrac{2}{3}\pi^3\left(\frac{(m_1m_2m_3)^3}{(m_1+m_2+m_3)}\right)^{1/2}(W-m_1-m_2-m_3)^4. \tag{A20}$$

Putting this into Eq. (A7) we obtain the cross section near threshold,

$$\sigma \simeq \frac{\alpha}{24\pi}|F^{3\pi}|^2\left(\frac{(m_1m_2m_3)^3}{(m_1+m_2+m_3)^9}\right)^{1/2}(W-m_1-m_2-m_3)^4. \tag{A21}$$

To estimate the $e^+e^- \to \gamma \to \pi^+\pi^-\pi^0$ cross section we use the experimental value of $|F^\pi| \simeq (\alpha/\pi)(3/2\mu)$ in Eq. (1) to obtain $|F^{3\pi}| \simeq (3e/4\pi^2)(1/\mu^3)$. Setting $m_1 = m_2 = m_3 = \mu =$ pion mass, we get

$$\sigma = 3.7\times10^{-10}\frac{1}{\mu^2}\left(\frac{W-3\mu}{\mu}\right)^4 = (0.7\times10^{-35}\,\text{cm}^2)\left(\frac{W-3\mu}{\mu}\right)^4.$$

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[1]This theorem has already been presented by S. L. Adler, B. W. Lee, S. B. Treiman, and A. Zee, Phys. Rev. D 4, 3497 (1971). It has been discovered independently by M. V. Terentiev, Soviet Phys. JETP Letters 14, 140 (1971).

[2]D. G. Sutherland, Phys. Letters 23, 384 (1966).

[3]S. L. Adler, Phys. Rev. 177, 2426 (1969).

[4]J. S. Bell and R. Jackiw, Nuovo Cimento 60, 47 (1969).

[5]An exposition of the theory of PCAC anomalies is given by S. L. Adler, in *Lectures on Elementary Particles and Quantum Field Theory,* 1970 Brandeis University Summer Institute in Theoretical Physics, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1971), Vol. 1.

[6]S. L. Adler and W. A. Bardeen, Phys. Rev. 182, 1517 (1969).

[7]W. A. Bardeen, Phys. Rev. 184, 1848 (1969); R. W. Brown, C.-C. Shih, and B.-L. Young, *ibid.* 186, 1491 (1969). All possible PCAC anomalies, including the ones discussed here, are contained in a general (and somewhat formidable) expression given by Bardeen. Many of these anomalies apparently have had to be rediscovered.

[8]The general expression of Bardeen has been reconstructed recently in a most elegant manner by J. Wess and B. Zumino, Phys. Letters 37B, 95 (1971).

[9]J. Schwinger, Phys. Rev. 82, 664 (1951). See also Y. Nambu, Progr. Theoret. Phys. 5, 82 (1950). This powerful technique has been recently revived and used to treat the $2\gamma \to 3\pi$ problem [R. Aviv and R. F. Sawyer, Phys. Rev. D 4, 451 (1971)] and photon splitting [S. L. Adler, Ann. Phys. (N.Y.) 67, 599 (1971).

[10]There is no anomaly in the Ward identity implicitly used here since pseudoscalar vertices are involved.

[11]S. Weinberg, Phys. Rev. Letters 17, 168 (1966).

[12]S. L. Adler, Phys. Rev. 137, B1022 (1965); 139, B1638 (1965).

[13]In any theory respecting gauge invariance, PCAC and assumption (c), c_1, c_2, and h must have the value we deduced. The σ model provides an example of how a dynamics thus constrained leads to these values. See Ref. 7 and T. F. Wong, Phys. Rev. Letters 27, 1617 (1971).

[14]T. P. Cheng and R. F. Dashen, Phys. Rev. D 4, 1561 (1971); S. L. Adler and F. J. Gilman, Phys. Rev. 152, 1460 (1966); S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento 40, 1171 (1965).

[15]It is likely that these two difficulties are related.

[16]An excellent review of the $\eta \to 3\pi$ problem is given by J. S. Bell and D. G. Sutherland, Nucl. Phys. B4, 315 (1968). See also K. Wilson, Phys. Rev. 179, 1499 (1969); E. S. Abers, D. A. Dicus, V. L. Teplitz, Phys. Rev. D 3, 485 (1971).

[17]R. Dashen, Phys. Rev. 182, 1245 (1969). Dashen's mass formula holds in the SU(3)⊗SU(3)-symmetric limit and follows from the assumption that J_μ^{em} is a U-spin singlet and assumption (c). The assumptions are not

mild, but the result is in violent agreement with experiment.

[18]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[19]K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters 16, 255 (1966). These authors dropped the "surface term" $k^\mu p^\nu P^{\text{CSGI}}_{\mu\nu\lambda}$ and therefore arrived at an incorrect relation between $\gamma \to \pi^+\pi^-\pi^0$ and $\pi^0 \to 2\gamma$. In an Erratum [Phys. Rev. Letters 16, 384 (1966)] they estimated the "surface term" by appealing to a vector-dominance model. Quite aside from its unreliability, this approach is contrary to the philosophy expressed in the present work.

[20]Adler and Bardeen (Ref. 6).

[21]The assertion has been verified explicitly to second order in Ref. 6 and by S. L. Adler, R. W. Brown, T. F. Wong, and B.-L. Young, Phys. Rev. D 4, 1787 (1971). We are aware that the validity of this assertion is not unchallenged.

[22]These statements are all predicated on the belief that any amplitude is completely described by the infinite collection of Feynman diagrams given in the Dyson series. At issue is the question whether or not nonperturbative effects are automatically included.

[23]For $\pi \to 2\gamma$, see J. Steinberger, Phys. Rev. 76, 1180 (1969). For $\gamma \to 3\pi$, see K. Itabashi, M. Kato, K. Nagakawa, and G. Takeda, Progr. Theoret. Phys. (Kyoto) 24, 529 (1960).

[24]$\sqrt{2} f = f_\pi$, in terms of which the π-π scattering length is given by $a_0 = (7/32\pi)(m_\pi/f_\pi{}^2)$. For details see, for example, S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. 41, 531 (1969).

[25]A. Donnachie and G. Shaw, Ann. Phys. (N.Y.) 37, 333 (1966).

[26]For previous theoretical work on $\gamma \to 3\pi$, see M. G. Miller, Phys. Rev. D 4, 769 (1971) and references cited therein. These calculations are contrary to the spirit of the present work.

[27]S. Okubo and B. Sakita, Phys. Rev. Letters 11, 50 (1963).

[28]Wess and Zumino (Ref. 8).

[29]G. Källén, *Elementary Particle Physics* (Addison-Wesley, Reading, Mass., 1971), p. 190.